Groups and Group Actions, Sheet 2, HT2020

Pudding

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

I'm not going to give full details/proofs for every question, but hopefully I'll give something useful against which you can compare your thinking.

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P1. What is the largest possible order of an element in S_5? In S_9? What is the smallest n for which an element of largest order in S_n must have three cycles (in disjoint cycle notation)?

We use disjoint cycle notation, because the order of a permutation written in disjoint cycle notation is the least common multiple of the lengths of the cycles.

In S_5, we can have an element of order 6, for example $(1\ 2\ 3)(4\ 5)$. By thinking about the possible cycle types, we find that this is the largest possible order.

In S_9, we can have an element of order 20, for example $(1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9)$. This is the largest possible.

In S_9, an element of largest order can have two cycles. We can check values up to 8. It is clear that if $n \leqslant 4$ then an element of largest order in S_n does not have three cycles, and we have seen that in S_5 there is an element of largest order with two cycles, so we concentrate on checking $6 \leqslant n \leqslant 8$.

$n = 6$: the largest possible order is 6, which can be obtained with $(1\ 2\ 3\ 4\ 5\ 6)$.

$n = 7$: the largest possible order is 12, which can be obtained with $(1\ 2\ 3\ 4)(5\ 6\ 7)$.

$n = 8$: the largest possible order is 15, which can be obtained with $(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$.

So the smallest n for which an element of largest order in S_n must have three cycles is at least 10.

Checking $n = 10$, we find that the largest order of an element of S_{10} is 30, which is achieved only with three cycles (for example $(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)(9\ 10)$).

P2. For each of the following sets X_i, determine whether every permutation in S_n can be written as a product of elements of X_i.

(i) $X_1 = \{(j\ j+1) : 1 \leqslant j < n\}$

(ii) $X_2 = \{(1\ k) : 1 < k \leqslant n\}$

(iii) $X_3 = \{(1\ 2), (1\ 2\ \ldots\ n)\}$.

(i) Every permutation in S_n can be written as a product of elements of X_1.

We already know that every permutation can be written as a product of transpositions, so it suffices to show that every transposition can be written as a product of elements of X_1.

We can build up, for example we can obtain transpositions of the form $(j\ j+2)$ via $(j\ j+1)(j+1\ j+2)(j\ j+1)$.

One nice way to think about this is as conjugation, see Q3 of the main course on this sheet.

(ii) Again, every permutation in S_n can be written as a product of elements of X_2.

Again it is enough to show that every transposition can be written as a product of elements of X_2, and $(j\ k) = (1\ j)(1\ k)(1\ j)$. (This is another instance of conjugation as in Q3.)

(iii) Perhaps surprisingly, every permutation in S_n can be written as a product of elements of X_3.

Conjugating $(1\ 2)$ by a suitable power of $(1\ 2\ \ldots\ n)$ allows us to write every element of X_1 as a product of elements of X_3, and then we are done by (i).

P3. If we choose a permutation in S_n at random, with all permutations being equally likely, what is the probability that our chosen permutation has exactly one 1-cycle?

To find the probability, we can find the number of permutations in S_n that have exactly one 1-cycle, and then divide by the number of permutations in S_n, that is, divide by $n!$.

There are n possible elements that could be in the 1-cycle. We then want permutations of the remaining $n-1$ elements that do not fix any elements. These are called *derangements*. One elegant way to count derangements is called the *inclusion-exclusion principle*, which you might have seen on Sheet 1 of the Probability course last term.