Relativistic QM for MMathPhys QFT MT 2020*

Professor John Wheater

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The QFT course will assume that you have already done an introductory course on Relativistic Quantum Mechanics (RQM) and so know a little about relativistic wave equations. These notes are provided in case you haven't, or you would like to do some revision. They are based on part of an optional 3rd year course that I give called *Advanced Quantum Mechanics*. The lectures for this course in academic year 2019-20 were recorded and are available at Weblearn. Once you are registered in the Oxford system you should have access to these recordings; if you have a problem please email me at john.wheater@physics.ox.ac.uk. The RQM part of the course starts about half way through the lecture given on 18 February 2020. You can learn all you need for the QFT course by watching the lectures and using these notes; the Homework exercises scattered through the text are worth doing – most are very simple.

The earlier lectures of the *Advanced Quantum Mechanics* are on non-relativistic scattering theory; if you have not previously done a course on this topic you might find those lectures interesting too, though at the moment I don't have notes available for them.

There are many good books on all this material. For an introduction to RQM I recommend *Modern Quantum Mechanics* by Sakurai, which is available on line from within the Oxford University library system or as a paperback. A more advanced book on RQM is the classic *Relativistic Quantum Mechanics* by Bjorken and Drell, which is available here.

If you want to read up on non-relativistic scattering theory then I recommend *Quantum Mechanics* by Schiff, which is available here.

1 Relativistic QM: The Klein-Gordon Equation

1.1 A relativistic wave equation

The Schrödinger Equation for a free particle is based on the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} \tag{1}$$

which is non-relativistic so the resulting wave equation is not Lorentz covariant. To make a covariant wave equation we must use the relativistic energy-momentum relationship

$$E^2 = \mathbf{p}^2 + m^2, \qquad (c = 1) \tag{2}$$

or, maybe,

$$E = \sqrt{\mathbf{p}^2 + m^2}. \tag{3}$$

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If we use (3) then we get an evil-looking wave equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\sqrt{-\hbar^2\boldsymbol{\nabla}^2 + m^2}\right)\psi. \tag{4}$$

It is not at all clear how the square root of a differential operator should be treated, so instead Schrodinger considered the second order wave equation built on (2)

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = (-\hbar^2\boldsymbol{\nabla}^2 + m^2)\psi \tag{5}$$

which is properly called the relativistic Schrodinger equation. However Schrödinger did not like the consequences and quietly dropped it; Klein and Gordon rehabilitated the equation and usually it is called the Klein-Gordon (or KG) equation. So what are the consequences that made Schrödinger uneasy?

The KG equation has plane wave solutions of the usual form

$$\psi(\mathbf{x}, t) = e^{-i(Et-\mathbf{p}.\mathbf{x})/\hbar}\,\psi_0, \tag{6}$$

where ψ_0 is a constant. Substituting this form into the KG differential equation we do indeed find that

$$E^2 = \mathbf{p}^2 + m^2, \qquad (c = 1). \tag{7}$$

Now there are two square roots, plus and minus, so the general solution for a state of momentum $\mathbf{p}$ is

$$\psi(\mathbf{x}, t) = a_{\mathbf{p}}e^{-i(E_{\mathbf{p}}t-\mathbf{p}.\mathbf{x})/\hbar} + b_{\mathbf{p}}e^{-i(-E_{\mathbf{p}}t-\mathbf{p}.\mathbf{x})/\hbar} \tag{8}$$

where $E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 + m^2}$, and $a_{\mathbf{p}}, b_{\mathbf{p}}$ are constants. We can have positive and negative energy states in contrast to the Schrödinger equation where states have $E = \frac{\mathbf{p}^2}{2m} > 0$. In fact it appears that we can have states of arbitrarily negative energy which raises the issue of existence of a ground (lowest energy) state. For example a charged KG particle would be able to emit photons and transition to a lower energy state *ad infinitum*. Also the KG equation does not *require* that ψ is complex as the Schrodinger equation does, so it is not clear that ψ is really a quantum mechanical wavefunction. It is instructive to calculate the conserved charge and current; we have

$$\psi^*(-\hbar^2\frac{\partial^2\psi}{\partial t^2}) - (-\hbar^2\frac{\partial^2\psi^*}{\partial t^2})\psi = \psi^*(-\hbar^2\boldsymbol{\nabla}^2\psi + m^2\psi) - (-\hbar^2\boldsymbol{\nabla}^2\psi^* + m^2\psi^*)\psi \tag{9}$$

so

$$-\hbar^2\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t} - \frac{\partial\psi^*}{\partial t}\psi)\right) = -\hbar^2\boldsymbol{\nabla}.\left(\psi^*\boldsymbol{\nabla}\psi - (\boldsymbol{\nabla}\psi^*)\psi\right). \tag{10}$$

It follows that there is a density ρ and a current $\mathbf{j}$ satisfying the conservation law

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}.\mathbf{j} = 0, \tag{11}$$

and given by

$$\begin{aligned} \rho &= \frac{i\hbar}{2m}\left(\psi^*\frac{\partial\psi}{\partial t} - \frac{\partial\psi^*}{\partial t}\psi)\right), \\ \mathbf{j} &= -\frac{i\hbar}{2m}\left(\psi^*\boldsymbol{\nabla}\psi - (\boldsymbol{\nabla}\psi^*)\psi\right). \end{aligned} \tag{12}$$

We have chosen the prefactor so that both quantities are real and the current is the same as the Schrodinger current in the non-relativistic limit. The charge and current are real but, because of the time derivative in its definition, our new ρ is not positive definite. Therefore the quantity $\int d^3\mathbf{x}\,\rho$ cannot be a probability; on the other hand $\psi^*\psi$, which could be a probability density as it is positive definite, is not conserved! We conclude that there are some problems with interpreting the KG equation as a quantum mechanical wave equation with the usual postulates of quantum mechanics.

1.2 Covariant form of the KG equation

Although the KG equation is certainly relativistic we haven't written it in a form which makes the Lorentz transformation properties manifest. It's very convenient to have a Lorentz covariant formulation both for the KG and for the Dirac equation. We define the metric and various four vectors as follows[1]

$$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) \tag{13}$$

$$x^\mu = (t,\mathbf{x}) \tag{14}$$

$$x_\mu = x^\nu g_{\mu\nu} = (t,-\mathbf{x}) \tag{15}$$

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = (\frac{\partial}{\partial t},-\boldsymbol{\nabla}) \tag{16}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t},\boldsymbol{\nabla}) \tag{17}$$

$$p^\mu = i\hbar\partial^\mu = (i\hbar\frac{\partial}{\partial t},-i\hbar\boldsymbol{\nabla}) = (E,\mathbf{p}) \tag{18}$$

By convention repeated indices – one upper and one lower are summed over – this process is called *contraction* of indices. We see that, for example, $x^\mu x_\mu = t^2 - \mathbf{x}.\mathbf{x}$ and $\partial^\mu\partial_\mu = \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2$. These quantities are Lorentz scalars because they have no uncontracted indices.

Homework 1.1: Compute $\partial^\nu x_\nu$, $\partial^\nu(x^\mu a_\mu)$ (here a_μ is independent of x_μ) and $x^\nu\partial_\nu(x^\mu x_\mu)$. Be sure to express your results in covariant notation.

In covariant form the KG equation in units with $\hbar = 1$, which we will use from now on, is

$$\left(\partial^\mu\partial_\mu + m^2\right)\psi = 0. \tag{19}$$

As the KG operator is Lorentz-invariant (ie takes the same form in different inertial frames), the wavefunction ψ is a Lorentz scalar.

Homework 1.2: Prove that $\partial^\mu\partial_\mu \to \partial'^\mu\partial'_\mu$ under a Lorentz boost along the z-axis.

The conserved four-current is

$$j^\mu = (\rho,\mathbf{j}) = \frac{i}{2m}\left(\psi^*\partial^\mu\psi - (\partial^\mu\psi^*)\psi\right) \tag{20}$$

and satisfies the conservation equation $\partial_\mu j^\mu = 0$.

Homework 1.3: Check that these expressions for the current reproduce (11) and (12). Compute $\partial_\mu j^\mu = 0$ explicitly in covariant notation and use the KG equation to show that it vanishes.

[1] We use the same conventions as *An Introduction to Quantum Field Theory* by Peskin and Schroeder.

1.3 KG equation in an electromagnetic field

The KG equation is coupled to an electromagnetic field by using the minimal coupling prescription

$$\partial_\mu \rightarrow \partial_\mu + ieA_\mu, \tag{21}$$

where $A_\mu = (V, \mathbf{A})$ is the four-potential and e the charge. The KG equation becomes

$$\left((\partial^\mu + ieA^\mu)(\partial_\mu + ieA_\mu) + m^2\right)\psi = 0. \tag{22}$$

Note that this equation is *gauge invariant* under the local transformation

$$\begin{aligned} \psi(x) &\rightarrow e^{ie\alpha(x)}\psi(x) \\ A_\mu(x) &\rightarrow A_\mu(x) - \partial_\mu\alpha(x) \end{aligned} \tag{23}$$

where $\alpha(x)$ is any real differentiable function. The law for A_μ is exactly the gauge transformation law for classical electromagnetism. It is useful to define the *covariant derivative* $D_\mu = \partial_\mu + ieA_\mu$ in terms of which the KG equation is

$$(D^\mu D_\mu + m^2)\psi = 0. \tag{24}$$

We can get a little more insight by considering the KG equation in a region of constant but non-zero scalar potential V

$$-\left(\frac{\partial}{\partial t} + ieV\right)^2 \psi = \left(-\frac{\partial^2}{\partial x^2} + m^2\right)\psi. \tag{25}$$

Substituting in the plane wave solution

$$\psi(\mathbf{x}, t) = e^{-i(Et - \mathbf{p}.\mathbf{x})}\,\psi_0 \tag{26}$$

we get

$$(E - eV)^2 = \mathbf{p}^2 + m^2. \tag{27}$$

So

$$E = eV \pm E_\mathbf{p} \tag{28}$$

where, as before,

$$E_\mathbf{p} = +\sqrt{\mathbf{p}^2 + m^2}. \tag{29}$$

We can interpret this as follows

1. The '+ve' energy solution with $E = eV + E_\mathbf{p}$ describes a particle of momentum $\mathbf{p}$, kinetic energy $E_\mathbf{p}$ and electric charge e in an electric potential V.

2. We note that for the '-ve' energy solution we have

$$-E = -eV + E_\mathbf{p} \tag{30}$$

and interpret it as a particle of momentum $\mathbf{p}$, kinetic energy $E_\mathbf{p}$ and electric charge $-e$. That is to say, we interpret it as an *anti-particle*; looking closely at the time dependence of the wave equation we can see that the anti-particle moves *backwards* in time.

If this is right then the wave function when $V = 0$,

$$\psi(\mathbf{x}, t) = a_{\mathbf{p}} e^{-i(E_{\mathbf{p}}t - \mathbf{p}.\mathbf{x})/\hbar} + b_{\mathbf{p}} e^{-i(-E_{\mathbf{p}}t - \mathbf{p}.\mathbf{x})/\hbar} \tag{31}$$

ought to be a mixture of particle and anti-particle! Let's compute ρ to check this out.

$$\begin{aligned}
\rho &= \frac{1}{2m}\left(\psi^*(i\partial_t)\psi - (i\partial_t\psi^*)\psi\right) && (32)\\
&= \frac{1}{2m}\big((a^* e^{iE_{\mathbf{p}}t} + b^* e^{-iE_{\mathbf{p}}t})E_{\mathbf{p}}(a e^{-iE_{\mathbf{p}}t} - b e^{iE_{\mathbf{p}}t}) && (33)\\
&\quad - E_{\mathbf{p}}(-a^* e^{iE_{\mathbf{p}}t} + b^* e^{-iE_{\mathbf{p}}t})(a e^{-iE_{\mathbf{p}}t} + b e^{iE_{\mathbf{p}}t})\big) && (34)\\
&= \frac{E_{\mathbf{p}}}{m}(a^* a - b^* b) && (35)
\end{aligned}$$

which is exactly what we would expect if ρ is actually the *charge density*!

Note that this is a purely heuristic discussion – we have not really solved the underlying problem, which is the existence or otherwise of a ground state. We'll come back to that when we've learnt about the Dirac equation.

1.4 KG Equation: angular momentum

The KG equation in a spherically symmetric potential is, from (25)

$$\left(i\frac{\partial}{\partial t} - eV(r)\right)^2 \psi = \left(-\boldsymbol{\nabla}^2 + m^2\right)\psi. \tag{36}$$

The $\boldsymbol{\nabla}^2$ operator can be re-written in terms of the radial derivative and the orbital angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$,

$$\boldsymbol{\nabla}^2 = \frac{1}{r^2}\frac{\partial^2}{\partial r^2} r^2 - \frac{1}{r^2}\mathbf{L}^2 \tag{37}$$

so we get

$$\left(i\frac{\partial}{\partial t} - eV(r)\right)^2 \psi = -\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2\psi) + \frac{1}{r^2}\mathbf{L}^2\psi + m^2\psi. \tag{38}$$

from which we see that a stationary state takes the form[2]

$$\psi = e^{-iEt} Y_{\ell,m}(\theta, \phi) R(r). \tag{40}$$

This shows that the KG equation describes a particle which can have orbital angular momentum but does not have spin.

2 Relativistic QM: The Dirac Equation

2.1 A first order wave equation

Historically the Dirac equation was introduced in an attempt to rectify the defects of the KG equation, in particular the absence of a current with positive definite density, and the presence

[2] $Y_{\ell,m}(\theta, \phi)$ are the spherical harmonics. They satisfy

$$\mathbf{L}^2 Y_{\ell,m} = \ell(\ell+1)Y_{\ell,m}, \quad L_z Y_{\ell,m} = mY_{\ell,m}, \quad -\ell \le m \le \ell, \quad \ell = 0, 1, 2, 3, \ldots \tag{39}$$

of negative energy modes. To obtain a density with no derivatives in it (which as we have seen leads to it not being positive definite) Dirac required that the wave equation should be first order in the time derivative, while respecting the usual relativistic energy momentum relationship for a free particle (keeping c explicit just for now)

$$E^2 = c^2\mathbf{p}^2 + m^2c^4. \tag{41}$$

But if the wave equation is first order in time it must be first order in spatial derivatives too, otherwise it could not be Lorentz covariant (as Lorentz transformations mix space and time coordinates). Dirac proposed that

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi = (c\,\boldsymbol{\alpha}.\hat{\mathbf{p}} + mc^2\beta)\psi \tag{42}$$

where we have temporarily put hats on operators and next need to establish what $\boldsymbol{\alpha}, \beta$ are. Now for a free particle of energy E and momentum $\mathbf{p}$ we must have a plane wave solution

$$\psi \;\; = \;\; e^{-\frac{i}{\hbar}(Et-\mathbf{p}.\mathbf{x})}\psi_0 \tag{43}$$

(but be aware we don't yet know what sort of quantity ψ_0 is except that it is space-time independent). Substituting in (42) gives

$$E\psi_0 = \hat{H}\psi_0 = (c\,\boldsymbol{\alpha}.\hat{\mathbf{p}} + mc^2\beta)\psi_0 \tag{44}$$

from which we get

$$E^2\psi_0 \;\; = \;\; \hat{H}^2\psi = (c^2(\mathbf{p}.\boldsymbol{\alpha})^2 + mc^3(\mathbf{p}.\boldsymbol{\alpha}\beta + \beta\mathbf{p}.\boldsymbol{\alpha}) + m^2c^4\beta^2)\psi_0. \tag{45}$$

This equation can only reproduce the relativistic energy-momentum relation (41) if

$$\begin{aligned} \alpha_i\alpha_j + \alpha_j\alpha_i &= 2\delta_{ij}, \\ \alpha_i\beta + \beta\alpha_i &= 0, \\ \beta^2 &= 1, \end{aligned} \tag{46}$$

where $i, j = 1, 2, 3$. Clearly these are not commuting objects so they must be matrices (and therefore the right hand sides of these three equations all multiply an identity matrix). It is easy to show that one choice of matrices satisfying these requirements is

$$\beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \tag{47}$$

where σ_i are the Pauli sigma matrices[3] and I is the 2×2 identity matrix. There are in fact many choices, and we will return to this question later, but note now that there are no sets of matrices *smaller* than 4×4 that will satisfy the requirements; the Pauli matrices satisfy very

[3]The Pauli sigma matrices are

$$\sigma_1 \;\; = \;\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{48}$$

They satisfy the relationships

$$\sigma_i\sigma_j + \sigma_j\sigma_i \;\; = \;\; 2\delta_{ij}, \quad \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k, \quad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{49}$$

where repeated indices are summed over; and $\delta_{ij} = 1$ if $i = j$, zero otherwise; and $\epsilon_{ijk} = 1$ if ijk is an even permutation of 123 and $\epsilon_{ijk} = -1$ if ijk is an odd permutation of 123.

similar anti-commutation rules but there are only three of them, and it is impossible to construct a fourth 2×2 matrix that anti-commutes with all the Pauli matrices.

Homework 2.1: Check that α_i and β do indeed satisfy (46). Prove that there is no 2×2 matrix that anticommutes with all the Pauli matrices. Prove that there are no 3×3 matrices satisfying (46).

The equation (42) together with the relationships (46) essentially defines a unique relativistic wave equation that is first order in time derivatives. In fact they show that there is a way of taking the square root in (4) but at the price of working in a space of 4×4 hermitian matrices rather than real numbers. We see that ψ_0 must actually be a four component constant object satisfying the equation (from now on we will revert to $c = 1$ units)

$$(\mathbf{p}.\boldsymbol{\alpha} + m\beta)\psi_0 = E\psi_0. \tag{50}$$

That is to say ψ_0 is the eigenvector of the matrix $\mathbf{p}.\boldsymbol{\alpha} + m\beta$ with eigenvalue E; ψ_0 is called a *spinor* (NB although it has four components it is *not* a four-vector), and, in the context of the Dirac equation, often a *Dirac spinor*. It is easy to check that the eigenvalue condition is $(E^2 - (\mathbf{p}^2 + m^2))^2 = 0$ so there are two eigenstates of positive $E = +\sqrt{\mathbf{p}^2 + m^2}$ and two of negative $E = -\sqrt{\mathbf{p}^2 + m^2}$. Despite the Dirac equation being first order in the time derivative, there are still negative energy solutions; as for the KG equation these negative energy solutions turn out to describe antiparticles, we will return to this in a while.

Homework 2.2: Show by explicit calculation of the determinant that

$$\mathrm{Det}(\mathbf{p}.\boldsymbol{\alpha} + m\beta - E) = (E^2 - (\mathbf{p}^2 + m^2))^2. \tag{51}$$

We can find the conserved charge and current by starting with

$$i\hbar\frac{\partial\psi}{\partial t} = (-i\hbar\boldsymbol{\nabla}.\boldsymbol{\alpha} + m\beta)\psi \tag{52}$$

and then taking the hermitian conjugate (note that α, β are hermitian matrices)

$$-i\hbar\frac{\partial\psi^\dagger}{\partial t} = (i\hbar(\boldsymbol{\nabla}\psi^\dagger).\boldsymbol{\alpha} + m\psi^\dagger\beta. \tag{53}$$

Multiplying the first by $\psi^\dagger$ on the left and the second by ψ on the right and subtracting gives

$$i\left(\psi^\dagger\frac{\partial\psi}{\partial t} + \frac{\partial\psi^\dagger}{\partial t}\psi\right) = -i\left(\psi^\dagger\boldsymbol{\nabla}.\boldsymbol{\alpha}\psi + (\boldsymbol{\nabla}\psi^\dagger).\boldsymbol{\alpha}\psi\right) \tag{54}$$

so $\rho = \psi^\dagger\psi$ and $\mathbf{j} = \psi^\dagger\boldsymbol{\alpha}\psi$ satisfy the conservation equation.

2.2 Angular momentum and spin

To understand the solutions of the Dirac equation we need to know what quantities are conserved so we need to find operators that commute with the free particle Hamiltonian

$$H = \boldsymbol{\alpha}.\mathbf{p} + m\beta. \tag{55}$$

In these calculations we have to keep track of two sets of non-commuting quantities. As well as the matrix relations (46), we have the usual quantum commutation rule $[p_i, x_j] = -i\hbar\delta_{ij}$; but crucially x_i and p_i commute with β and α_i.

Of course linear momentum is conserved

$$[H, p_i] = [\alpha_k p_k + m\beta, p_i] = \alpha_k[p_k, p_i] = 0. \tag{56}$$

The orbital angular momentum operator is as usual

$$\begin{aligned} \mathbf{L} &= \mathbf{x} \times \mathbf{p} \\ L_i &= \epsilon_{ijk} x_j p_k, \end{aligned} \tag{57}$$

and then

$$\begin{aligned} [L_i, H] &= \alpha_l [\epsilon_{ijk} x_j p_k, p_l] \\ &= \alpha_l \epsilon_{ijk} [x_j, p_l], p_k \\ &= i\hbar \epsilon_{ijk} \alpha_j p_k \neq 0 \end{aligned} \tag{58}$$

so $\mathbf{L}$ is not conserved; we have to add something to it to get a conserved quantity. Consider

$$\mathbf{S} = \frac{\hbar}{2} \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \tag{59}$$

which by construction commutes with β but not $\boldsymbol{\alpha}$ so

$$\begin{aligned} [S_i, H] &= \frac{\hbar}{2} [\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \begin{pmatrix} -\sigma_k p_k & 0 \\ 0 & \sigma_k p_k \end{pmatrix}] \\ &= \frac{\hbar}{2} \begin{pmatrix} -[\sigma_i, \sigma_k] p_k & 0 \\ 0 & [\sigma_i, \sigma_k] p_k \end{pmatrix} \\ &= \frac{\hbar}{2} \begin{pmatrix} -2i\epsilon_{ikj} \sigma_j p_k & 0 \\ 0 & 2i\epsilon_{ikj} \sigma_j p_k \end{pmatrix} \\ &= i\hbar \epsilon_{ikj} \alpha_k p_j. \end{aligned} \tag{60}$$

Therefore

$$[L_i + S_i, H] = 0 \tag{61}$$

$$\tag{62}$$

and it is the *total angular momentum operator* $\mathbf{J} = \mathbf{L} + \mathbf{S}$ that commutes with H. Note that $\mathbf{S}$ by itself satisfies the usual commutation algebra for spin-$\frac{1}{2}$; so particles at rest (for which $\mathbf{L}$ must be zero) simply have spin-$\frac{1}{2}$. This explains why there are two solutions of positive energy, and two solutions of negative energy; each has two spin states – in the rest frame simply spin-up and spin-down. A stationary ($\mathbf{p} = 0$) state therefore has four independent solutions, ψ_E^s, labelled by the sign of the energy E and the spin component S,

$$\begin{aligned} H\psi_+^\pm = +m\psi_+^\pm, &\quad S_z \psi_+^\pm = \pm\frac{\hbar}{2}\psi_+^\pm; \\ H\psi_-^\pm = -m\psi_-^\pm, &\quad S_z \psi_-^\pm = \pm\frac{\hbar}{2}\psi_-^\pm. \end{aligned} \tag{63}$$

Homework 2.3: Show that $\mathbf{S}$ satisfies the usual angular momentum rules for spin quantum number $S = \frac{1}{2}$ ie the eigenvalues of $\mathbf{S}^2$ and S_z are $\frac{1}{2}(\frac{1}{2}+1)\hbar^2$ and $\pm\frac{1}{2}\hbar$ respectively.

2.3 Covariant form of the Dirac equation

The formulation of the Dirac equation that we have been looking at does not look covariant; space derivatives are multiplied by αs but time derivatives are not, and the mass appears multiplied by β. Actually the Dirac equation *is* covariant; but it doesn't look it and it is very often convenient to write it in a more covariant form. Multiplying (42) through by β (and from now on taking $\hbar = 1$) we get

$$
\begin{aligned}
0 &= \left(-i\beta\frac{\partial}{\partial t} - i\beta\boldsymbol{\alpha}.\boldsymbol{\nabla} + m\right)\psi \\
&= \left(-i\gamma^{\mu}\partial_{\mu} + m\right)\psi \qquad (64)
\end{aligned}
$$

where

$$\gamma^{\mu} = (\beta, \beta\boldsymbol{\alpha}) \qquad (65)$$

are called the gamma-matrices. It is straightforward to show, using the definition of the γs and the anticommutation properties of $(\boldsymbol{\alpha}, \beta)$, that

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}. \qquad (66)$$

Homework 2.4: Show directly from the anticommutation properties (46) that the gamma-matrices do indeed satisfy (66).

In the representation (47) the γs are given by

$$\text{Rep 1}: \quad \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \qquad (67)$$

There is an infinite number of representations of the gamma-matrices and it turns out that particular forms are more or less convenient for different applications. Another representation that is very convenient is

$$\text{Rep 2}: \quad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \qquad (68)$$

Most of the time, and especially in quantum field theory, (64) is the form in which the Dirac equation is written. Sets of matrices satisfying the relationships (66) are called Clifford Algebras and are an important topic in mathematical physics.

Homework 2.5: Show that the current conservation equation (54) can be written as $\partial_{\mu}j^{\mu}$=0 with $j^{\mu} = \psi^{\dagger}\gamma^0\gamma^{\mu}\psi$.

2.4 Plane wave spinor solutions

We are now in a position to find the constant spinor solution for an eigenstate of 4-momentum

$$\psi = c^{-ip_{\mu}x^{\mu}}\psi_0(\mathbf{p}) \qquad (69)$$

– note that it will turn out to depend on $\mathbf{p}$ which we have emphasized by making the moment an argument. Substituting into the Dirac Equation we get

$$(-p_{\mu}\gamma^{\mu} + m)\psi_0(\mathbf{p}) = 0. \qquad (70)$$

It's convenient to define $\not{p} = p_{\mu}\gamma^{\mu}$ so we can also write this as

$$(-\not{p} + m)\psi_0(\mathbf{p}) = 0. \qquad (71)$$

To proceed we need to choose a representation of the gamma matrices; it turns out that to analyse the spin in the rest frame it is best to use Rep 2 (68) which gives

$$0 = \begin{pmatrix} -E + m & \mathbf{p}.\boldsymbol{\sigma} \\ -\mathbf{p}.\boldsymbol{\sigma} & E + m \end{pmatrix}\psi_0(\mathbf{p}). \qquad (72)$$

Clearly we can write the four-component spinor in the form

$$\psi_0(\mathbf{p}) = \begin{pmatrix} u_A \\ u_B \end{pmatrix} \tag{73}$$

where $u_{A,B}$ are two component objects. Then

$$\begin{aligned} 0 &= (-E+m)u_A + \mathbf{p}.\boldsymbol{\sigma} u_B & (74) \\ 0 &= -\mathbf{p}.\boldsymbol{\sigma} u_A + (E+m)u_B. & (75) \end{aligned}$$

Homework 2.6: Show that (74) and (75) are equivalent provided $E = \pm E_{\mathbf{p}}$.
Looking at positive energy solutions, ie $E = +E_{\mathbf{p}}$, and using (75) we have

$$\psi_+^s(\mathbf{p}) = u^s(\mathbf{p}) = const \begin{pmatrix} \xi^s \\ \frac{\mathbf{p}.\boldsymbol{\sigma}}{E_{\mathbf{p}}+m}\xi^s \end{pmatrix}, \tag{76}$$

where ξ^s is a normalized two-component spinor. If $E = -E_{\mathbf{p}}$ similar reasoning leads to

$$\psi_-^s(\mathbf{p}) = v^s(\mathbf{p}) = const \begin{pmatrix} -\frac{\mathbf{p}.\boldsymbol{\sigma}}{E_{\mathbf{p}}+m}\xi^s \\ \xi^s \end{pmatrix}. \tag{77}$$

For a particle at rest, $\mathbf{p} = 0$, the lower half of the spinors $u^{\pm}(\mathbf{p})$ vanishes. It is useful to choose ξ^s to be an eigenstate of σ_z ie

$$\sigma_z \xi^+ = \xi^+, \quad \sigma_z \xi^- = -\xi^-. \tag{78}$$

Then $u^{\pm}(\mathbf{p})$ are eigenstates of S_z with eigenvalues $\pm\frac{\hbar}{2}$; they represent spin-half particles with spin up and down in the rest-frame respectively. Similarly the upper half of $v^{\pm}(\mathbf{p})$ vanishes and they represent spin-half anti-particles with spin up and down in the rest-frame respectively.
Homework 2.7: Show that the four spinors $u^{\pm}(\mathbf{p}), v^{\pm}(\mathbf{p})$ are mutually orthogonal.

We showed earlier that $\psi^\dagger\psi$ is a density and thus the zero-th component of a four-vector. It is therefore conventional to normalise Dirac spinors so that

$$u^{s\dagger}(\mathbf{p})u^s(\mathbf{p}) = v^{s\dagger}(\mathbf{p})v^s(\mathbf{p}) = 2E_{\mathbf{p}} \tag{79}$$

The $E_{\mathbf{p}}$ is exactly the 0-component of a 4-vector, the 2 is purely convention. We get finally[4]

$$\begin{aligned} u^s(\mathbf{p}) &= \sqrt{E_{\mathbf{p}}+m} \begin{pmatrix} \xi^s \\ \frac{\mathbf{p}.\boldsymbol{\sigma}}{E_{\mathbf{p}}+m}\xi^s \end{pmatrix} \\ v^s(\mathbf{p}) &= \sqrt{E_{\mathbf{p}}+m} \begin{pmatrix} -\frac{\mathbf{p}.\boldsymbol{\sigma}}{E_{\mathbf{p}}+m}\xi^s \\ \xi^s \end{pmatrix}. & (80) \end{aligned}$$

When $\mathbf{p} \neq 0$, it is $J_z = L_z + S_z$ that commutes with H so of course $u^{\pm}(\mathbf{p}), v^{\pm}(\mathbf{p})$ are no longer eigenstates of S_z. However we observe that since $\mathbf{p}$ and $\mathbf{J}$ commute with H so does $\mathbf{p}.\mathbf{J}$; but $\mathbf{p}.\mathbf{L} = 0$ so this implies that the spin projection operator $\mathbf{p}.\mathbf{S}$, ie the component of spin along $\mathbf{p}$, commutes with H. Therefore it is often convenient to choose the two-component spinors to be eigenstates of $\mathbf{p}.\mathbf{S}$ rather than S_z and we define $\xi^{\pm}(\mathbf{p})$ with the properties

$$\begin{aligned} \boldsymbol{\sigma}.\mathbf{p}\,\xi^{\pm}(\mathbf{p}) &= \pm|\mathbf{p}|\,\xi^{\pm}(\mathbf{p}) \\ \xi^{\pm}(\mathbf{p})^\dagger \xi^{\pm}(\mathbf{p}) &= 1. & (81) \end{aligned}$$

[4]NB In quantum field theory it is conventional to call the anti-particle spinor defined here $v^{\pm}(-\mathbf{p})$.

Homework 2.8: Find explicit expressions for $\xi^{\pm}(\mathbf{p})$ in terms of the components (p_x, p_y, p_z). Note that by changing $\mathbf{p} \to -\mathbf{p}$ you can see from (81) that $\xi^{\pm}(-\mathbf{p}) = \xi^{\mp}(\mathbf{p})$; check that your expressions have this property.

Clearly if particles are massless we cannot study them in the rest-frame. It's helpful to use Rep 1 (67) which gives

$$0 = \begin{pmatrix} 0 & -(E - \mathbf{p}.\boldsymbol{\sigma}) \\ -(E + \mathbf{p}.\boldsymbol{\sigma}) & 0 \end{pmatrix} \psi_0(\mathbf{p}) \tag{82}$$

We see that the upper and lower components of the Dirac spinor are not mixed and that solutions take the form

$$\psi_0(\mathbf{p}) = \begin{pmatrix} \xi^s(\mathbf{p}) \\ 0 \end{pmatrix}, \text{or } \psi_0(\mathbf{p}) = \begin{pmatrix} 0 \\ \xi^s(\mathbf{p}) \end{pmatrix} \tag{83}$$

so really they are two-component spinors. The origin of this is that if $m = 0$ we do not need the β matrix, just the α_i and since there are only three of them we can choose $\alpha_i = \sigma_i$. The resulting two-component spinors are called *Weyl spinors*; the eigenvalue of $\hat{\mathbf{p}}.\boldsymbol{\sigma}$ is usually called the *helicity* of the state.

We can also find Dirac spinors in Rep 1 (67) by repeating the steps above. Substituting a plane wave state into the Dirac Equation we get

$$0 = \begin{pmatrix} m & -(E - \mathbf{p}.\boldsymbol{\sigma}) \\ -(E + \mathbf{p}.\boldsymbol{\sigma}) & m \end{pmatrix} \psi^s_E(\mathbf{p}) \tag{84}$$

where as usual $E = \pm E_{\mathbf{p}}$. We then decompose the four-component spinor into spin projection two-component spinors (81)

$$\psi^s_E(\mathbf{p}) = \begin{pmatrix} A\,\xi^s(\mathbf{p}) \\ B\,\xi^s(\mathbf{p}) \end{pmatrix} \tag{85}$$

and substituting into (84) gives

$$\begin{aligned} 0 &= mA - (E - s|\mathbf{p}|)B \\ 0 &= -(E + s|\mathbf{p}|)A + mB. \end{aligned} \tag{86}$$

Again these equations are equivalent to each other. Normalizing to $2E_{\mathbf{p}}$ we finally obtain

1. the positive energy spinors

$$u^{\pm}(\mathbf{p}) = \begin{pmatrix} \sqrt{E_{\mathbf{p}} \mp |\mathbf{p}|}\,\xi^{\pm}(\mathbf{p}) \\ \sqrt{E_{\mathbf{p}} \pm |\mathbf{p}|}\,\xi^{\pm}(\mathbf{p}) \end{pmatrix} \tag{87}$$

2. and the negative energy spinors (see footnote 4)

$$v^{\pm}(-\mathbf{p}) = \begin{pmatrix} \sqrt{E_{\mathbf{p}} \pm |\mathbf{p}|}\,\xi^{\pm}(\mathbf{p}) \\ -\sqrt{E_{\mathbf{p}} \mp |\mathbf{p}|}\,\xi^{\pm}(\mathbf{p}) \end{pmatrix} \tag{88}$$

2.5 Properties under Lorentz transformations

We have seen that $\psi^\dagger\psi$ is not a Lorentz scalar. So what combination *is* a scalar? The answer is $\psi^\dagger\gamma^0\psi$. Computing explicitly in Rep 1 (67) (87) (88) for the positive energy spinors we get

$$u^{\pm}(\mathbf{p})^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} u^{\pm}(\mathbf{p}) = 2\sqrt{E_{\mathbf{p}} \mp |\mathbf{p}|}\sqrt{E_{\mathbf{p}} \pm |\mathbf{p}|} \tag{89}$$

$$= 2m \tag{90}$$

Similarly we find that

$$v^{\pm}(\mathbf{p})^{\dagger}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} v^{\pm}(\mathbf{p}) = -2\sqrt{E_{\mathbf{p}} \mp |\mathbf{p}|}\sqrt{E_{\mathbf{p}} \pm |\mathbf{p}|} \tag{91}$$

$$= -2m \tag{92}$$

Of course the result is explicitly a Lorentz scalar. The combination $\psi^{\dagger}\gamma_0$ is ubiquitous and usually called $\bar{\psi}$.

Homework 2.9: Check these relationships in Rep 2.

2.6 Dirac Equation in an external E.M. field I

As for the KG equation, the Dirac spin-half particle is coupled to an electromagnetic field by using the minimal coupling prescription

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu \tag{93}$$

where $A_\mu = (V, \mathbf{A})$ is the vector potential and e the charge. The Dirac equation becomes

$$(-i\not{D} + m)\psi = 0 \tag{94}$$

Again this equation is *gauge invariant* under the local transformation

$$\begin{aligned} \psi(x) &\to e^{ie\alpha(x)}\psi(x) \\ A_\mu(x) &\to A_\mu(x) - \partial_\mu\alpha(x). \end{aligned} \tag{95}$$

We first look at scattering from a potential step $A_\mu = (V, 0)$ for $z > 0$, and $A_\mu = 0$ for $z < 0$.

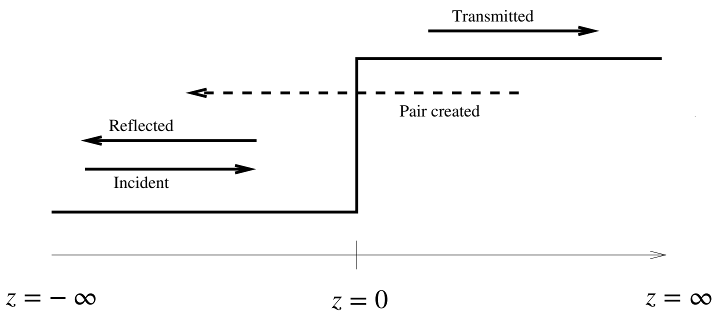

Figure 1: The potential step. The labelled arrowed lines show the particle *currents* discussed on page 13.

Consider a Dirac particle of momentum $\mathbf{p} = (0, 0, p_z)$, energy $E = E_{\mathbf{p}}$ and $S_z = +\frac{1}{2}$ incident from $z = -\infty$, Fig 1. Note that $\mathbf{S}$ commutes with γ_0 so the spin will be conserved in this process and we only have to consider spin $+\frac{1}{2}$ states throughout; we set $\xi_s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ everywhere. Considering the usual plane wave solutions we have energy conservation at the barrier and the energy-momentum relations are

$$\begin{aligned} E^2 &= p_z^2 + m^2, \quad z < 0 \\ (E - eV)^2 &= {p'_z}^2 + m^2, \quad z \geq 0. \end{aligned} \tag{96}$$

The boundary conditions are simply continuity of the wave-function at the interface; there is no condition on the derivative because the Hamiltonian is first order in $\mathbf{p}$. Otherwise the calculation is very similar to the non-relativistic case; equating ingoing plus reflected to transmitted waves at $z = 0$,

$$e^{ip_z z} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ 0 \end{pmatrix} + r e^{-ip_z z} \begin{pmatrix} 1 \\ 0 \\ \frac{-p_z}{E+m} \\ 0 \end{pmatrix} \Bigg|_{z=0} = A e^{ip'_z z} \begin{pmatrix} 1 \\ 0 \\ \frac{p'_z}{E+m-V} \\ 0 \end{pmatrix} \Bigg|_{z=0} \tag{97}$$

which gives two equations to determine r and A. As usual the physical information is in the current j_3 which is given by

$$j_3 = \psi^\dagger \alpha_3 \psi = \psi^\dagger \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \psi \tag{98}$$

For $z < 0$ this gives

$$j_3 = \frac{2p_z}{E+m}(1 - r^* r) \tag{99}$$

and for $z > 0$

$$j_3 = \frac{2p'_z}{E - eV + m} A^* A \tag{100}$$

if p'_z is real and $j_3 = 0$ if p'_z is imaginary.

Eliminating A from the boundary conditions (97) gives

$$\frac{1-r}{1+r} = \frac{p'_z(E+m)}{p_z(E - eV + m)} = \kappa \tag{101}$$

so

$$r = \frac{1-\kappa}{1+\kappa}. \tag{102}$$

There are three regimes (see Fig 1)

1. If $E > eV + m$ then the particle has sufficient kinetic energy $E_K = E - m > eV$ to overcome the barrier; p'_z is real and $1 > \kappa > 0$. The reflection and transmission coefficients are given by

$$R = \frac{j_{reflected}}{j_{incident}} = \left(\frac{1-\kappa}{1+\kappa}\right)^2, \qquad T = \frac{j_{transmitted}}{j_{incident}} = \frac{4\kappa}{(1+\kappa)^2} \tag{103}$$

so $R < 1$ and $T > 0$ which is again what we would expect as now the particle has enough kinetic energy to get over the barrier. It is easy to check that $R + T = 1$.

2. If $eV - m < E < eV + m$ then the particle has insufficient kinetic energy $E_K = E - m < eV$ to overcome the barrier; then p'_z is imaginary, as is κ, therefore $|r|^2 = 1$ and there is total reflection, $R = 1$. The particle does not have enough kinetic energy $E_K = E - m$ to overcome the barrier height eV. This is as we would expect.

3. But if $m < E < eV - m$ (which can only happen if the barrier is high enough $eV > 2m$) then p'_z is again real and, assuming $p'_z > 0$, $\kappa < 0$; so (103) tells us that and $R > 1, T < 0$ – but still $R + T = 1$. This can only be understood if we assume that there must be pair creation of particles and anti-particles at the boundary. Particles of energy E can only inhabit the $z < 0$ region; however anti-particles see a potential *drop* of V from $z < 0$ to $z > 0$ because they have the opposite charge to particles. So if pairs are created at the boundary the particles must have momentum $-\mathbf{p}$ and move to the left while the antiparticles must have momentum $+\mathbf{p}$ and move to the right. The left moving particles augment the totally reflected incident particles so $R > 1$. The stream of right moving antiparticles constitutes a current which is equivalent to a stream of left moving particles ie $j_{transmitted} < 0$ so $T < 0$. Of course this explanation is not totally convincing; we have forced changing particle number on a formalism that requires fixed particle number.

Homework 2.10: What is the significance of the fact that regime (3) above only occurs when $eV > 2m$?

Homework 2.11: Compute the scattering of a KG particle off the potential step in Fig 1. Remember that because the KG equation is second order in spatial derivatives, the first derivative of the wave function must be continuous at the step.

3 Dirac Equation in an external E.M. field II

In this section we'll look at some of the other consequences of the Dirac equation for a particle in an E.M. field. Starting from the Dirac equation

$$(-i\not{D} + m)\psi = 0 \tag{104}$$

and acting on the left with $(i\not{D} + m)$ we find

$$\left(\gamma^\mu\gamma^\nu D_\mu D_\nu + m^2\right)\psi = 0. \tag{105}$$

Now

$$\begin{aligned}\gamma^\mu\gamma^\nu &= \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) + \frac{1}{2}[\gamma^\mu, \gamma^\nu] \\ &= g^{\mu\nu} + \Sigma^{\mu\nu}\end{aligned} \tag{106}$$

where we have introduced $\Sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$; so (105) can be written

$$(D^\mu D_\mu + m^2)\psi + \Sigma^{\mu\nu} D_\mu D_\nu \psi = 0. \tag{107}$$

The first term is the Klein Gordon operator in the presence of an E.M. field – it is the same as for spinless particles. The second term is the modification for spin-$\frac{1}{2}$ particles. Note that $\Sigma_{\mu\nu}$ is antisymmetric on its two indices but that $D_\mu D_\nu \psi$ is *not* symmetric because the derivatives ∂_μ act on A_μ as well as on ψ; we can write this as

$$\begin{aligned}0 &= (D^\mu D_\mu + m^2)\psi + \frac{1}{2}\Sigma^{\mu\nu}[D_\mu, D_\nu]\psi \\ &= (D^\mu D_\mu + m^2)\psi + ie\frac{1}{2}\Sigma^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi \\ &= (D^\mu D_\mu + m^2)\psi + ie\frac{1}{2}\Sigma^{\mu\nu} F_{\mu\nu}\psi\end{aligned} \tag{108}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor. Note that the new term only depends on the physical field $F_{\mu\nu}$ which is gauge invariant, and not on the vector potential which is not.

To see the implications of the new term let's look at the situation with a static uniform $\mathbf{B}$ field. First choose $\mathbf{A} = (-By, 0, 0)$ so $\mathbf{B} = \text{curl}\mathbf{A} = (0, 0, B)$. Then $A_\mu = (0, -By, 0, 0)$ and

$$F_{12} = -F_{21} = \partial_1 A_2 - \partial_2 A_1 = \partial_x 0 - \partial_y(By) = B. \tag{109}$$

Using

$$\Sigma^{12} = -\Sigma^{21} = \frac{1}{2}(\gamma^1\gamma^2 - \gamma^2\gamma^1) = 2iS_3 \tag{110}$$

we get

$$0 = (D^\mu D_\mu + m^2)\psi - 2eBS_3\psi. \tag{111}$$

Homework 3.1: Check (110).

By translation and rotation invariance of the spatial coordinates the result for general $\mathbf{B}$ must be

$$0 = (D^\mu D_\mu + m^2)\psi - 2e\mathbf{B}.\mathbf{S}\psi. \tag{112}$$

so this looks like a magnetic dipole interaction but to read off the magnetic dipole moment we need to look at the system in the non-relativistic limit. In this limit $E = m + \epsilon$ with $\frac{\epsilon}{m} \ll 1$ and we set

$$\psi = e^{-it(m+\epsilon)}\phi(\mathbf{x}) \tag{113}$$

which gives

$$0 = (-(m+\epsilon)^2 - (\boldsymbol{\nabla} + ie\mathbf{A})^2 + m^2)\phi(\mathbf{x}) - 2e\mathbf{B}.\mathbf{S}\phi(\mathbf{x}) \tag{114}$$

or, neglecting terms of order $\frac{\epsilon^2}{m}$,

$$\epsilon\phi(\mathbf{x}) = -\frac{1}{2m}(\boldsymbol{\nabla} + ie\mathbf{A})^2\phi(\mathbf{x}) - \frac{e}{m}\mathbf{B}.\mathbf{S}\phi(\mathbf{x}). \tag{115}$$

So we see that the particle has a magnetic dipole moment $\boldsymbol{\mu} = \frac{e}{m}\mathbf{S} = 2\mu_B\mathbf{S}$ where we have introduced the Bohr magneton $\mu_B = \frac{e}{2m}$. The magnetic dipole moment arising from orbital motion is $\boldsymbol{\mu} = \mu_B\mathbf{L}$; the extra factor 2, usually called the g-factor, arising in the spin case is one of the original triumphs of the Dirac equation – experimentally this number is extremely close to 2 for the electron and the discrepancy is explained by Quantum Electrodynamics.

Homework 3.2: Restore the $\hbar$ factors in (115).

Homework 3.3: Show that the term linear in $\mathbf{A}$ in (115) leads to an interaction term $-\frac{e}{2m}\mathbf{B}.\mathbf{L}\phi(\mathbf{x})$ and thus confirm that the g-factor associated with orbital motion is indeed 1. Hint: use (and first check!) the fact that for a constant magnetic field $\mathbf{B}$ the vector potential is given by $\mathbf{A} = \frac{1}{2}\mathbf{x} \times \mathbf{B}$, and then make some vector operator manipulations.

Using the vector potential $\mathbf{A} = (-By, 0, 0)$ in (115) we can solve the eigenvalue problem for ϵ. Setting

$$\phi(\mathbf{x}) = e^{i(p_x x + p_z z)}\phi_S(y) \tag{116}$$

where $S_z\phi_S(y) = S\phi_S(y)$ we get

$$\epsilon\phi_S(y) = \frac{1}{2m}\left((p_x - eBy)^2 - \frac{d^2}{dy^2} + p_z^2\right)\phi_S(y) - \frac{e}{m}BS\phi_S(y) \tag{117}$$

Now choose $y = y' + p_x/(eB)$ to get

$$\epsilon\phi_S(y') = \left(-\frac{1}{2m}\frac{d^2}{dy'^2} + \frac{(eB)^2}{2m}y'^2 + \frac{p_z^2}{2m} - \frac{e}{m}BS\right)\phi_S(y') \tag{118}$$

which shows that this is just a shifted SHO problem with frequency $\omega = \frac{eB}{m}$. We see that

$$\epsilon = \frac{p_z^2}{2m} + (n + \frac{1}{2} - S_z)\omega, \quad n = 0, 1, 2, \ldots; S = \pm\frac{1}{2} \tag{119}$$

Homework 3.4: The $S\omega$ contribution in (119) is exactly the interaction energy of a magnetic dipole $2S\mu_B$ lying parallel to a magnetic field B. Because of the way we've done the calculation it's not so clear whether our result is really dependent only on physical fields and not directly on the vector potential. To check this, repeat the calculation with $\mathbf{A} = (-\frac{1}{2}By, \frac{1}{2}Bx, 0)$. Discuss the gauge dependence of ϵ and of ϕ_S.

4 Relativistic Quantum Mechanics in the Round

Relativistic Quantum Mechanics is a necessary step but it cannot be the end of the story. There are some successes:

1. It is Lorentz covariant. (We haven't explored this in much depth here; see André Lukas's lectures in the MMathPhys for a thorough treatment of the Lorentz transformation properties of spinors.)

2. It naturally contains anti-particles with the opposite charge to the particles but the same mass.

3. It explains why the spin of electrons couples to an external magnetic field with twice the strength of the orbital angular momentum coupling.

4. It explains why the spin-orbit coupling Hamiltonian in a single electron atom or ion takes the form it does (we haven't looked at this at all but it is straightforward to derive).

But also there are some failings that are inherent in the approach:

1. The negative energy states remain a problem; despite reinterpreting them as something to do with anti-particles we do not have a way of calculating a total energy for the system which is really bounded below. – something that is certainly required by experiment.

2. Particles and antiparticles can annihilate (experimental fact) so the number of particles is not conserved. The formalism of a wave function with the Born interpretation necessarily describes a fixed number of particles so Relativistic QM is bound to fail in circumstances where pair creation can occur.

The next step is to develop Quantum Field Theory which is able to describe varying particle number, describes particles and anti-particles, and has a true ground state. It's worth noting that real relativistic systems are not the only variable-particle-number systems. We now understand that many quasi-particle descriptions of condensed matter systems are also like this and can also be described by quantum field theories (which are typically not Lorentz invariant but Galilean invariant).