In this sheet you may assume standard properties of $\sin, \cos, \exp, \log$, *as established in lectures and on Sheet 3.*

1. Discuss the existence of the following improper integrals.

(i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}}$, (ii) $\displaystyle\int_0^1 \frac{dx}{\sin x}$, (iii) $\displaystyle\int_0^\infty \frac{dx}{1+x^{3/2}}$, (iv) $\displaystyle\int_2^\infty \frac{dx}{x\log x}$.

2. Let $m, n \geqslant 0$ be nonnegative integers. Show that

$$\lim_{\varepsilon\to 0} \varepsilon^{m+1}(\log\varepsilon)^n = 0$$

(*Hint: you may find it helpful to set* $\varepsilon = e^{-t}$ *and to consider the series expansion of* e^t.)

Using induction on n or otherwise, show that the improper integral

$$\int_0^1 x^m (\log x)^n dx$$

exists, and give a closed-form expression for it.

3. Suppose that $f : [1,\infty) \to \mathbb{R}$ is a continuous function with the property that $f(x) \to 0$ as $x \to \infty$.

(i) Show that $\lim_{X\to\infty} \frac{1}{X}\int_1^X f(x)dx = 0$.

(ii) Does the improper integral $\int_1^\infty \frac{f(x)}{x}dx = \lim_{X\to\infty}\int_1^X \frac{f(x)}{x}dx$ necessarily exist?

4. Show that $\frac{1}{x^x}$ is continuous on $[0,1]$, and that its integral on this range is equal to $\sum_{n=1}^\infty \frac{1}{n^n}$. *Hint: write* $\frac{1}{x^x} = \sum_{n=0}^\infty \frac{1}{n!}(-x\log x)^n$.

5. Does the improper integral $\int_{2\pi}^\infty \frac{\sin x}{x}dx$ exist?

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