

Can You Do These? Preparing for QFT

Here are a few questions on classical and non-relativistic quantum mechanics. The questions are taken from recent OU Physics exam papers – hence the funny numbers! If you can do them then you know enough about these topics for the QFT course. If you need to do a little bit of revision then most text books on quantum mechanics and classical mechanics will provide what you need. I like:

Quantum Physics by Gasiorowicz, available as a free pdf download, see in particular the chapter on Operator Methods; and

Classical Mechanics by Kibble and Berkshire, also available as a free pdf download, see in particular the chapters on Lagrangian Mechanics and Hamiltonian Mechanics.

The QFT course will also assume that you have done a basic course on relativistic quantum mechanics. If you haven't, or need to revise the topic, then read the notes *Relativistic QM for MMathPhys* which are also available here on the Course Materials tab.

2. What is a constant of motion? State and prove Noether's theorem. What is the significance of symmetries of the Lagrangian?

Consider a system of three particles, of masses m_1 , m_2 and m_3 , moving and interacting in three-dimensional space (x, y, z) . The position of each particle is labelled by $\mathbf{r}_i \equiv (x_i, y_i, z_i)$, for $i = 1, 2, 3$. The Lagrangian describing the system is

$$L = \sum_{i=1}^3 \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 - \sum_{i=1}^3 \frac{1}{2} m_i \omega^2 x_i^2 - \cos(\lambda(m_1 - m_2)t) U_{12}(y_1 - y_2) - U_{13}(y_1 - y_3) - V(z_1 - z_2)$$

You may regard U_{12} , U_{13} and V as generic functions with no special properties. ω and λ are constants that are independent of the masses m_i .

- (a) In the case that $m_1 \neq m_2 \neq m_3$, identify and describe the conserved quantities.
- (b) How many additional conserved quantities exist in the cases that, first, $m_1 = m_2$ and secondly, $m_1 = m_2 = m_3$?
- (c) In the latter case where all masses are equal, write explicit expressions for these additional conserved quantities and interpret them.

7. The Hamiltonian of a one-dimensional harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 = \frac{1}{2} \hbar \omega (a a^\dagger + a^\dagger a)$$

where all the symbols have their usual meanings, and a is the lowering operator

$$a = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{m\omega}{\hbar}} x + i \frac{p}{\sqrt{\hbar m \omega}} \right).$$

The action of a on the energy eigenket $|n\rangle$ is $a|n\rangle = \sqrt{n}|n-1\rangle$ for all $n \geq 0$. Find expressions for x and p in terms of a and $a^\dagger$. The oscillator is in the state

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left(|n-1\rangle e^{-i(n-\frac{1}{2})\omega t} + |n\rangle e^{-i(n+\frac{1}{2})\omega t} \right).$$

Calculate the expectation values of H , x and p for this state.

Calculate the amplitude of oscillation of a classical oscillator of this frequency and energy $E = \langle \psi(t) | H | \psi(t) \rangle$ and show that it differs from your result for $\langle \psi(t) | x | \psi(t) \rangle$ by a factor independent of n .

Find an expression for the expectation value of x^2 in the state $|\psi(t)\rangle$. Explain briefly why this state has a time-independent value for $\langle \psi(t) | x^2 | \psi(t) \rangle$, whereas a superposition of more than two consecutive energy eigenstates would not.

3. Starting with the action principle, derive the Euler-Lagrange equations,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0.$$

Define the canonical momentum, p_i , and the Hamiltonian, $H(\mathbf{q}, \mathbf{p}, t)$. State Hamilton's equations and derive them using the action principle.

Explain what is meant by *phase space*, and state and prove Liouville's theorem.

Illustrate Liouville's theorem by explicitly verifying it for motions in the phase space of the harmonic oscillator Hamiltonian,

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}.$$