Groups and Group Actions, Sheet 1, HT2020
Pudding

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

I'm not going to give full details/proofs for every question, but hopefully I'll give something useful against which you can compare your thinking.

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P1. Let G be a finite group of even order. Must it contain an element of order 2?

The answer turns out to be yes.

Here is a hint for a possible strategy.

Each element of G has an inverse. Sometimes g is its own inverse (for example when $g = e$). When it isn't, we can pair it up with its inverse.

P2.

(i) Is there a group in which no non-identity element is its own inverse?

(ii) Is there a group in which every non-identity element is its own inverse?

(i) Yes. For example, consider C_5. Here every non-identity element has order 4, and hence is not its own inverse.

(ii) Yes. For example, consider C_2. In fact there are many more examples. We'll return to groups in which every non-identity element is its own inverse on a future sheet, but you might like to start exploring their properties now.

P3. How many ways are there to complete the following grid so that it is the Cayley table of a group?

$*$	e	a	b	c
e				
a				
b				
c				

We know what the first row and column must be, so we can fill in some entries immediately.

$*$	e	a	b	c
e	e	a	b	c
a	a			
b	b			
c	c			

There are then three possibilities for $a * a$, namely b, c and e (note that it cannot be a, because each element must appear exactly once in each row and column).
Case 1: $a * a = b$. Then we find the remaining entries are all determined (using the 'every element in every row and column' property), and we get the table

$*$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

This does correspond to a group table. Perhaps the easiest way to see this is to note that it's the group table of C_4 (with $a^2 = b$ and $a^3 = c$)—this is probably more convenient than checking associativity by hand.
Case 2: $a * a = c$. As in Case 1, it turns out that there are no more decisions to make. We get the table

$*$	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

which again corresponds to C_4, this time with $a^2 = c$ and $a^3 = b$.
Case 3: $a * a = e$. Then we can fill in some entries, but when we reach

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c		
c	c	b		

we have to make another decision.
Case 3a: $a * a = e$ and $b * b = e$. Then everything else is determined (there is no more choice), and we get

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

which corresponds to $C_2 \times C_2$.
Case 3b: $a * a = e$ and $b * b = a$. Then again everything else is determined, leading to

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

which is the group table of C_4, with b as a generator and $b^2 = a$, $b^3 = c$.

So there are just two possible groups of order 4, up to isomorphism, and they are both Abelian.