**Analysis II: Continuity and Differentiability  Sheet 1  HT 2020**

**1**. Suppose that $f$, $g$ are real-valued functions defined on some interval $(a, b)$ containing the point $x_0$, and that $\lim_{x\to x_0} f(x) = k$ and $\lim_{x\to x_0} g(x) = l$.

(a) If $k > 0$, prove that there is a positive number $\delta > 0$ such that $f(x) \geq \frac{1}{2}k$ whenever $0 < |x - x_0| < \delta$.

(b) Prove that: $\lim_{x\to x_0} (f(x) + g(x)) = k + l$; $\lim_{x\to x_0} f(x)g(x) = kl$ and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{k}{l}$ provided $l \neq 0$.

(c) Prove that if $f(x) < g(x)$ for all $x \in (x_0 - \delta, x_0 + \delta)$ (for some $\delta > 0$) then $k \leq l$. In this case is it true that $k < l$?

[*Hint: mimic what was done for limits of sequences.*]

**2**. Prove carefully that if $f(x) \to y_0$ as $x \to x_0$, $g(y) \to l$ as $y \to y_0$, and if $g(y_0) = l$, then $g(f(x)) \to l$ as $x \to x_0$.

**3**. (a) Find a function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at the points of set $\{1/n : n \in \mathbb{Z}_+\} \cup \{0\}$ but is continuous everywhere else.

(b) Find a function $g : \mathbb{R} \to \mathbb{R}$ which is discontinuous at the points of set $\{1/n : n \in \mathbb{Z}_+\}$ but is continuous everywhere else.

**4**. (a) Prove that if $f$ is continuous at $x_0$, then $|f|$ is continuous at $x_0$. Is the converse statement also true? That is $|f|$ is continuous at $x_0$, is $f$ necessarily continuous at $x_0$?

(b) For $x$, $y \in \mathbb{R}$, show that $\max\{x, y\} = \frac{1}{2}(x + y + |x - y|)$, and using the relation that $\min\{x, y\} = -\max\{-x, -y\}$ derive a similar expression for $\min\{x, y\}$.

(c) Prove that if $f$ and $g$ are continuous at $x_0$, then $\max\{f, g\}$ and $\min\{f, g\}$ are continuous at $x_0$.

**5**. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying

$$f(x + y) = f(x) + f(y) \qquad \text{for any } x, y \in \mathbb{R} .$$

Prove that $f(x) = cx$ (for every $x \in \mathbb{R}$) for some constant $c$.

*[Hint: First show that $f(0) = 0$, $f(-x) = -f(x)$ for every $x$, and use induction to show that $f(nx) = nf(x)$, then show that $f(rx) = rf(x)$ for any rational number $r = n/m$.]*

**6**. Suppose that $E \subseteq \mathbb{R}$ is bounded, non-empty, and suppose that $E$ possesses the following property: if $x, y \in E$ and $z \in \mathbb{R}$ is between $x$ and $y$ (that is $x \leq z \leq y$ or $y \leq z \leq x$), then $z \in E$. Show carefully that there are two real numbers $a \leq b$ such that $E = (a, b)$, $(a, b]$, $[a, b)$ or $[a, b]$.