Analysis II: Continuity and Differentiability   Sheet 8   HT 2020

1. Suppose that the real-valued function f is such that the $(n-1)^{\text{th}}$ derivative of f exists and is continuous on $[0,h]$ (where $h>0$) and the n-th derivative exists on $(0,h)$. Consider the function $G:[0,h]\to\mathbb{R}$ defined by

$$G(t)=F(t)-\left(\frac{h-t}{h}\right)^{p}F(0),$$

where $F:[0,h]\to\mathbb{R}$ is given by

$$F(t)=f(h)-f(t)-(h-t)f'(t)-\cdots-\frac{(h-t)^{n-1}}{(n-1)!}f^{(n-1)}(t)$$

and p is a constant. By considering the derivative of G and choosing p appropriately, prove that there exist θ_1, θ_2 such that

$$f(h)=f(0)+hf'(0)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0)+S_n$$

where

$$S_n=\frac{h^n}{n!}f^{(n)}\left(\theta_1 h\right)=\frac{h^n}{(n-1)!}(1-\theta_2)^{n-1}f^{(n)}\left(\theta_2 h\right).$$

2. Assume that $f:\mathbb{R}\to\mathbb{R}$ is such that both f' and f'' exist for all $x\in\mathbb{R}$. Taylor's Theorem tells us that, for each $a,h\in\mathbb{R}$ there is a $\theta\in(0,1)$ such that

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2}f''(a+\theta h)\ .$$

Assume further that on the interval $[0,2]$ the inequalities $|f(x)|\le 1$ and $|f''(x)|\le 1$ hold.

Write down the Taylor expansions of $f(0)$ and $f(2)$ about the point $x\in[0,2]$, using the above form of Taylor's Theorem, with a remainder involving f''. Hence prove that for all $x\in[0,2]$ we have $|f'(x)|\le 2$.

3. Compute the Taylor expansion about 0 for $(1+x)^{-1/2}$, and use it to evaluate

$$\sum_{n=0}^{\infty}\binom{2n}{n}\left(-\frac{6}{25}\right)^{n}\ .$$

4. By using Taylor's Theorem and the Identity Theorem, prove that

$$\sqrt{1+x}=1+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n}\frac{(1-\frac{1}{2})(2-\frac{1}{2})\cdots\big((n-1)-\frac{1}{2}\big)}{(n-1)!}x^n$$

for $-1<x\le 1$.

5. Suppose that f is twice differentiable on $[a, b]$, and $f'(a) = f'(b) = 0$, show that there is $\xi \in (a, b)$ such that

$$|f''(\xi)| \geq \frac{4}{(b-a)^2} |f(b) - f(a)| \; .$$

[*Hint: By the triangle inequality to obtain*

$$|f(b) - f(a)| \leq |f(b) - f((b+a)/2)| + |f(a) - f((b+a)/2)|.$$

Apply Taylor's formula to f *at* a *and* b].