ANALYSIS I: Problem Sheet 1

Real Numbers, Arithmetic, Order, (Real) Modulus.

1. Prove, *from the given Axioms for the Real Numbers*, that, for real numbers a, b, c, d:

   (a) $a(bc) = c(ba)$;

   (b) $-(a + b) = (-a) + (-b)$;

   (c) if $ab = ac$ and $a \neq 0$, then $b = c$;

   (d) if $a < b$ and $c < d$ then $a + c < b + d$;

   (e) if $a \leqslant b$ and $c \leqslant d$ then $a + c = b + d$ only if $a = b$ and $c = d$.

   [*Try to write out detailed answers, justifying each line of your argument by appealing to one of the axioms.*]

2. Prove the following assertions, for real numbers a, b, c:

   (a) if $a < b$, then $ac > bc$ if and only if $c < 0$;

   (b) $a^2 + b^2 = 0$ if and only if $a = b = 0$;

   (c) $a^3 < b^3$ if and only if $a < b$.

   [*Less detailed answers are required than in Q.* 1, *but you should try to justify each step using axioms or results which have already been proved from the axioms.*]

3. (a) Prove that $(a^m)^{-1} = (a^{-1})^m$, for all $a \in \mathbb{R} \setminus \{0\}$ $(m = 1, 2, \ldots)$.

   (b) Prove that $a^{k+1} = a^k a$ for $a \neq 0$ and $k = -1, -2, -3, \ldots$.

   (c) Derive the law of indices: $a^m a^n = a^{m+n}$ for $a \neq 0$ and $m, n \in \mathbb{Z}$.

4. [*In this question you may use familiar results about arithmetic and order in the real numbers. Here (and from now on) you are not expected to justify each line of your argument by citing axioms or properties derived from the axioms.*] For $n = 1, 2, 3, \ldots$, let

$$a_n := \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad b_n := \left(1 + \frac{1}{n}\right)^{n+1}.$$

   (a) Show that the inequality $a_n \leqslant a_{n+1}$ can be rearranged as

$$\left(\frac{n\,(n+2)}{(n+1)^2}\right)^{n+1} \geqslant \frac{n}{n+1}.$$

   By applying Bernoulli's inequality to the left-hand side, verify this inequality.

   (b) Show that $b_{n+1} \leqslant b_n$ for all n.

   (c) Note that $a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n$ for all n. Deduce that $a_n < 3$ for all n.

   [*The significance of the result in this exercise will be revealed later.*]

turn over/ ...

5. Define $\max(a, b)$, the *maximum* of two real numbers a and b, saying which axiom(s) show that your specification is well-defined. Using your definition prove that $\max(a, b) = \frac{1}{2}(a+b)+\frac{1}{2}|a-b|$, and write down an analogous formula for $\min(a, b)$.

Points to ponder [*For thinking through, by yourself or in discussion. Written answers not sought.*]

A. Why, in the arithmetic of $\mathbb{R}$, must division by 0 be forbidden?

B. What does Q. 2(c) imply about the possibility of real solutions to the equation $x^2 + 1 = 0$?

C. Should ∞ (infinity) be regarded as a real number?