# Supersymmetry and supergravity

Lecture 12

# SUSY gauge theories with matter

- So far we have studied models with a collection of chiral multiplets, and models constructed with vector multiplets only. We have to combine the two to get a general SUSY gauge theory
- At the moment we are interested in renormalizable models. In particular, all fields have canonical kinetic terms
- SUSY gauge theories are very rich on their own, and this formalism is necessary to start to apply SUSY to questions related to the SM

# The building blocks in isolation

Matter fields = chiral multiplets $(X^i, \psi^i_\alpha, F^i)$

$$\delta X^i = \sqrt{2}\,\xi\,\psi^i \;, \qquad \delta\psi^i = i\sqrt{2}\,(\sigma^\mu\,\bar\xi)_\alpha\,\partial_\mu X^i + \sqrt{2}\,F^i\,\xi_\alpha \;, \qquad \delta F^i = i\sqrt{2}\,\bar\xi\,\bar\sigma^\mu\,\partial_\mu\psi^i$$

$$\mathscr{L}_{\mathrm{kin}} = -\,\partial^\mu \bar X_i\,\partial_\mu X^i + i\,\partial_\mu\bar\psi_i\,\bar\sigma^\mu\,\psi^i + \bar F_i\,F^i$$

$$\mathscr{L}_W = F^i\,W_i - \frac{1}{2}\,W_{ij}\,\psi^i\,\psi^j + \mathrm{h.c.}$$

Gauge fields = vector multiplet $(A^a_\mu, \lambda^a_\alpha, D^a)$ $\quad a$ = adjoint index

$$\delta A^a_\mu = i\,\bar\xi\,\bar\sigma_\mu\,\lambda^a + \mathrm{h.c.} \quad , \; \delta\lambda^a_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F^a{}_{\mu\nu} + i\,D^a\,\xi_\alpha \;, \quad \delta D^a = \bar\xi\,\bar\sigma^\mu\,D_\mu\lambda^a + \mathrm{h.c.}$$

$$\mathscr{L}_{\mathrm{vec}} = -\frac{1}{4}\,F^{a\mu\nu}\,F_{a\mu\nu} + i\,D_\mu\bar\lambda^a\,\bar\sigma^\mu\,\lambda_a + \frac{1}{2}\,D^a\,D_a$$

Goal: Take a model with chirals that has global symmetry $G$ and gauge the symmetry, coupling to a vector multiplet

# Chiral models with global continuous symm's

Matter fields = chiral multiplets $(X^i, \psi^i_\alpha, F^i)$

We identify the index $i$ with the index of a representation of a global flavor symmetry group $G$. The generators of $G$ are $T_a$ and obey

$$[T_a, T_b] = i f_{ab}{}^c\, T_c$$

Adjoint indices are raised/lowered with $\delta_{ab}$. Then $f_{abc}$ is real and totally antisymmetric.

We consider a representation of $G$ by hermitian matrices $(t_a)^i{}_j$. For instance

$$\delta_{\text{flavor}} X^i = i\,\Lambda_0^a\,(t_a)^i{}_j\,X^j \quad , \qquad \delta_{\text{flavor}} \overline{X}_i = -\,i\,\Lambda_0^a\,\overline{X}_j\,(t_a)^j{}_i$$

where $\Lambda_0^a$ is a constant real parameter of the infinitesimal flavor transformation. All fields in $(X^i, \psi^i_\alpha, F^i)$ have the same transformation law.

NB: our notation works for a simple non-Abelian group, but also for a $U(1)$ factor. In the latter case the matrix $(t_a)^i{}_j$ is the identity times the charge of the field,

for a $U(1)$: $\quad \delta_{\text{flavor}} X^i = i\,\Lambda_0^a\, q[X^i]\,X^i \quad , \qquad \delta_{\text{flavor}} \overline{X}_i = -\,i\,\Lambda_0^a\, q[X^i]\,\overline{X}_i \qquad$ (no sum on $i$)

# Chiral models with global continuous symm's

$$\delta_{\text{flavor}} X^i = i\,\Lambda_0^a\,(t_a)^i{}_j\,X^j \quad , \qquad \delta_{\text{flavor}} \overline{X}_i = -\,i\,\Lambda_0^a\,\overline{X}_j\,(t_a)^i{}_j$$

$$\mathscr{L}_{\text{kin}} = -\,\partial^\mu \overline{X}_i\,\partial_\mu X^i + i\,\partial_\mu \overline{\psi}_i\,\overline{\sigma}^\mu\,\psi^i + \overline{F}_i\,F^i$$

$$\mathscr{L}_W = F^i\,W_i - \frac{1}{2}\,W_{ij}\,\psi^i\,\psi^j + \text{h.c.}$$

The kinetic Lagrangian is automatically invariant. $\mathscr{L}_W$ is invariant provided that the superpotential is invariant under a flavor transformation:

$$0 = \delta_{\text{flavor}} W(X) = W_i\,\delta_{\text{flavor}} X^i = i\,\Lambda_0^a\,W_i\,(t_a)^i{}_j\,X^j \quad \Rightarrow \quad W_i\,(t_a)^i{}_j\,X^j = 0$$

# Gauging the flavor symmetry

As usual, we gauge the flavor symmetry by promoting the transformation parameter to be an arbitrary function of spacetime: we go from

$$\delta_{\text{flavor}} X^i = i\,\Lambda_0^a\,(t_a)^i{}_j\,X^j \quad , \qquad \delta_{\text{flavor}} \overline{X}_i = -\,i\,\Lambda_0^a\,\overline{X}_j\,(t_a)^i{}_j$$

to

$$\delta_{\text{gauge}} X^i = -\,i\,g\,\Lambda^a\,(t_a)^i{}_j\,X^j \quad , \quad \delta_{\text{gauge}} \overline{X}_i = i\,g\,\Lambda^a\,\overline{X}_j\,(t_a)^i{}_j \quad , \quad \partial_\mu \Lambda \neq 0$$

The coupling constant $g$ is inserted for convenience. Ordinary derivatives of $X^i$ must be replaced by gauge-covariant derivative. They are constructed with a gauge vector $A_\mu^a$ in the adjoint of $G$:

$$D_\mu X^i = \partial_\mu X^i + i\,g\,A_\mu^a\,(t_a)^i{}_j\,X^j \quad , \quad D_\mu \overline{X}_i = \partial_\mu \overline{X}_i - i\,g\,A_\mu^a\,\overline{X}_j\,(t_a)^j{}_i$$

# SUSY variations after gauging

When the global flavor symmetry $G$ is gauged, the fields $(X^i, \psi^i_\alpha, F^i)$ acquire a gauge redundancy. This modifies their off-shell SUSY transformations. The full set of off-shell SUSY variations is

$$\delta X^i = \sqrt{2}\, \xi\, \psi^i$$

$$\delta \psi^i = i\sqrt{2}\, (\sigma^\mu\, \bar{\xi})_\alpha\, D_\mu X^i + \sqrt{2}\, F^i\, \xi_\alpha$$

$$\delta F^i = i\sqrt{2}\, \bar{\xi}\, \bar{\sigma}^\mu\, D_\mu \psi^i + 2\, i\, g\, (\bar{\xi}\, \bar{\lambda}^a)\, (t_a)^i{}_j\, X^j$$

$$\delta A^a_\mu = i\, \bar{\xi}\, \bar{\sigma}_\mu\, \lambda^a + \mathrm{h.c.}$$

$$\delta \lambda^a_\alpha = (\sigma^{\mu\nu}\, \xi)_\alpha\, F^a{}_{\mu\nu} + i\, D^a\, \xi_\alpha$$

$$\delta D^a = \bar{\xi}\, \bar{\sigma}^\mu\, D_\mu \lambda^a + \mathrm{h.c.}$$

Modifications:

1. partial derivatives in the chiral multiplet variations are replaced by gauge-cov der's
2. there is a new term in the variation of $F$

# The new term in the variation of $F$

The superspace formalism gives a derivation of the previous SUSY variations. Even without superspace we can have an intuition for the origin of the new term in the variation of $F$. It is needed to get the gauge-cov. version of off-shell closure of the SUSY algebra:

$$\delta_1\delta_2 A^a_\mu - \delta_2\delta_1 A^a_\mu = -2\,i\,(\xi_1\,\sigma^\nu\,\bar{\xi}_2 - \xi_2\,\sigma^\nu\,\bar{\xi}_1)\,F^a{}_{\nu\mu}$$

$$\delta_1\delta_2\Phi - \delta_2\delta_1\Phi = -2\,i\,(\xi_1\,\sigma^\nu\,\bar{\xi}_2 - \xi_2\,\sigma^\nu\,\bar{\xi}_1)\,D_\nu\Phi$$

where $\Phi$ stands for any field that transforms tensorially under a gauge transformation (in the adjoint rep for fields in the vector multiplet; in the rep with $i$ indices for the chiral multiplets)

# The new term in the variation of $F$

A sketch of the check:

$$\delta_1\delta_2 F^i = i\sqrt{2}\,\bar\xi_2\,\bar\sigma^\mu\,\delta_1 D_\mu\psi^i + 2\,i\,g\,(\bar\xi_2\,\delta_1\bar\lambda^a)\,(t_a)^i{}_j\,X^j + 2\,i\,g\,(\bar\xi_2\,\bar\lambda^a)\,(t_a)^i{}_j\,\delta_1 X^j$$

We get a novel term from the SUSY variation of the gauge field inside $D_\mu\psi^i$:

$$\delta_1 D_\mu\psi^i_\alpha \supset i\,g\,\delta_1 A^a_\mu\,(t_a)^i{}_j\,\psi^j_\alpha = i\,g\,(i\,\bar\xi_1\,\bar\sigma_\mu\,\lambda^a - i\,\bar\lambda^a\,\bar\sigma_\mu\,\xi_1)\,(t_a)^i{}_j\,\psi^j_\alpha$$

which gives

$$i\sqrt{2}\,\bar\xi_2\,\bar\sigma^\mu\,\delta_1 D_\mu\psi^i \supset -\,g\sqrt{2}\,(\bar\xi_2\,\bar\sigma^\mu\,\psi^j)\,(t_a)^i{}_j\,(i\,\bar\xi_1\,\bar\sigma_\mu\,\lambda^a - i\,\bar\lambda^a\,\bar\sigma_\mu\,\xi_1)$$

To cancel this 4-Fermi term we need the contribution from

$$2\,i\,g\,(\bar\xi_2\,\bar\lambda^a)\,(t_a)^i{}_j\,\delta_1 X^j = 2\sqrt{2}\,i\,g\,(\bar\xi_2\,\bar\lambda^a)\,(t_a)^i{}_j\,(\xi_1\,\psi^j)$$

$$\delta X^i = \sqrt{2}\,\xi\,\psi^i \qquad \delta A^a_\mu = i\,\bar\xi\,\bar\sigma_\mu\,\lambda^a + \mathrm{h.c.}$$

$$\delta\psi^i = i\sqrt{2}\,(\sigma^\mu\,\bar\xi)_\alpha\,D_\mu X^i + \sqrt{2}\,F^i\,\xi_\alpha \qquad \delta\lambda^a_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F^a{}_{\mu\nu} + i\,D^a\,\xi_\alpha$$

$$\delta F^i = i\sqrt{2}\,\bar\xi\,\bar\sigma^\mu\,D_\mu\psi^i + 2\,i\,g\,(\bar\xi\,\bar\lambda^a)\,(t_a)^i{}_j\,X^j \qquad \delta D^a = \bar\xi\,\bar\sigma^\mu\,D_\mu\lambda^a + \mathrm{h.c.}$$

# The full SUSY Lagrangian

The full Lagrangian of a SUSY gauge theory is the sum of several pieces:

- YM term and supersymmetrization:

$$\mathscr{L}_{\text{vec}} = -\frac{1}{4}\, F^{a\mu\nu}\, F_{a\mu\nu} + i\, D_\mu \bar{\lambda}^a\, \bar{\sigma}^\mu\, \lambda_a + \frac{1}{2}\, D^a\, D_a$$

- Kinetic terms for chirals, written with gauge-cov der's

$$\mathscr{L}_{\text{kin}} = -\, D^\mu \bar{X}_i\, D_\mu X^i + i\, D_\mu \bar{\psi}_i\, \bar{\sigma}^\mu\, \psi^i + \bar{F}_i\, F^i$$

- Extra terms that are required after gauging:

$$\mathscr{L}_{\text{coupl}} = i\sqrt{2}\, g\, \left[\bar{X}_i\, (t_a)^i{}_j\, \psi^j\, \lambda^a - \bar{\lambda}^a\, \bar{\psi}_i\, (t_a)^i{}_j\, X^j\right] + g\, D^a\, \bar{X}_i\, (t_a)^i{}_j\, X^j$$

- Terms that come from the superpotential (if any: this part is optional)

$$\mathscr{L}_W = F^i\, W_i - \frac{1}{2}\, W_{ij}\, \psi^i\, \psi^j + \text{h.c.}$$

- The superpotential must be gauge-invariant: $W_i\, (t_a)^i{}_j\, X^j = 0$
- Optionally: Fayet-Iliopoulos terms

# Fayet-Iliopoulos terms

There is one extra class of couplings that are compatible with SUSY:

$$\mathscr{L}_{\mathrm{FI}} = p_a D^a$$

where $p_a$'s are constants. Is this term gauge invariant?

$$\delta_{\mathrm{gauge}}(p_a D^a) = p_a \delta_{\mathrm{gauge}} D^a \propto p_a f^{abc} \Lambda_b D_c$$

In order for this term to be allowed, the constants $p_a$ must obey

$$f_{ab}{}^{c} p_c = 0$$

The superspace analysis shows that if this is true, then $\mathscr{L}_{\mathrm{FI}}$ is also supersymmetric.

The constant $p_a$ associated to a generator $T_a$ can be non-zero only if $T_a$ never appears in the commutator of two other generators. In our setup $T_a$ is either inside a simple non-Abelian factor of the gauge group, or it is a $U(1)$ factor. We find that

the FI constant $p_a$ can be non-zero only if $T_a$ is the generator of a $U(1)$ factor

# Supersymmetry and supergravity

Lecture 13

# The full SUSY Lagrangian

The full Lagrangian of a SUSY gauge theory is the sum of several pieces:

- YM term and supersymmetrization:

$$\mathscr{L}_{\text{vec}} = -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} + i\, D_\mu \bar{\lambda}^a \bar{\sigma}^\mu \lambda_a + \frac{1}{2} D^a D_a$$

- Kinetic terms for chirals, written with gauge-cov der's

$$\mathscr{L}_{\text{kin}} = -\, D^\mu \bar{X}_i\, D_\mu X^i + i\, D_\mu \bar{\psi}_i\, \bar{\sigma}^\mu \psi^i + \bar{F}_i F^i$$

- Extra terms that are required after gauging:

$$\mathscr{L}_{\text{coupl}} = i\sqrt{2}\, g \left[\bar{X}_i (t_a)^i{}_j \psi^j \lambda^a - \bar{\lambda}^a \bar{\psi}_i (t_a)^i{}_j X^j\right] + g\, D^a \bar{X}_i (t_a)^i{}_j X^j$$

- Terms that come from a gauge-inv superpotential (if any)

$$\mathscr{L}_W = F^i W_i - \frac{1}{2} W_{ij} \psi^i \psi^j + \mathrm{h.c.} \qquad W_i (t_a)^i{}_j X^j = 0$$

- FI terms for the $U(1)$ factors of the gauge group: $\mathscr{L}_{\text{FI}} = p_a D^a$

# Integrating out the auxiliary fields

$$\mathscr{L}_{\text{vec}} = -\frac{1}{4}\,F^{a\mu\nu}\,F_{a\mu\nu} + i\,D_\mu\bar{\lambda}^a\,\bar{\sigma}^\mu\,\lambda_a + \frac{1}{2}\,D^a\,D_a$$

$$\mathscr{L}_{\text{kin}} = -\,D^\mu\bar{X}_i\,D_\mu X^i + i\,D_\mu\bar{\psi}_i\,\bar{\sigma}^\mu\,\psi^i + \bar{F}_i\,F^i$$

$$\mathscr{L}_{\text{coupl}} = i\sqrt{2}\,g\,\left[\bar{X}_i\,(t_a)^i{}_j\,\psi^j\,\lambda^a - \bar{\lambda}^a\,\bar{\psi}_i\,(t_a)^i{}_j\,X^j\right] + g\,D^a\,\bar{X}_i\,(t_a)^i{}_j\,X^j$$

$$\mathscr{L}_W = F^i\,W_i - \frac{1}{2}\,W_{ij}\,\psi^i\,\psi^j + \mathrm{h.c.} \qquad\qquad \mathscr{L}_{\text{FI}} = p_a\,D^a$$

The EOMs of the auxiliary fields are

$$F^i = -\,\bar{W}^i(\bar{X})\ , \qquad \bar{F}^i = -\,W_i(X)\ , \qquad D_a = -\,g\,\bar{X}_i\,(t_a)^i{}_j\,X^j - p_a$$

These fields enter the Lagrangian algebraically and quadratically. They are integrated out exactly via their classical EOMs.

# Integrating out the auxiliary fields

$$\mathscr{L}_{\text{vec}} = -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} + i D_\mu \overline{\lambda}^a \overline{\sigma}^\mu \lambda_a + \frac{1}{2} D^a D_a$$

$$\mathscr{L}_{\text{kin}} = - D^\mu \overline{X}_i D_\mu X^i + i D_\mu \overline{\psi}_i \overline{\sigma}^\mu \psi^i + \overline{F}_i F^i$$

$$\mathscr{L}_{\text{coupl}} = i\sqrt{2}\, g \left[\overline{X}_i (t_a)^i{}_j \psi^j \lambda^a - \overline{\lambda}^a \overline{\psi}_i (t_a)^i{}_j X^j\right] + g D^a \overline{X}_i (t_a)^i{}_j X^j$$

$$\mathscr{L}_W = F^i W_i - \frac{1}{2} W_{ij} \psi^i \psi^j + \mathrm{h.c.} \qquad \mathscr{L}_{\text{FI}} = p_a D^a$$

We get the Lagrangian

$$\mathscr{L} = \text{(YM and kin terms)} + i\sqrt{2}\, g \left[\overline{X}_i (t_a)^i{}_j \psi^j \lambda^a - \overline{\lambda}^a \overline{\psi}_i (t_a)^i{}_j X^j\right]$$

$$-\frac{1}{2} W_{ij} \psi^i \psi^j - \frac{1}{2} \overline{W}^{ij} \overline{\psi}_i \overline{\psi}_j - V(X, \overline{X})$$

where the scalar potential is the sum of an "F-term" and a "D-term":

$$V(X, \overline{X}) = \overline{F}_i(X) F^i(\overline{X}) + \frac{1}{2} D_a(X, \overline{X}) D^a(X, \overline{X})$$

$$F^i = - \overline{W}^i(\overline{X}) \ , \qquad \overline{F}^i = - W_i(X) \ , \quad D_a = - g \overline{X}_i (t_a)^i{}_j X^j - p_a$$

# SUSY vacua

The scalar potential is the sum of an "F-term" and a "D-term":

$$V(X,\overline{X}) = \overline{F}_i(X)\,F^i(\overline{X}) + \frac{1}{2}\,D_a(X,\overline{X})\,D^a(X,\overline{X})$$

$$F^i = -\,\overline{W}^i(\overline{X}) \quad , \qquad \overline{F}^i = -\,W_i(X) \quad , \qquad D_a = -\,g\,\overline{X}_i\,(t_a)^i{}_j\,X^j - p_a$$

As expected in a SUSY theory, $V$ is non-negative.

- In a generic vacuum the scalars $X^i$'s have (covariantly) constant VEVs that are at a stationary point of $V$
- In a SUSY vacuum, we must have $V=0$ and therefore we must set to zero all F-terms and all D-terms:

$$\frac{\partial W}{\partial X^i} = 0 \quad , \qquad g\,\overline{X}_i\,(t_a)^i{}_j\,X^j + p_a = 0$$

- We can see that SUSY is unbroken from the variations of the fermions:

$$\delta\psi^i = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\,D_\mu X^i + \sqrt{2}\,F^i\,\xi_\alpha \quad , \qquad \delta\lambda^a_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F^a{}_{\mu\nu} + i\,D^a\,\xi_\alpha$$

- Depending on the model, SUSY vacua might not exist! Spontaneous SUSY breaking

# Example: SQED

Our first example is the supersymmetric version of QED with a massive electron. To construct it we start from a model with two chiral superfields $(X^+, \psi^+_\alpha, F^+)$ and $(X^-, \psi^-_\alpha, F^-)$ with canonical kinetic terms and superpotential

$$W = m\, X^+ X^-$$

This model is invariant under a global flavor $U(1)$ symmetry, under which $(X^+, \psi^+_\alpha, F^+)$ and $(X^-, \psi^-_\alpha, F^-)$ have opposite charges $\pm 1$.

We gauge this global symmetry with a $U(1)$ vector multiplet. Since the gauge group is Abelian, we can add an FI term.

In our general notation, $a$ takes only one value, while $i = \pm$ and

$$(t_a)^{+}{}_{+} = +1 \;\;,\;\; (t_a)^{+}{}_{+} = -1 \;\;, \qquad (t_a)^{\pm}{}_{\mp} = 0$$

# Example: SQED

The full Lagrangian reads (after eliminating the aux fields)

$$
\begin{aligned}
\mathscr{L} = &\ \text{(YM and kin terms)} \\
&+ i\sqrt{2}\, g \left[\overline{X^+}\, \psi^{+j} \lambda - \bar{\lambda}\, \overline{\psi^+}\, X^+\right] + i\sqrt{2}\, g \left[-\overline{X^-}\, \psi^{-j} \lambda + \bar{\lambda} \overline{\psi^-}\, X^-\right] \\
&- m\, \psi^+ \psi^- - \overline{m}\, \overline{\psi^+}\, \overline{\psi^-} - V
\end{aligned}
$$

where

$$
V = |m|^2 |X^+|^2 + |m|^2 |X^-|^2 + \frac{1}{2} \left[g \left(\overline{X^+}\, X^+ - \overline{X^-}\, X^-\right) + p\right]^2
$$

The F-term and D-term equations are

$$
\overline{F^{\pm}} = -m X^{\mp} = 0 \ , \quad F^{\pm} = -\overline{m}\, \overline{X^{\mp}} = 0 \ , \qquad D = -g\, [\overline{X^+}\, X^+ - \overline{X^-}\, X^-] - p = 0
$$

If the FI parameter $p$ is non-zero, we cannot have a SUSY vacuum. Let's set $p = 0$.

# Example: SQED

$$\mathscr{L} = \text{(YM and kin terms)} - m\,\psi^+\,\psi^- - \overline{m}\,\overline{\psi^+}\,\overline{\psi^-} - V$$
$$+ i\sqrt{2}\,g\left[\overline{X^+}\,\psi^{+j}\,\lambda - \bar{\lambda}\,\overline{\psi^+}\,X^+\right] + i\sqrt{2}\,g\left[-\overline{X^-}\,\psi^{-j}\,\lambda + \bar{\lambda}\,\overline{\psi^-}\,X^-\right]$$
$$V = |\,m\,|^2\,|\,X^+\,|^2 + |\,m\,|^2\,|\,X^-\,|^2 + \frac{1}{2}\,g^2\left(\overline{X^+}\,X^+ - \overline{X^-}\,X^-\right)^2$$

Suppose $m = |\,m\,|\,e^{i\alpha}$. With the redefinitions

$$(X^+, \psi^+_\alpha, F^+) \to (X^+, \psi^+_\alpha, F^+)\,e^{-i\alpha/2} \quad , \qquad (X^-, \psi^-_\alpha, F^-) \to (X^-, \psi^-_\alpha, F^-)\,e^{-i\alpha/2}$$

we can get rid of the phase of $m$. We can then assume $m$ is real and positive. We collect the two Weyl spinor into a 4-component Dirac spinor and we find a standard mass term

$$\Psi = \begin{pmatrix} \psi^+_\alpha \\ (\overline{\psi^-})^{\dot{\alpha}} \end{pmatrix} \quad , \qquad \overline{\Psi} = \Psi^\dagger\,i\,\gamma^0 = \left(\psi^{-\alpha} \quad (\overline{\psi^+})_{\dot{\alpha}}\right) , \qquad \overline{\Psi}\,\Psi = \psi^-\,\psi^+ + \overline{\psi^+}\,\overline{\psi^-}$$

The 4-component Dirac spinor $\Psi$ plays the role of the electron. The coupling $g$ is the electron charge.

# Intermezzo: fermion masses

If we take a generic SUSY gauge theory and we expand around some VEV for the scalars, we will generically originate mass terms of the fermions, of the form

$$\mathscr{L} \supset -\mathscr{M}_{ij}\psi^i\psi^j - \overline{\mathscr{M}}^{ij}\overline{\psi}_i\overline{\psi}_j$$

The symmetric matrix $\mathscr{M}_{ij}$ can receive contributions from explicit mass couplings from the superpotential, as well as contributions from the VEVs of scalars.

Fact of life: for any complex symmetric matrix $\mathscr{M}$, a unitary matrix $\mathscr{U}$ exists such that

$$\mathscr{M}_\Delta = \mathscr{U}^T\,\mathscr{M}\,\mathscr{U}$$

is diagonal with real non-negative entries, $\mathscr{M}_\Delta = \mathrm{diag}(m_1, m_2, \ldots)$. The eigenvalues $m_1$, $m_2$, …, are the physical masses of the fermions in the model. It is worth noting that $m_1^2$, $m_2^2$, …, are the eigenvalues of $\mathscr{M}^\dagger\,\mathscr{M}$ (which is a positive hermitian matrix). This observation is often useful in comparing the masses of fermions and bosons.

# Example: SQCD

Our next example is the supersymmetric version of QCD with a generic number $N_c$ of colors and $N_f$ of flavors. We consider a model with massless quarks. Ordinary QCD is non-chiral. A given flavor of quark is usually described by a 4-component Dirac spinor $\Psi^I$ where $I$ is a fund index of $SU(N_c)$. In 2-component language, we find two independent Weyl spinors, one in the fund, the other in the antifund,

$$\Psi^I = \begin{pmatrix} {\psi^I}_\alpha \\ \epsilon^{\dot\alpha\dot\beta}\,(\widetilde{\psi}_{I\beta})^* \end{pmatrix}$$

This observation motivates the content of SQCD:

- a vector superfield in the adjoint of $SU(N_c)$
- $N_f$ identical copies of a chiral supermultiplet in the fund rep of $SU(N_c)$
- $N_f$ identical copies of a chiral supermultiplet in the antifund rep of $SU(N_c)$

We consider a model with $W = 0$. We cannot turn on any FI terms.

# Example: SQCD

The scalars in the chiral multiplets are usually called “squarks” and often denoted $Q$ instead of $X$. Our notation is as follows:

chiral multiplets in the fund of $SU(N_c)$ : $\quad (Q^I{}_{\hat{I}}, \psi_\alpha{}^I{}_{\hat{I}}, F^I{}_{\hat{I}})$

chiral multiplets in the antifund of $SU(N_c)$ : $\quad (\widetilde{Q}^{\hat{I}'}{}_I, \overline{\psi}_\alpha{}^{\hat{I}'}{}_I, \widetilde{F}^{\hat{I}'}{}_I)$

where

$I = 1,\ldots,N_c$ is a fund/antifund index of $SU(N_c)$

$\hat{I} = 1,\ldots,N_f$ labels the copies of the chirals in the fund of $SU(N_c)$

$\hat{I}' = 1,\ldots,N_f$ labels the copies of the chirals in the antifund of $SU(N_c)$

# Example: SQCD

It is convenient to introduce an index-free matrix notation:

$$[Q]_{I\hat{I}} = Q^{I}{}_{\hat{I}} \quad , \quad [\widetilde{Q}]_{\hat{I}'I} = \widetilde{Q}^{\hat{I}'}{}_{I} \qquad , \quad \text{and similarly for all other fields}$$

A finite $SU(N_c)$ gauge transf in this notation is

$$\text{gauge:} \quad Q \to U\,Q \quad , \quad Q^\dagger \to Q^\dagger\,U^{-1} \; , \qquad \widetilde{Q} \to \widetilde{Q}\,U^{-1} \, , \qquad \widetilde{Q}^\dagger \to U\,\widetilde{Q}^\dagger \quad , \quad U \in SU(N_c)$$

It turns out that all terms in the Lagrangian are invariant under a chiral $SU(N_f) \times SU(N_f)$ global symmetry. The first factor acts on the $Q$'s only ($\hat{I}$ indices):

$$\text{global on } Q\text{'s:} \qquad Q \to Q\,V^{-1} \quad , \quad Q^\dagger \to V\,Q^\dagger \; , \quad \widetilde{Q} \to \widetilde{Q} \, , \quad \widetilde{Q}^\dagger \to \widetilde{Q}^\dagger \quad , \quad V \in SU(N_f)$$

The  first factor acts on the $\widetilde{Q}$'s only ($\hat{I}'$ indices):

$$\text{global on } \widetilde{Q}\text{'s:} \qquad Q \to Q \quad , \quad Q^\dagger \to Q^\dagger \; , \quad \widetilde{Q} \to V'\,\widetilde{Q} \, , \quad \widetilde{Q}^\dagger \to \widetilde{Q}^\dagger\,V'^{-1} \quad , \quad V' \in SU(N_f)$$

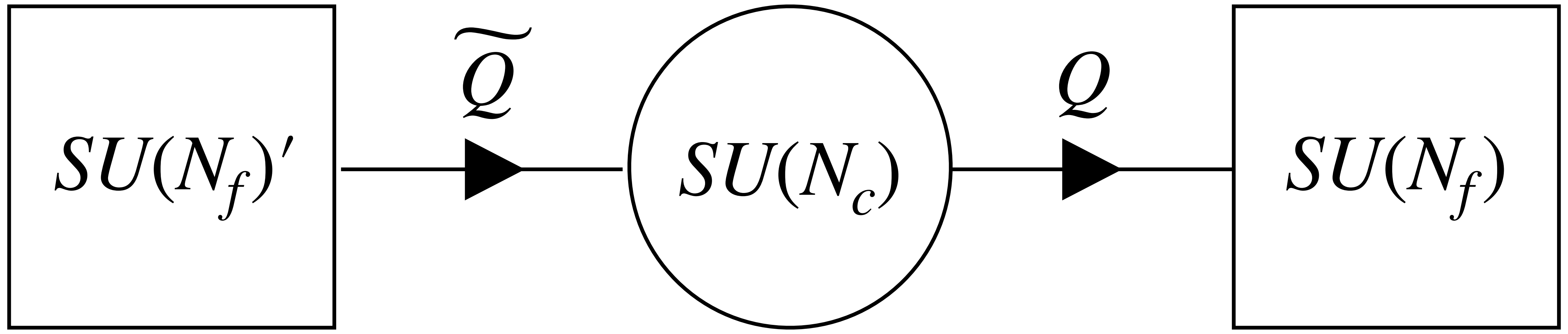
This $SU(N_f) \times SU(N_f)$ is the analog of the chiral symmetry of ordinary QCD with massless quarks.

# Example: SQCD

It is customary to collect the content of the model in a table with representations

| | $SU(N_c)$ | $SU(N_f)$ | $SU(N_f)'$ |
|---|---|---|---|
| ${Q^I}_{\widehat{I}}$ | $\square$ | $\overline{\square}$ | $\bullet$ |
| ${\widetilde{Q}^{\widehat{I}'}}_I$ | $\overline{\square}$ | $\bullet$ | $\square$ |
| ${\overline{Q}_I}^{\widehat{I}}$ | $\overline{\square}$ | $\square$ | $\bullet$ |
| ${\overline{\widetilde{Q}}_{\widehat{I}'}}^I$ | $\square$ | $\bullet$ | $\overline{\square}$ |

There is also a pictorial way based on “quiver diagrams”



# Example: SQCD

$$[Q]_{I\hat{I}} = Q^I{}_{\hat{I}} \quad , \quad [\widetilde{Q}]_{\hat{I}'I} = \widetilde{Q}^{\hat{I}'}{}_I$$

The D-term in a general model is $-D_a = \overline{X}_i\,(t_a)^i{}_j\,X^j$. This term splits into two because $X$ stands both for the $Q$'s and the $\widetilde{Q}$'s:

from the $Q$'s: $\quad \overline{X}_i\,(t_a)^i{}_j\,X^j \supset \overline{Q}_I{}^{\hat{I}}\,(t_a)^I{}_J\,Q^J{}_{\hat{I}} = \mathrm{tr}(Q^\dagger\,t_a\,Q)$

from the $\widetilde{Q}$'s: $\quad \overline{X}_i\,(t_a)^i{}_j\,X^j \supset \overline{\widetilde{Q}}_{\hat{I}'}{}^I\,(t_a^{\mathrm{anti}})_I{}^J\,\widetilde{Q}^{\hat{I}'}{}_J = \widetilde{Q}^{\hat{I}'}{}_J\,(-t_a)^J{}_I\,\overline{\widetilde{Q}}_{\hat{I}'}{}^I = -\,\mathrm{tr}(\widetilde{Q}\,t_a\,\widetilde{Q}^\dagger)$

To see the relation between $(t_a^{\mathrm{anti}})_I{}^J$ to $(t_a)^I{}_J$ we can use for example

$$\delta_{\mathrm{gauge}}\widetilde{Q}^{\hat{I}'}{}_I = -\,i\,\Lambda^a\,\widetilde{Q}^{\hat{I}'}{}_J\,(t_a)^J{}_I \equiv i\,\Lambda^a\,(t_a^{\mathrm{anti}})_I{}^J\,\widetilde{Q}^{\hat{I}'}{}_J$$

Notice that $\mathrm{tr}(Q^\dagger\,t_a\,Q)$ and $\mathrm{tr}(\widetilde{Q}\,t_a\,\widetilde{Q}^\dagger)$ are manifestly invariant under $SU(N_f) \times SU(N_f)$ :

$$Q \to Q\,V^{-1} \quad , \quad \widetilde{Q} \to V'\,\widetilde{Q}$$

# Example: SQCD

The D-term relations are

SUSY vacua: $$\mathrm{tr}(Q^\dagger\, t_a\, Q) - \mathrm{tr}(\widetilde{Q}\, t_a\, \widetilde{Q}^\dagger) = 0$$

These equations can have a non-trivial space of solutions (the “moduli space” of the model). NB: this is a classical analysis, which receives interesting quantum corrections.

# Supersymmetry and supergravity

Lecture 14

# Superspace: the idea

- SUSY variations and actions can in principle be found by trial-and-error, but it can be cumbersome and computationally demanding
- We would like a formalism in which SUSY is **manifest**
- Superspace is such a formalism
- Ordinary space with ordinary bosonic coordinates $x^\mu$ is enlarged with fictitious "fermionic coordinates" $\theta_\alpha$, $\bar{\theta}_{\dot{\alpha}}$. These are Grassmann numbers
- We will interpret a SUSY transformation as the effect of a translation along the fermionic coordinates $\theta_\alpha$, $\bar{\theta}_{\dot{\alpha}}$
- Using a suitable notion of integration in superspace, we will construct actions that are manifestly invariant under "translations in the fermionic directions"

# Warm-up: ordinary space as a coset

- Let us consider the ordinary Poincaré Lie algebra:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\,\eta_{\mu\rho}\,J_{\nu\sigma} - i\,\eta_{\nu\rho}\,J_{\mu\sigma} - i\,\eta_{\mu\sigma}\,J_{\nu\rho} + i\,\eta_{\nu\sigma}\,J_{\mu\rho}$$

$$[J_{\mu\nu}, P_\rho] = i\,\eta_{\mu\rho}\,P_\nu - i\,\eta_{\nu\rho}\,P_\mu \quad , \qquad\qquad [P_\mu, P_\nu] = 0$$

- We use the notation $ISO(1,3)$ for the Poincaré group, while $SO(1,3)$ denotes its Lorentz subgroup (no translations). The respective Lie algebras are denoted $\mathfrak{iso}(1,3)$ and $\mathfrak{so}(1,3)$
- The exponential map take an arbitrary element of the Lie algebra to an element of the corresponding Lie group:

$$-x^\mu\,P_\mu + \frac{1}{2}\,\omega^{\mu\nu}\,J_{\mu\nu} \in \mathfrak{iso}(1,3) \;\; , \qquad \exp\Big(-i\,x^\mu\,P_\mu + \frac{1}{2}\,i\,\omega^{\mu\nu}\,J_{\mu\nu}\Big) \in ISO(1,3)$$

$$\frac{1}{2}\,\omega^{\mu\nu}\,J_{\mu\nu} \in \mathfrak{so}(1,3) \;\; , \qquad \exp\Big(\frac{1}{2}\,i\,\omega^{\mu\nu}\,J_{\mu\nu}\Big) \in SO(1,3)$$

- The quantities $x^\mu$, $\omega^{[\mu\nu]}$ are real parameters. Our conventions are motivated by thinking of $P_\mu$, $J_{\mu\nu}$ as Hermitian operators, which are taken by $\exp(i\ldots)$ to unitary operators

# Warm-up: ordinary space as a coset

- In general, given a group $G$ and a subgroup $H$ of $G$, we can define the coset $G/H$ by taking equivalence classes in $G$ under the equiv. relation defined by action of $H$ from the right:

$$g \sim g\, h \,, \qquad g \in G \;\;, \qquad h \in H$$

- We apply this construction to $G = ISO(1,3)$, $H = SO(1,3)$. The coset is Minkowski space

$$\mathbb{R}^{1,3} \cong ISO(1,3)/SO(1,3)$$

- We will not give a formal proof. Intuitively, the quotient makes the $\frac{1}{2}\,\omega^{\mu\nu}\,J_{\mu\nu}$ part "pure gauge", so that we can parametrize an element of the coset using a standard representative

$$G(x) = \exp(-i\, x^\mu\, P_\mu)$$

- The parameters $x^\mu$ are coordinates in Minkowski space

# Left action on the coset

- Let us choose a fixed element $g_0 \in G$ and let us consider the action of $g_0$ on the standard representative $G(x)$, $g_0^{-1}\, G(x)$ (the inverse is for later convenience)
- The element $g_0^{-1}\, G(x) \in G$ belongs to a unique equivalence class in $G/H$, which has a unique standard representative $G(x')$ for some coordinates $x'$. Explicitly:

$$g_0^{-1}\, G(x) = G(x')\, h(x, g_0) \quad , \quad h(x, g_0) \in H$$

- We can think of $h(x, g_0)$ as a “compensating gauge transformation” that restores the standard form of $G(x)$ after we act with $g_0$
- Each $g_0$ determines a “motion” in the space with coordinates $x$, going from $x$ to $x'$
- If $g_0$ is the identity plus and infinitesimal piece, it determines an infinitesimal variation $\delta x$ (we can think of it as a vector field on the coset)

# Example: $g_0$ is a translation

$$g_0^{-1}\, G(x) = G(x')\, h(x, g_0) \quad , \quad h(x, g_0) \in H$$

We can write

$$g_0 = \exp(-i\, a^\mu\, P_\mu)$$

and therefore

$$g_0^{-1}\, G(x) = \exp(i\, a^\mu\, P_\mu)\, \exp(-i\, x^\nu\, P_\nu) = \exp(-i\, (x-a)^\nu\, P_\nu)$$

This case is simple: we do not need any compensating $h(x, g_0) \in H$. The induced motion is simply a translation

$$g_0 = \exp(-i\, a^\mu\, P_\mu) \ : \quad x'^\mu = x^\mu - a^\mu$$

# Example: $g_0$ is a Lorentz transf

$$g_0^{-1}\, G(x) = G(x')\, h(x, g_0) \quad , \quad h(x, g_0) \in H$$

We can write $g_0 = \exp(\frac{i}{2}\,\lambda^{\mu\nu}\, J_{\mu\nu})$ and therefore $g_0^{-1}\, G(x) = \exp(-\frac{i}{2}\,\lambda^{\mu\nu}\, J_{\mu\nu})\, \exp(-i\, x^\nu\, P_\nu)$ .

For simplicity we work to linear order in the parameters $\lambda^{\mu\nu}$. Recall the Baker-Campbell-Hausdorff formula

$$\exp(A)\, \exp(B) = \exp\Big(A + B + \frac{1}{2}[A, B] + \frac{1}{12}\,[A, [A, B]] + \dots\Big)$$

Using the JP commutator, we find

$$g_0^{-1}\, G(x) = \exp(-\frac{i}{2}\,\lambda^{\mu\nu}\, J_{\mu\nu} - i\, x^\nu\, P_\nu + \frac{i}{2}\,(\lambda\, x)^\nu\, P_\nu + \dots)$$

This quantity should be cast in the form $G(x')\, h(x, g_0)$, so we compare it

$$\exp(-i\, x'^\nu\, P_\nu)\, \exp\Big(-\frac{i}{2}\,\widetilde{\lambda}^{\mu\nu}\, J_{\mu\nu}\Big) = \exp(-\frac{i}{2}\,\widetilde{\lambda}^{\mu\nu}\, J_{\mu\nu} - i\, x'^\nu\, P_\nu + \frac{i}{2}\,(\widetilde{\lambda}\, x')^\nu\, P_\nu + \dots)$$

Here $\widetilde{\lambda}^{\mu\nu}$ are params of the compensating $H$ transformation, and $x'^\mu$ are the coordinates of the transformed point on the coset. Comparing the arguments of the exp's at leading order in $\lambda^{\mu\nu}$ we discover

$$\widetilde{\lambda}^{\mu\nu} = \lambda^{\mu\nu} \quad , \qquad (x' - x)^\mu = -\,(\lambda\, x)^\mu$$

# Scalar fields

In the "active transformation" perspective, the transformation law of a scalar field on the coset is

$$\Phi'(x') = \Phi(x) \quad , \qquad g_0^{-1}\, G(x) = G(x') \mod H$$

For an infinitesimal transformation, this gives

$$\Phi(x+\delta) + \delta\Phi(x) = \Phi(x) \ , \qquad \delta\Phi(x) = -\,\delta x^\mu\, \partial_\mu \Phi(x)$$

In the example of a translation:

$$g_0 = \exp(-i\, a^\mu\, P_\mu) \quad , \ \delta x^\mu = -\, a^\mu \quad , \qquad \delta\Phi = a^\mu\, \partial_\mu \Phi = i\, a^\mu\, (-i\, \partial_\mu)\Phi$$

In the example of a Lorentz transformation:

$$g_0 = \exp(\tfrac{i}{2}\, \lambda^{\mu\nu}\, J_{\mu\nu}) \quad , \ \delta x^\mu = -\, \lambda^\mu{}_\nu\, x^\nu \quad , \qquad \delta\Phi = \lambda^{\mu\nu}\, x_\nu\, \partial_\mu \Phi = -\tfrac{i}{2}\, \lambda^{\mu\nu}\, (2\, i\, x_\nu\, \partial_\mu)\Phi$$

We read off a set of differential operators that satisfy the same algebra as the Poincaré generators:

$$\mathbf{P}_\mu = -\, i\, \partial_\mu \quad , \quad \mathbf{J}_{\mu\nu} = -\, i\, (x_\mu\, \partial_\nu - x_\nu\, \partial_\mu)$$

$$[\mathbf{J}_{\mu\nu}, \mathbf{J}_{\rho\sigma}] = i\, \eta_{\mu\rho}\, \mathbf{J}_{\nu\sigma} + \dots \ , \ [\mathbf{J}_{\mu\nu}, \mathbf{P}_\rho] = i\, \eta_{\mu\rho}\, \mathbf{P}_\nu + \dots$$

# Superspace

We define (flat) superspace as (super-Poincaré)/(Lorentz). Since the $J_{\mu\nu}$ generators are "pure gauge", our standard form of a coset representative is

$$G(x, \theta, \overline{\theta}) = \exp(-i\, x^{\mu}\, P_{\mu} + i\, \theta^{\alpha}\, Q_{\alpha} + i\, \overline{\theta}_{\dot{\alpha}}\, \overline{Q}^{\dot{\alpha}})$$

The motion in superspace generated by an element $g_0$ of the super-Poincaré group is defined according to the general formula

$$g_0^{-1}\, G(x, \theta, \overline{\theta}) = G(x', \theta', \overline{\theta}') \mod SO(1{,}3)$$

An ordinary translation is easy because $[P, P] = 0$, $[P, Q] = 0$, $[P, \overline{Q}] = 0$

$$g_0 = \exp(-ia^{\mu}\, P_{\mu}) \quad , \quad x' = x - a \quad , \quad \theta' = \theta \quad , \quad \overline{\theta}' = \overline{\theta}$$

# Supertranslations

$$g_0^{-1}\, G(x,\theta,\overline{\theta}) = G(x',\theta',\overline{\theta}') \quad \text{mod } SO(1,3)$$

Let us now consider $g_0 = \exp(i\xi\, Q + i\,\overline{\xi}\,\overline{Q})$. We use BCH

$$\exp(A)\ \exp(B) = \exp\Big(A + B + \frac{1}{2}[A,B] + \frac{1}{12}\,[A,[A,B]] + \dots\Big)$$

to simplify

$$\exp(-i\xi\, Q - i\,\overline{\xi}\,\overline{Q})\ \exp(-i\,x^\mu\, P_\mu + i\,\theta\, Q + i\,\overline{\theta}\,\overline{Q})$$

Only the first commutator term is non-zero, all the nested commutators vanish. The above quantity is exactly equal to

$$\exp(-i\xi\, Q - i\,\overline{\xi}\,\overline{Q} - i\,x^\mu\, P_\mu + i\,\theta\, Q + i\,\overline{\theta}\,\overline{Q} + \frac{1}{2}\,[-i\xi\, Q - i\,\overline{\xi}\,\overline{Q}, i\,\theta\, Q + i\,\overline{\theta}\,\overline{Q}])$$

The SUSY algebra gives $[\xi\, Q, \overline{\theta}\,\overline{Q}] = 2\,\xi\,\sigma^\mu\,\overline{\theta}\,P_\mu$

# Supertranslations

In the end we find the quantity

$$\exp(-i\,x'^{\mu}\,P_{\mu} + i\,\theta'\,Q + i\,\overline{\theta}'\,\overline{Q})$$

where

$$\theta'^{\alpha} = \theta^{\alpha} - \xi^{\alpha} \quad , \qquad \overline{\theta}'_{\dot\alpha} = \overline{\theta}_{\dot\alpha} - \overline{\xi}_{\dot\alpha}\,, \qquad x'^{\mu} = x^{\mu} - (i\,\theta\,\sigma^{\mu}\,\overline{\xi} - i\,\xi\,\sigma^{\mu}\,\overline{\theta})$$

Remarks:

- We do not need any compensating $H$ transformation
- We have not assumed that the $\xi$, $\overline{\xi}$ params are infinitesimal
- A translation in $\theta$ or $\overline{\theta}$ induces a translation in regular space: this is the superspace version of the fundamental relation $\{Q, \overline{Q}\} \sim P$

# Superfields

By definition, a superfield is a function $\Phi(x, \theta, \overline{\theta})$ that transforms under the action of super-Poincaré according to the scalar transformation law

$$\Phi'(x', \theta', \overline{\theta}') = \Phi(x, \theta, \overline{\theta})$$

For an infinitesimal transformation $x' = x + \delta x$, $\theta' = \theta + \delta\theta$, $\overline{\theta}' = \overline{\theta} + \delta\overline{\theta}$ we have

$$\Phi'(x, \theta, \overline{\theta}) = \Phi(x, \theta, \overline{\theta}) + \delta\Phi(x, \theta, \overline{\theta})$$

$$\delta\Phi = -\,\delta x^\mu\,\partial_\mu\Phi - \delta\theta^\alpha\,\frac{\partial}{\partial\theta^\alpha}\Phi - \delta\overline{\theta}_{\dot{\alpha}}\,\frac{\partial}{\partial\overline{\theta}_{\dot{\alpha}}}\Phi$$

For an ordinary translation:

$$g_0 = \exp(-ia^\mu P_\mu)\ :\qquad \delta x^\mu = -\,a^\mu\ ,\ \delta\theta = 0,\ \ \delta\overline{\theta} = 0$$

$$\delta\Phi = a^\mu\,\partial_\mu\Phi = i\,a^\mu\,(-i\,\mathbf{P}_\mu)\Phi$$

# Differential op's implementing SUSY

$$\delta\Phi = -\,\delta x^\mu\,\partial_\mu\Phi - \delta\theta^\alpha\,\frac{\partial}{\partial\theta^\alpha}\Phi - \delta\bar\theta_{\dot\alpha}\,\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\Phi$$

For a supertranslation:

$$g_0 = \exp(i\xi\,Q + i\,\bar\xi\,\bar Q)\ :\qquad \delta\theta^\alpha = -\,\xi^\alpha\quad,\qquad \delta\bar\theta_{\dot\alpha} = -\,\bar\xi_{\dot\alpha}\,,\qquad \delta x^\mu = -\,(i\,\theta\,\sigma^\mu\,\bar\xi - i\,\xi\,\sigma^\mu\,\bar\theta)$$

We plug this into $\delta\Phi$ and we cast the result in the form

$$\delta\Phi = (-i\,\xi^\alpha\,\mathbf{Q}_\alpha - i\,\bar\xi_{\dot\alpha}\,\overline{\mathbf{Q}}^{\dot\alpha})\,\Phi\qquad \text{where}\quad \mathbf{Q}_\alpha = i\left(\frac{\partial}{\partial\theta^\alpha} - i\,(\sigma^\mu\,\bar\theta)_\alpha\,\partial_\mu\right)\quad,\qquad \overline{\mathbf{Q}}^{\dot\alpha} = i\left(\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i\,(\theta\,\sigma^\mu)^{\dot\alpha}\,\partial_\mu\right)$$

Remarks:

- we have used $(\theta\,\sigma^\mu)_{\dot\alpha}\,\bar\xi^{\dot\alpha} = -\,\bar\xi^{\dot\alpha}\,(\theta\,\sigma^\mu)_{\dot\alpha} = +\,\bar\xi_{\dot\alpha}\,(\theta\,\sigma^\mu)^{\dot\alpha}$
- to compare to Wess-Bagger, notice that our $\mathbf{Q}_\alpha$, $\overline{\mathbf{Q}}^{\dot\alpha}$ have an extra i compared to WB, and use $(\theta\,\sigma^\mu)^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\,(\theta\,\sigma^\mu)_{\dot\beta} = -\,(\theta\,\sigma^\mu)_{\dot\beta}\,\epsilon^{\dot\beta\dot\alpha}$
- we also have $(-i\,\xi^\alpha\,\mathbf{Q}_\alpha - i\,\bar\xi_{\dot\alpha}\,\overline{\mathbf{Q}}^{\dot\alpha})_{\text{here}} = (\xi^\alpha\,Q_\alpha + \bar\xi_{\dot\alpha}\,\bar Q^{\dot\alpha})_{\text{WB}}$ so our $\delta\Phi$ is the same as theirs

# Differential op's implementing SUSY

$$\mathbf{P}_\mu = -\,i\,\partial_\mu\,, \qquad \mathbf{Q}_\alpha = i\left(\frac{\partial}{\partial\theta^\alpha} - i\,(\sigma^\mu\,\bar\theta)_\alpha\,\partial_\mu\right)\,, \qquad \overline{\mathbf{Q}}^{\dot\alpha} = i\left(\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i\,(\theta\,\sigma^\mu)^{\dot\alpha}\,\partial_\mu\right)$$

The coset formalism guarantees that the differential operators $\mathbf{P}_\mu$, $\mathbf{Q}_\alpha$, $\overline{\mathbf{Q}}_{\dot\alpha}$ satisfy the same SUSY algebra as the abstract generators:

$$[\mathbf{P}_\mu, \mathbf{Q}_\alpha] = 0\,, \qquad [\mathbf{P}_\mu, \overline{\mathbf{Q}}_{\dot\alpha}] = 0$$

$$\{\mathbf{Q}_\alpha, \mathbf{Q}_\beta\} = 0\,, \qquad \{\overline{\mathbf{Q}}_{\dot\alpha}, \overline{\mathbf{Q}}_{\dot\beta}\} = 0\,, \qquad \{\mathbf{Q}_\alpha, \overline{\mathbf{Q}}_{\dot\beta}\} = 2\,(\sigma^\mu)_{\alpha\dot\beta}\,\mathbf{P}_\mu$$

One can also check these relations explicitly.

NB: when we lower the index on $\overline{\mathbf{Q}}^{\dot\alpha}$ we get

$$\overline{\mathbf{Q}}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\,\overline{\mathbf{Q}}^{\dot\beta} = i\left(\epsilon_{\dot\alpha\dot\beta}\,\frac{\partial}{\partial\bar\theta_{\dot\beta}} + i\,(\theta\,\sigma^\mu)_{\dot\alpha}\,\partial_\mu\right) = i\left(-\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\,(\theta\,\sigma^\mu)_{\dot\alpha}\,\partial_\mu\right)$$

We will see later in more detail how differentiation wrt Grassman variables works.

# Extra: cosets and diff op's in general

Let us consider a general coset $G/H$ with a choice of standard representatives $G(z)$ for a set of coordinates $z^M$. The left action of $G$ on the coset determines a motion in $G/H$ according to

$$g_0^{-1}\, G(z) = G(z') \mod H$$

If $g_0$ is infinitesimal, we can write

$$g_0 = \mathbb{I} + i\,\alpha^A\, T_A \quad , \qquad z^M \to z^M + \alpha^A\, V_A{}^M(z)$$

In this way we associate to each generator $T_A$ of $\mathfrak{g} = \mathrm{Lie}(G)$ a vector field $V_A{}^M(z)$ on $G/H$.

# Extra: cosets and diff op's in general

The Lie bracket of the vector fields ${V_A}^M(z)$ gives a representation of the abstract algebra: if $i\,\alpha_3^A\,T_A$ is the abstract commutator of $i\,\alpha_1^A\,T_A$ and $i\,\alpha_2^A\,T_A$,

$$i\,\alpha_3^A\,T_A = [i\,\alpha_1^A\,T_A, i\,\alpha_2^B\,T_B]$$

then the vector field $\alpha_3^A\,{V_A}^M(z)$ associated to $i\,\alpha_3^A\,T_A$ is the Lie bracket of those of $i\,\alpha_1^A\,T_A$ and $i\,\alpha_2^A\,T_A$,

$$\alpha_3^A\,{V_A}^M = \alpha_1^A\,{V_A}^N\,\partial_N(\alpha_2^B\,{V_B}^M) - \alpha_2^A\,{V_A}^N\,\partial_N(\alpha_1^B\,{V_B}^M)$$

# Extra: cosets and diff op's in general

To see this: define $g_{1,2} = \mathbb{I} + i\,\alpha^A_{1,2}\,T_A$ and observe that

$$g_0 = g_1^{-1}\,g_2^{-1}\,g_1\,g_2 = \mathbb{I} + [i\,\alpha_1^A\,T_A, i\,\alpha_2^B\,T_B] + \dots$$

Because of the inverse in our definition $g_0^{-1}\,G(z) = G(z') \mod H$, acting with $g_0$ means considering the motion induced by $g_1^{-1}$, then $g_2^{-1}$, then $g_1$, finally $g_2$,

$$z' = z - \alpha_1^A\,V_A(z) \;\;,\;\; z'' = z' - \alpha_2^A\,V_A(z')$$

$$z''' = z'' + \alpha_1^A\,V_A(z'')\,,\;\; z'''' = z''' + \alpha_2^A\,V_A(z''')$$

Keep only terms at most linear in $\alpha_{1,2}$ and get

$$z'''' - z = \alpha_1^A\,{V_A}^N\,\partial_N(\alpha_2^B\,{V_B}^M) - \alpha_2^A\,{V_A}^N\,\partial_N(\alpha_1^B\,{V_B}^M)$$

# Extra: cosets and diff op's in general

The transformation law of a scalar field ("active pov") $\Phi'(z') = \Phi(z)$ gives

$$\delta\Phi = -\,\delta z^M\,\partial_M\Phi = -\,\alpha^A\,V_A{}^M\,\partial_M\Phi \equiv -\,i\,\alpha^A\,\mathbf{T}_A\Phi \qquad \text{where by def} \qquad \mathbf{T}_A = -\,i\,V_A{}^M\,\partial_M$$

These operators obey the same algebra as the abstract generators: if we have

$$i\,\alpha_3^A\,T_A = [i\,\alpha_1^A\,T_A, i\,\alpha_2^B\,T_B]$$

then we have

$$i\,\alpha_3^A\,\mathbf{T}_A\Phi = i\,\alpha_1^A\,\mathbf{T}_A\big(i\,\alpha_2^B\,\mathbf{T}_B\Phi\big) - i\,\alpha_2^A\,\mathbf{T}_A\big(i\,\alpha_1^B\,\mathbf{T}_B\Phi\big)$$

This is easy to see using the fact that

$$\alpha_3^A\,V_A{}^M = \alpha_1^A\,V_A{}^N\,\partial_N(\alpha_2^B\,V_B{}^M) - \alpha_2^A\,V_A{}^N\,\partial_N(\alpha_1^B\,V_B{}^M)$$

NB: we have given a "purely bosonic" argument, but if one is careful not to change the order of generators/params this extends to cosets built with supergroups/Lie superalgebras.

# Supersymmetry and supergravity

Lecture 15

# Calculus with Grassmann variables

- We need to set our conventions about manipulating Grassmann variables to work with superspace
- Let's start with a single Grassmann variable $\eta$
- Any function of $\eta$ is understood as a power series. It truncates because $\eta\,\eta = 0$ :

$$f(\eta) = f_0 + \eta\, f_1$$

- If $f_0$ is Grassmann-even, then $f_1$ is Grassmann-odd, and vice versa (so the order $\eta\, f_1$ is important when $f_1$ is Grassmann-odd)
- The differential operator $\partial/(\partial\eta)$ acts <u>from the left</u> and satisfies

$$\frac{\partial}{\partial\eta}\eta = 1 \qquad \text{for example:} \quad \frac{\partial}{\partial\eta} f(\eta) = f_1$$

# Calculus with Grassmann variables

- If we have two Grassmann variables $\theta^\alpha = (\theta^1, \theta^2)$ then we have

$$\frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta$$

- For example $\frac{\partial}{\partial\theta^1}$ does not act on $\theta^2$, but we pick up a minus sign when $\frac{\partial}{\partial\theta^1}$ moves past $\theta^2$,

$$\frac{\partial}{\partial\theta^1}(\theta^2\,\theta^1) = -\,\theta^2\,\frac{\partial}{\partial\theta^1}\theta^1 = -\,\theta^2$$

- By definition, $\frac{\partial}{\partial\theta_\alpha}$ is defined by acting from the left and picking up $\theta_\alpha$ factors

$$\frac{\partial}{\partial\theta_\alpha}\theta_\beta = \delta_\beta{}^\alpha$$

# Calculus with Grassmann variables

- How are $\partial/(\partial\theta^\alpha)$ and $\partial/(\partial\theta_\alpha)$ related? Let's consider the quantity $f(\theta) = \theta^\alpha \chi_\alpha$ where $\chi_\alpha$ does not depend on $\theta$'s. Our definitions give

$$\frac{\partial}{\partial\theta^\beta} f(\theta) = \delta_\beta{}^\alpha \chi_\alpha = \chi_\beta$$

- On the other hand we can also write $f(\theta) = -\theta_\alpha \chi^\alpha$. Since we are not changing the order of $\theta$ and $\chi$ it doesn't matter if $\chi$ is a boson or a fermion. We then have

$$\frac{\partial}{\partial\theta_\beta} f(\theta) = -\chi^\beta$$

- Since $\chi^\beta = \epsilon^{\beta\gamma}\chi_\gamma$ we have verified the identity

$$\epsilon^{\alpha\beta} \frac{\partial}{\partial\theta^\beta} = -\frac{\partial}{\partial\theta_\alpha}$$

- For applications to superspace we need both $\theta$'s and $\bar{\theta}$'s. The definitions for derivatives wrt $\bar{\theta}$ follow the same conventions

# Calculus with Grassmann variables

- Integration over Grassmann variables is a formal operation (aka **Berezin integral**) determined by linearity, translational invariance, integration by parts. In the case of a single variable, we demand

$$\int d\eta[f(\eta)\, c_1 + g(\eta)\, c_2] = \left(\int d\eta\, f(\eta)\right) c_1 + \left(\int d\eta\, g(\eta)\right) c_2$$

$$\int d\eta\, f(\eta - \eta_0) = \int d\eta\, f(\eta) \qquad , \qquad \int d\eta\, \frac{\partial}{\partial \eta} f(\eta) = 0$$

- One verifies that integration wrt to $\eta$ is the same as differentiation wrt to $\eta$

$$\int d\eta\, \eta = 1$$

- Similar remarks apply to the case of several variables

# Calculus with Grassmann variables

- In superspace we have four Grassmann variables $\theta^{\alpha=1,2}$ and $\overline{\theta}^{\dot{\alpha}=1,2}$ and we set by definition

$$\int d^2\theta = \int \frac{1}{2}\, d\theta^1\, d\theta^2 \quad , \qquad \int d^2\overline{\theta} = \int \frac{1}{2}\, d\overline{\theta}^2\, d\overline{\theta}^1$$

- This is useful because it implies

$$\int d^2\theta\,(\theta^\alpha\,\theta_\alpha) = 1 \quad , \quad \int d^2\overline{\theta}\,(\overline{\theta}_{\dot{\alpha}}\,\overline{\theta}^{\dot{\alpha}}) = 1 \quad , \quad \int d^2\theta\, d^2\overline{\theta}\,(\theta^\alpha\,\theta_\alpha)\,(\overline{\theta}_{\dot{\beta}}\,\overline{\theta}_{\dot{\beta}}) = 1$$

- For example, $\theta^\alpha\,\theta_\alpha = \theta^\alpha\,\epsilon_{\alpha\beta}\,\theta^\beta = -\,\theta^1\,\theta^2 + \theta^2\,\theta^1 = 2\,\theta^2\,\theta^1$ because $\epsilon_{12} = -\,1$ and therefore

$$\int \frac{1}{2}\, d\theta^1\, d\theta^2\,(2\,\theta^2\,\theta^1) = 1$$

- NB: we have $d^2\theta \propto \epsilon_{\alpha\beta}\, d\theta^\alpha\, d\theta^\beta$ which shows that this measure is Lorentz invariant; similarly for the measure $d^2\overline{\theta}$

# Taylor expansion of a generic superfield

- Let us consider an arbitrary function $\mathcal{S}(x,\theta,\bar\theta)$ of regular spacetime and the four Grassmann variables $\theta^{\alpha=1,2}$ and $\bar\theta^{\dot\alpha=1,2}$ . The possible monomials that can appear in the Taylor expansion of $\mathcal{S}(x,\theta,\bar\theta)$ in the odd coordinates are

$$\theta^\alpha\,,\bar\theta_{\dot\alpha}\ \ ,\quad \theta\,\theta\equiv\theta^\alpha\,\theta_\alpha\,,\quad \bar\theta\,\bar\theta\equiv\bar\theta_{\dot\alpha}\,\bar\theta^{\dot\alpha}\ \ ,\quad \theta^\alpha\,\bar\theta^{\dot\beta}\propto(\bar\sigma^\mu)^{\dot\beta\alpha}\,(\theta\,\sigma_\mu\,\bar\theta)\,,\quad (\theta\,\theta)\,\bar\theta_{\dot\alpha}\ \ ,\ \ (\bar\theta\,\bar\theta)\ \theta^\alpha\,,\ \ (\theta\,\theta)\,(\bar\theta\,\bar\theta)$$

- Inspired by the Wess-Bagger parametrization of a real superfield we write

$$\begin{aligned}\mathcal{S}(x,\theta,\bar\theta) = C(x) + i\,\theta^\alpha\,\chi_\alpha(x) - i\,\bar\theta_{\dot\alpha}\,\bar\chi^{\dot\alpha}(x)&\\ +\frac{1}{2}\,i\,(\theta\,\theta)\,M(x)-\frac{1}{2}\,i\,(\bar\theta\,\bar\theta)\,\overline{M}(x) - (\theta\,\sigma^\mu\,\bar\theta)\,v_\mu(x)&\\ +\,i\,(\theta\,\theta)\,\bar\theta_{\dot\alpha}\left[\bar\lambda^{\dot\alpha}(x)+\frac{1}{2}\,i\,(\bar\sigma^\mu)^{\dot\alpha\beta}\,\partial_\mu\chi_\beta(x)\right] - i\,(\bar\theta\,\bar\theta)\,\theta^\alpha\left[\lambda_\alpha(x)+\frac{1}{2}\,i\,(\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu\bar\chi^{\dot\beta}(x)\right]&\\ +\frac{1}{2}\,(\theta\,\theta)\,(\bar\theta\,\bar\theta)\left[D(x)+\frac{1}{2}\,\partial^\mu\,\partial_\mu C(x)\right]&\end{aligned}$$

- The coeff of $(\theta\,\theta)\,\bar\theta_{\dot\alpha}$ is written as $\bar\lambda^{\dot\alpha}+\frac{1}{2}\,i\,(\bar\sigma^\mu)^{\dot\alpha\beta}\,\partial_\mu\chi_\beta$ rather than just $\bar\lambda^{\dot\alpha}$ for later convenience, and similarly with the coeffs of $(\bar\theta\,\bar\theta)\,\theta^\alpha$ and $(\bar\theta\,\bar\theta)\,(\theta\,\theta)$

# Taylor expansion of a generic superfield

$$
\begin{aligned}
\mathcal{S}(x,\theta,\bar\theta) = {} & C(x) + i\,\theta^\alpha\,\chi_\alpha(x) - i\,\bar\theta_{\dot\alpha}\,\bar\chi^{\dot\alpha}(x) \\
& + \frac{1}{2}\,i\,(\theta\,\theta)\,M(x) - \frac{1}{2}\,i\,(\bar\theta\,\bar\theta)\,\bar M(x) - (\theta\,\sigma^\mu\,\bar\theta)\,v_\mu(x) \\
& + i\,(\theta\,\theta)\,\bar\theta_{\dot\alpha}\left[\bar\lambda^{\dot\alpha}(x) + \frac{1}{2}\,i\,(\bar\sigma^\mu)^{\dot\alpha\beta}\,\partial_\mu\chi_\beta(x)\right] - i\,(\bar\theta\,\bar\theta)\,\theta^\alpha\left[\lambda_\alpha(x) + \frac{1}{2}\,i\,(\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu\bar\chi^{\dot\beta}(x)\right] \\
& + \frac{1}{2}\,(\theta\,\theta)\,(\bar\theta\,\bar\theta)\left[D(x) + \frac{1}{2}\,\partial^\mu\,\partial_\mu C(x)\right]
\end{aligned}
$$

Remarks:

- In the simplest case $\mathcal{S}(x,\theta,\bar\theta)$ does not carry any indices. Then the component fields $C$, $M$, $\bar M$, $D$ are Lorentz scalars, $\chi$, $\bar\chi$, $\lambda$, $\bar\lambda$ are spinors, and $v_\mu$ is a vector. $C$, $M$, $\bar M$, $D$, $v_\mu$ have the same statistics, and similarly for $\chi$, $\bar\chi$, $\lambda$, $\bar\lambda$
- If there is no gauge-invariance, a generic $\mathcal{S}(x,\theta,\bar\theta)$ without spinor indices describes 16 + 16 real dof's off-shell.

# Taylor expansion of a generic superfield

$$\begin{aligned}\mathcal{S}(x,\theta,\bar{\theta}) = C(x) + i\,\theta^\alpha\,\chi_\alpha(x) - i\,\bar{\theta}_{\dot{\alpha}}\,\bar{\chi}^{\dot{\alpha}}(x) \\ +\frac{1}{2}\,i\,(\theta\,\theta)\,M(x) - \frac{1}{2}\,i\,(\bar{\theta}\,\bar{\theta})\,\overline{M}(x) - (\theta\,\sigma^\mu\,\bar{\theta})\,v_\mu(x) \\ +\,i\,(\theta\,\theta)\,\bar{\theta}_{\dot{\alpha}}\left[\bar{\lambda}^{\dot{\alpha}}(x) + \frac{1}{2}\,i\,(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\,\partial_\mu\chi_\beta(x)\right] - i\,(\bar{\theta}\,\bar{\theta})\,\theta^\alpha\left[\lambda_\alpha(x) + \frac{1}{2}\,i\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\partial_\mu\bar{\chi}^{\dot{\beta}}(x)\right] \\ +\frac{1}{2}\,(\theta\,\theta)\,(\bar{\theta}\,\bar{\theta})\left[D(x) + \frac{1}{2}\,\partial^\mu\,\partial_\mu C(x)\right]\end{aligned}$$

Remarks:

- A generic $\mathcal{S}(x,\theta,\bar{\theta})$ is complex, and all the component fields are complex, too. A **real superfield** satisfies $\mathcal{S}(x,\theta,\bar{\theta})^* = \mathcal{S}(x,\theta,\bar{\theta})$ which translates to

$$C^* = C \;\;,\;\; (\chi_\alpha)^* = \bar{\chi}_{\dot{\alpha}} \;\;,\;\; M^* = \overline{M}\,,\;\; (v_\mu)^* = v_\mu\,,\;\; (\lambda_\alpha)^* = \bar{\lambda}_{\dot{\alpha}} \;\;,\;\; D^* = D$$

- We can use the above Taylor expansion also if $\mathcal{S}(x,\theta,\bar{\theta})$ carries spinor indices, $\mathcal{S}_{\alpha\ldots\dot{\alpha}\ldots}(x,\theta,\bar{\theta})$. In that case all the component fields carry the extra set of indices $_{\alpha\ldots\dot{\alpha}\ldots}$ and transform under the Lorentz group accordingly

# SUSY variations from Taylor expansion

- The function $\mathcal{S}(x,\theta,\bar\theta)$ is a superfield if it transforms under SUSY according to

$$\delta\mathcal{S} = (-i\,\xi^\alpha\,\mathbf{Q}_\alpha - i\,\bar\xi_{\dot\alpha}\,\overline{\mathbf{Q}}^{\dot\alpha})\,\mathcal{S} \qquad \text{where} \quad \mathbf{Q}_\alpha = i\left(\frac{\partial}{\partial\theta^\alpha} - i\,(\sigma^\mu\,\bar\theta)_\alpha\,\partial_\mu\right) \quad , \qquad \overline{\mathbf{Q}}^{\dot\alpha} = i\left(\frac{\partial}{\partial\bar\theta_{\dot\alpha}} + i\,(\theta\,\sigma^\mu)^{\dot\alpha}\,\partial_\mu\right)$$

- One computes $(-i\,\xi^\alpha\,\mathbf{Q}_\alpha - i\,\bar\xi_{\dot\alpha}\,\overline{\mathbf{Q}}^{\dot\alpha})\,\mathcal{S}$ and compares the result with

$$\delta\mathcal{S}(x,\theta,\bar\theta) = \delta C(x) + i\,\theta^\alpha\,\delta\chi_\alpha(x) - i\,\bar\theta_{\dot\alpha}\,\bar\delta\chi^{\dot\alpha}(x) + \ldots$$

- For example, to read off the variation of $C(x)$ we need the terms with no $\theta$'s or $\bar\theta$'s. The terms $(\sigma^\mu\,\bar\theta)_\alpha\,\partial_\mu$ and $(\theta\,\sigma^\mu)^{\dot\alpha}\,\partial_\mu$ inside $\mathbf{Q}_\alpha$, $\overline{\mathbf{Q}}^{\dot\alpha}$ don't contribute, we only need

$$\xi^\alpha\,\frac{\partial}{\partial\theta^\alpha}\left(i\,\theta^\beta\,\chi_\beta\right) = i\,\xi^\alpha\,\chi_\alpha \quad , \qquad \bar\xi_{\dot\alpha}\,\frac{\partial}{\partial\bar\theta_{\dot\alpha}}\left(-\,i\,\bar\theta_{\dot\beta}\,\bar\chi^{\dot\beta}\right) = -\,i\,\bar\xi_{\dot\alpha}\,\bar\chi^{\dot\alpha}$$

  to conclude

$$\delta C = i\,\xi\,\chi - i\,\bar\xi\,\bar\chi$$

- Finding the variations of all other component fields is tedious but straightforward.

# SUSY variations from Taylor expnasion

Here is the result (from Cyril Closset's note):

$$\delta C = i\,\xi\,\chi - i\,\overline{\xi}\,\overline{\chi}\ ,$$
$$\delta\chi_\alpha = \chi_\alpha\,M + (\sigma^\mu\,\overline{\xi})_\alpha\,(\partial_\mu C + i\,v_\mu)\ ,$$
$$\delta\overline{\chi}_{\dot\alpha} = \overline{\xi}_{\dot\alpha}\,\overline{M} + (\xi\,\sigma^\mu)_{\dot\alpha}\,(\partial_\mu C - i\,v_\mu)\ ,$$
$$\delta M = 2\,i\,\overline{\xi}\,\overline{\sigma}^\mu\,\partial_\mu\chi + 2\,\overline{\xi}\,\overline{\lambda}\ ,$$
$$\delta\overline{M} = 2\,i\,\xi\,\sigma^\mu\,\partial_\mu\overline{\chi} + 2\,\xi\,\lambda\ ,$$
$$\delta v_\mu = i\,\xi\,\sigma_\mu\,\overline{\lambda} + i\,\overline{\xi}\,\overline{\sigma}_\mu\,\lambda + \xi\,\partial_\mu\chi + \overline{\xi}\,\partial_\mu\overline{\chi}$$
$$\delta\lambda_\alpha = i\,\xi_\alpha\,D + 2\,(\sigma^{\mu\nu}\,\xi)_\alpha\,\partial_\mu v_\nu\ ,$$
$$\delta\overline{\lambda}_{\dot\alpha} = -i\,\overline{\xi}_{\dot\alpha}\,D - 2\,(\overline{\xi}\,\overline{\sigma}^{\mu\nu})_{\dot\alpha}\,\partial_\mu v_\nu\ ,$$
$$\delta D = -\xi\,\sigma^\mu\,\partial_\mu\overline{\lambda} + \overline{\xi}\,\overline{\sigma}^\mu\,\partial_\mu\lambda$$

Remarks:

- This is a linear representation of the SUSY algebra on a set of $x$-dependent fields
- This rep is highly **reducible**
- The task: find **constraints** on $\mathcal{S}(x,\theta,\overline{\theta})$ that are supersymmetric and do not impose EOMs on the $x$-dependent fields (because we want an off-shell formalism)
- Simplest example: the reality conditions

$$C^* = C\ ,\ (\chi_\alpha)^* = \overline{\chi}_{\dot\alpha}\ ,\quad M^* = \overline{M}\,,$$
$$(v_\mu)^* = v_\mu\,,\quad (\lambda_\alpha)^* = \overline{\lambda}_{\dot\alpha}\ ,\ D^* = D$$

that define a real superfield

# The D-component and invariant functionals

$$\mathcal{S}(x,\theta,\overline{\theta}) = C(x) + i\,\theta^\alpha\,\chi_\alpha(x) - i\,\overline{\theta}_{\dot\alpha}\,\overline{\chi}^{\dot\alpha}(x)$$
$$+\frac{1}{2}\,i\,(\theta\,\theta)\,M(x) - \frac{1}{2}\,i\,(\overline{\theta}\,\overline{\theta})\,\overline{M}(x) - (\theta\,\sigma^\mu\,\overline{\theta})\,v_\mu(x)$$
$$+\,i\,(\theta\,\theta)\,\overline{\theta}_{\dot\alpha}\left[\overline{\lambda}^{\dot\alpha}(x) + \frac{1}{2}\,i\,(\overline{\sigma}^\mu)^{\dot\alpha\beta}\,\partial_\mu\chi_\beta(x)\right] - i\,(\overline{\theta}\,\overline{\theta})\,\theta^\alpha\left[\lambda_\alpha(x) + \frac{1}{2}\,i\,(\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu\overline{\chi}^{\dot\beta}(x)\right]$$
$$+\frac{1}{2}\,(\theta\,\theta)\,(\overline{\theta}\,\overline{\theta})\left[D(x) + \frac{1}{2}\,\partial^\mu\,\partial_\mu C(x)\right]$$

The SUSY variation of the component field $D$ in the $(\theta\,\theta)\,(\overline{\theta}\,\overline{\theta})$ component is important, because it turns out to be a total spacetime derivative:

$$\delta D = \partial_\mu\Big( -\xi\,\sigma^\mu\overline{\lambda} + \overline{\xi}\,\overline{\sigma}^\mu\,\lambda\Big)$$

This is expected, because $D$ has the largest mass dimension. The shift by $\frac{1}{2}\,\partial^\mu\,\partial_\mu C$ does not change the fact that

$$\delta\Big(\mathcal{S}(x,\theta,\overline{\theta})\Big|_{\theta\theta\overline{\theta}\overline{\theta}}\Big) = \partial_\mu(\ldots)$$

# The D-component and invariant functionals

This observation gives us a recipe to construct SUSY invariant actions:

- Take any real superfield $V(x,\theta,\bar{\theta})$ , $V(x,\theta,\bar{\theta})^* = V(x,\theta,\bar{\theta})$
- Extract its $(\theta\,\theta)\,(\bar{\theta}\,\bar{\theta})$ component
- Integrate it over ordinary spacetime: $\int d^4x\, V(x,\theta,\bar{\theta})\Big|_{\theta\theta\bar{\theta}\bar{\theta}}$

We need $V(x,\theta,\bar{\theta})$ to be real because the action should be real.

# The D-component and invariant functionals

A more suggestive way of writing the same quantity is

$$\int d^4x\, V(x,\theta,\overline{\theta})\Big|_{\theta\theta\overline{\theta}\overline{\theta}} = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, V(x,\theta,\overline{\theta}) \qquad \text{because} \qquad \int d^2\theta\, d^2\overline{\theta}\,(\theta\,\theta)\,(\overline{\theta}\,\overline{\theta}) = 1$$

SUSY is manifest in this language. We can see it in two equivalent ways:

1. The integrand transforms as a scalar field and the measure is invariant

$$V'(x',\theta',\overline{\theta}') = V(x,\theta,\overline{\theta}) \qquad \text{and} \qquad d^4x'\, d^2\theta'\, d^2\overline{\theta}' = d^4x\, d^2\theta\, d^2\overline{\theta}$$

2. An infinitesimal SUSY variation is implemented as a diff op and can be cast as a tot der

$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\,\mathbf{Q}_\alpha V = i\int d^4x\, d^2\theta\, d^2\overline{\theta}\left(\frac{\partial}{\partial\theta^\alpha} - i\,(\sigma^\mu\,\overline{\theta})_\alpha\,\partial_\mu\right)V = i\int d^4x\, d^2\theta\, d^2\overline{\theta}\,\frac{\partial}{\partial\theta^\alpha}V + \int d^4x\, d^2\theta\, d^2\overline{\theta}\,\partial_\mu[(\sigma^\mu\,\overline{\theta})_\alpha\, V]$$

and similarly for $\overline{\mathbf{Q}}^{\dot{\alpha}}$

# Aside: invariance of the volume form

For ordinary bosonic change of coordinates, the volume form transforms with the determinant of the Jacobian

$$d^4x' = d^4x \ \det \frac{\partial x'^\mu}{\partial x^\nu}$$

The analog notion in superspace involves the "superdeterminant" of the Jacobian

$$d^4x'\, d^2\theta'\, d^2\overline{\theta}' = d^4x\, d^2\theta\, d^2\overline{\theta}\, \mathrm{sdet} \frac{\partial(x'^\mu, \theta^\alpha, \overline{\theta}^{\dot{\alpha}})}{\partial(x^\nu, \theta^\beta, \overline{\theta}^{\dot{\beta}})}$$

The superdeterminant is defined in block-matrix notation as

$$\mathrm{sdet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\det A}{\det(D - C\,A^{-1}\,B)} = \frac{\det(A - B\,D^{-1}\,C)}{\det D}$$

Here the square matrix $A$ maps bosons to bosons, the square matrix $D$ maps fermions to fermions, while the rectangular matrices $B$, $C$ interchange them.

# Aside: invariance of the volume form

We are interested in a supertranslation

$$\theta'^{\alpha} = \theta^{\alpha} - \xi^{\alpha} \quad , \qquad \bar{\theta}'_{\dot\alpha} = \bar{\theta}_{\dot\alpha} - \bar{\xi}_{\dot\alpha} \, , \qquad x'^{\mu} = x^{\mu} - (i\,\theta\,\sigma^{\mu}\,\bar{\xi} - i\,\xi\,\sigma^{\mu}\,\bar{\theta})$$

We find

$$\frac{\partial(x'^{\mu}, \theta^{\alpha}, \bar{\theta}^{\dot\alpha})}{\partial(x^{\nu}, \theta^{\beta}, \bar{\theta}^{\dot\beta})} = \begin{pmatrix} \delta^{\mu}_{\nu} & -i\,(\sigma^{\mu})_{\beta\dot\gamma}\,\bar{\xi}^{\dot\gamma} & -i\,\xi^{\gamma}\,(\sigma^{\mu})_{\gamma\dot\beta} \\ 0 & \delta^{\alpha}_{\beta} & 0 \\ 0 & 0 & \delta^{\dot\alpha}_{\dot\beta} \end{pmatrix}$$

The $A$, $B$, $C$, $D$ blocks are

$$A = (\delta^{\mu}_{\nu}) \quad , \qquad B = \begin{pmatrix} -i\,(\sigma^{\mu})_{\beta\dot\gamma}\,\bar{\xi}^{\dot\gamma} & -i\,\xi^{\gamma}\,(\sigma^{\mu})_{\gamma\dot\beta} \end{pmatrix} \, , \qquad C = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \, , \qquad D = \begin{pmatrix} \delta^{\alpha}_{\beta} & a \\ 0 & \delta^{\dot\alpha}_{\dot\beta} \end{pmatrix}$$

and thus

$$\mathrm{sdet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\det A}{\det(D - C\,A^{-1}\,B)} = \frac{\det(A - B\,D^{-1}\,C)}{\det D} = 1$$

# Supersymmetry and supergravity

Lecture 16

# SUSY covariant derivatives

- A superfield is a function $\mathcal{S}(x, \theta, \bar{\theta})$ such that $\mathcal{S}'(x', \theta', \bar{\theta}') = \mathcal{S}(x, \theta, \bar{\theta})$ under the action of super-Poincaré. Infinitesimally, the SUSY variation is generated by a differential operator

$$\delta\mathcal{S} = (-i\,\xi^\alpha\,\mathbf{Q}_\alpha - i\,\bar{\xi}_{\dot{\alpha}}\,\overline{\mathbf{Q}}^{\dot{\alpha}})\,\mathcal{S}$$

$$\mathbf{Q}_\alpha = i\left(\frac{\partial}{\partial\theta^\alpha} - i\,(\sigma^\mu\,\bar{\theta})_\alpha\,\partial_\mu\right) \quad , \qquad \overline{\mathbf{Q}}^{\dot{\alpha}} = i\left(\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} + i\,(\theta\,\sigma^\mu)^{\dot{\alpha}}\,\partial_\mu\right)$$

- We know that $\mathbf{P}_\mu = -\,i\,\partial_\mu$ commutes with $\mathbf{Q}_\alpha$, $\overline{\mathbf{Q}}_{\dot{\alpha}}$ and therefore the ordinary derivative of a superfield is still a superfield,

$$\delta(\partial_\mu\mathcal{S}) = \partial_\mu(\delta\mathcal{S}) = \partial_\mu\,(-i\,\xi^\alpha\,\mathbf{Q}_\alpha - i\,\bar{\xi}_{\dot{\alpha}}\,\overline{\mathbf{Q}}^{\dot{\alpha}})\,\mathcal{S} = (-i\,\xi^\alpha\,\mathbf{Q}_\alpha - i\,\bar{\xi}_{\dot{\alpha}}\,\overline{\mathbf{Q}}^{\dot{\alpha}})\,\partial_\mu\mathcal{S}$$

- The partial derivatives $\partial/\partial\theta^\alpha$ , $\partial/\partial\bar{\theta}^{\dot{\alpha}}$ do <u>not</u> anticommute with $\mathbf{Q}_\alpha$, $\overline{\mathbf{Q}}_{\dot{\alpha}}$, so the partial fermionic derivative of a superfield is not a superfield,

$$\delta(\tfrac{\partial}{\partial\theta^\beta}\mathcal{S}) = \tfrac{\partial}{\partial\theta^\beta}(\delta\mathcal{S}) = \tfrac{\partial}{\partial\theta^\beta}\,(-i\,\xi^\alpha\,\mathbf{Q}_\alpha - i\,\bar{\xi}_{\dot{\alpha}}\,\overline{\mathbf{Q}}^{\dot{\alpha}})\,\mathcal{S} \neq (-i\,\xi^\alpha\,\mathbf{Q}_\alpha - i\,\bar{\xi}_{\dot{\alpha}}\,\overline{\mathbf{Q}}^{\dot{\alpha}})\,\tfrac{\partial}{\partial\theta^\beta}\mathcal{S}$$

# SUSY covariant derivatives

- We want to construct suitable covariant derivatives $D_\alpha$, $\overline{D}_{\dot\alpha}$ that anticommute with $\mathbf{Q}_\alpha$, $\overline{\mathbf{Q}}_{\dot\alpha}$
- Recall that $\mathbf{Q}_\alpha$, $\overline{\mathbf{Q}}_{\dot\alpha}$ were constructed by considering the left action of the super-Poincaré group on the coset representative $\exp(-i\,x^\mu\,P_\mu + i\,\theta\,Q + i\,\overline{\theta}\,\overline{Q})$: $\exp(-i\xi\,Q - i\,\overline{\xi}\,\overline{Q})\ \exp(-i\,x^\mu\,P_\mu + i\,\theta\,Q + i\,\overline{\theta}\,\overline{Q})$ induces the motion

$$\theta'^\alpha = \theta^\alpha - \xi^\alpha \quad , \qquad \overline{\theta}'_{\dot\alpha} = \overline{\theta}_{\dot\alpha} - \overline{\xi}_{\dot\alpha}\,, \qquad x'^\mu = x^\mu - (i\,\theta\,\sigma^\mu\,\overline{\xi} - i\,\xi\,\sigma^\mu\,\overline{\theta})$$

- The can also consider the <u>right action</u> of $\exp(-i\xi\,Q - i\,\overline{\xi}\,\overline{Q})$ on the coset representative. (This should not be confused with the quotient by $H$ from the right.)
- In applying the BCH formula, the sign of the $[A, B]$ term is flipped, so the right action induces the motion

$$\theta'^\alpha = \theta^\alpha - \xi^\alpha \quad , \qquad \overline{\theta}'_{\dot\alpha} = \overline{\theta}_{\dot\alpha} - \overline{\xi}_{\dot\alpha}\,, \qquad x'^\mu = x^\mu + (i\,\theta\,\sigma^\mu\,\overline{\xi} - i\,\xi\,\sigma^\mu\,\overline{\theta})$$

- Collecting factors of $\xi^\alpha$, $\overline{\xi}_{\dot\alpha}$ we find the operators

$$\frac{\partial}{\partial\theta^\alpha} + i\,(\sigma^\mu\,\overline{\theta})_\alpha\,\partial_\mu \quad , \qquad \frac{\partial}{\partial\overline{\theta}_{\dot\alpha}} - i\,(\theta\,\sigma^\mu)^{\dot\alpha}\,\partial_\mu$$

# SUSY covariant derivatives

$$\frac{\partial}{\partial\theta^\alpha} + i\,(\sigma^\mu\,\overline{\theta})_\alpha\,\partial_\mu \quad , \qquad \frac{\partial}{\partial\overline{\theta}_{\dot\alpha}} - i\,(\theta\,\sigma^\mu)^{\dot\alpha}\,\partial_\mu$$

- After lowering the index on the second expression, we find the differential operators

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\,(\sigma^\mu\,\overline{\theta})_\alpha\,\partial_\mu \qquad \overline{D}_{\dot\alpha} = -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - i\,(\theta\,\sigma^\mu)_{\dot\alpha}\,\partial_\mu$$

- Left actions and right actions commute, so we automatically have (or we check brute-force that)

$$\{D_\alpha, \mathbf{Q}_\beta\} = \{D_\alpha, \overline{\mathbf{Q}}_{\dot\beta}\} = \{\overline{D}_{\dot\alpha}, \mathbf{Q}_\beta\} = \{\overline{D}_{\dot\alpha}, \overline{\mathbf{Q}}_{\dot\beta}\} = 0$$

- The covariant derivative of a superfield is a superfield:

$$\delta(D_\beta\mathcal{S}) = D_\beta(\delta\mathcal{S}) = D_\beta\,(-i\,\xi^\alpha\,\mathbf{Q}_\alpha - i\,\overline{\xi}_{\dot\alpha}\,\overline{\mathbf{Q}}^{\dot\alpha})\,\mathcal{S} = (-i\,\xi^\alpha\,\mathbf{Q}_\alpha - i\,\overline{\xi}_{\dot\alpha}\,\overline{\mathbf{Q}}^{\dot\alpha})\,D_\beta\mathcal{S}$$

# Comment: torsion in flat superspace

- While the partial derivatives $\partial/\partial\theta^\alpha$ , $\partial/\partial\bar{\theta}^{\dot{\alpha}}$ anticommute, the covariant derivatives satisfy non-trivial anticommutation relations:

$$\{D_\alpha, D_\beta\} = 0 \quad , \quad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0 \quad , \quad \{D_\alpha, \bar{D}_{\dot{\beta}}\} = -2i\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\partial_\mu$$

- In ordinary geometry, the fact that two covariant derivatives do not commute when acting on a scalar field is a signal that the space has **torsion** (the familiar Levi-Civita connection is torsionless by definition)

$$[\nabla_\mu, \nabla_\nu] f = -T_{\mu\nu}{}^{\rho}\,\partial_\rho f$$

- One can develop a notion of differential geometry for superspace and show that the flat superspace (super-Poincaré)/(Lorentz) has no curvature but non-zero torsion

# Constraining superfields

Recall the expression of a generic complex scalar superfield

$$
\begin{aligned}
\mathcal{S}(x,\theta,\overline{\theta}) = {} & C(x) + i\,\theta^\alpha\,\chi_\alpha(x) - i\,\overline{\theta}_{\dot{\alpha}}\,\overline{\chi}^{\dot{\alpha}}(x) \\
& + \frac{1}{2}\,i\,(\theta\,\theta)\,M(x) - \frac{1}{2}\,i\,(\overline{\theta}\,\overline{\theta})\,\overline{M}(x) - (\theta\,\sigma^\mu\,\overline{\theta})\,v_\mu(x) \\
& + i\,(\theta\,\theta)\,\overline{\theta}_{\dot{\alpha}}\left[\overline{\lambda}^{\dot{\alpha}}(x) + \frac{1}{2}\,i\,(\overline{\sigma}^\mu)^{\dot{\alpha}\beta}\,\partial_\mu\chi_\beta(x)\right] - i\,(\overline{\theta}\,\overline{\theta})\,\theta^\alpha\left[\lambda_\alpha(x) + \frac{1}{2}\,i\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\partial_\mu\overline{\chi}^{\dot{\beta}}(x)\right] \\
& + \frac{1}{2}\,(\theta\,\theta)\,(\overline{\theta}\,\overline{\theta})\left[D(x) + \frac{1}{2}\,\partial^\mu\,\partial_\mu C(x)\right]
\end{aligned}
$$

- It has 16 + 16 dof's. SUSY is manifest, but we have too many fields. More precisely, we have a linear rep of SUSY, but it is highly reducible.
- We need to find suitable constraints on $\mathcal{S}(x,\theta,\overline{\theta})$ that do not spoil manifest SUSY and reduce the number of component fields in order to match known off-shell multiplets
- The constraints should not restrict the $x$-dependence of the component fields

# Chiral superfields

A chiral superfield $\Phi(x, \theta, \overline{\theta})$ is defined by the condition

$$\overline{D}_{\dot{\alpha}}\Phi = 0$$

while its complex conjugate is an antichiral superfield, in the sense that

$$D_{\alpha}\overline{\Phi} = 0$$

A direct but tedious way of finding the most general chiral superfield would be to start from the expansion of $\Phi(x, \theta, \overline{\theta})$, use it to compute $\overline{D}_{\dot{\alpha}}\Phi$, and set to zero all terms with different $\theta$, $\overline{\theta}$ structures. There is a more illuminating approach using the abstract definition of $\overline{D}_{\dot{\alpha}}$ in terms of the right action on the coset representative

# Chiral superspace

Our standard representative of the element in the coset is

$$\exp(-i\, x^\mu\, P_\mu + i\,\theta\, Q + i\,\overline{\theta}\,\overline{Q})$$

To study chiral superfields it is convenient to choose a different representative, parametrized by coordinates $(y^\mu, \vartheta, \overline{\vartheta})$ as

$$\exp(i\,\vartheta\, Q)\ \exp(-i\, y^\mu\, P_\mu)\ \exp(i\,\overline{\vartheta}\,\overline{Q})$$

Using BKH we can relate the two sets of coordinates:

$$y^\mu = x^\mu + i\,\theta\,\sigma^\mu\,\overline{\theta} \quad , \quad \vartheta = \theta \ , \quad \overline{\vartheta} = \overline{\theta}$$

NB: there is also a notion antichiral superspace, based on the coset representative $\exp(i\,\widehat{\overline{\vartheta}}\,\overline{Q})\ \exp(-i\,\widehat{y}^{\,\mu}\,P_\mu)\ \exp(i\,\widehat{\vartheta}\,Q)$. It is the natural set of coordinates for an antichiral superfield. All remarks we make about chiral superspace/superfields have analogs for the antichiral counterparts.

# Chiral superspace

The advantage of the new representative is that $\overline{D}_{\dot\alpha}$ acts very simply. Recall that this diff op is defined via the action of $\exp(-i\,\bar\xi\,\overline{Q})$ from the right. This is immediate with the new coset representative:

$$\exp(i\,\vartheta\,Q)\,\exp(-i\,y^\mu\,P_\mu)\,\exp(i\,\bar\vartheta\,\overline{Q})\,\exp(-i\,\bar\xi\,\overline{Q}) = \exp(i\,\vartheta\,Q)\,\exp(-i\,y^\mu\,P_\mu)\,\exp(i\,(\bar\vartheta-\bar\xi)\,\overline{Q})$$

This relation implies

$$\overline{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\vartheta^{\dot\alpha}}$$

The other cov der is found computing

$$\exp(i\,\vartheta\,Q)\,\exp(-i\,y^\mu\,P_\mu)\,\exp(i\,\bar\vartheta\,\overline{Q})\,\exp(-i\,\xi\,Q)$$

With the help of BCH one finds

$$D_\alpha = \frac{\partial}{\partial\vartheta^\alpha} + 2\,i\,(\sigma^\mu\,\bar\vartheta)_\alpha\,\frac{\partial}{\partial y^\mu}$$

# Chiral superspace

One can also translate the diff op's that implement a SUSY variation in the new coors.

Summary: in terms of the new coordinates $(y, \vartheta, \bar{\vartheta})$ we have

$$\mathbf{Q}_\alpha = i\,\frac{\partial}{\partial\vartheta^\alpha}\;, \qquad \overline{\mathbf{Q}}_{\dot\alpha} = i\Big[-\frac{\partial}{\partial\bar\vartheta^{\dot\alpha}} + 2\,i\,(\vartheta\,\sigma^\mu)_{\dot\alpha}\frac{\partial}{\partial y^\mu}\Big]\;, \qquad \mathbf{P}_\mu = -\,i\,\frac{\partial}{\partial y^\mu}$$

$$D_\alpha = \frac{\partial}{\partial\vartheta^\alpha} + 2\,i\,(\sigma^\mu\,\bar\vartheta)_\alpha\frac{\partial}{\partial y^\mu}\quad, \qquad \overline{D}_{\dot\alpha} = -\,\frac{\partial}{\partial\bar\vartheta^{\dot\alpha}}$$

A chiral superfield $\overline{D}_{\dot\alpha}\Phi = 0$ is simply any superfield that does not depend on $\bar\vartheta$. If the operators $\mathbf{Q}_\alpha$, $\overline{\mathbf{Q}}_{\dot\alpha}$ act on a function of $y$ and $\vartheta$ only, the resulting quantity is also a function of $y$ and $\vartheta$ only. A chiral superfield satisfies

$$\Phi'(y', \vartheta') = \Phi(y, \vartheta)$$

# Chiral superfield expansion

In the new coordinates $(y, \vartheta, \bar{\vartheta})$ a chiral superfield is simply any function of $y$ and $\vartheta$, but not $\bar{\vartheta}$. Its Taylor expansion is of the form

$$\Phi(y, \vartheta, \bar{\vartheta}) = X(y) + \sqrt{2}\, \vartheta^{\alpha}\, \psi_{\alpha}(y) + (\vartheta\, \vartheta)\, F(y)$$

Recalling that

$$y^{\mu} = x^{\mu} + i\, \theta\, \sigma^{\mu}\, \bar{\theta} \quad , \quad \vartheta = \theta \;\; , \quad \bar{\vartheta} = \bar{\theta}$$

one can find the expansion in the original superspace coords:

$$\Phi(x, \theta, \bar{\theta}) = X(x) + \sqrt{2}\, \theta^{\alpha}\, \psi_{\alpha}(x) + (\theta\, \theta)\, F(x)$$
$$+\, i\, \theta\, \sigma^{\mu}\, \bar{\theta}\, \partial_{\mu} X(x) + \frac{1}{4}\, (\theta\, \theta)\, (\bar{\theta}\, \bar{\theta})\, \partial^{\mu}\, \partial_{\mu} X - \frac{i}{\sqrt{2}}\, (\theta\, \theta)\, \partial_{\mu} \psi(x)\, \sigma^{\mu}\, \bar{\theta}$$

# Chiral superfield expansion

NB: $\Phi(y, \vartheta, \bar{\vartheta})$ is the same as

$$\begin{aligned}
\mathcal{S}(x,\theta,\bar{\theta}) = {} & C(x) + i\,\theta^{\alpha}\,\chi_{\alpha}(x) - i\,\bar{\theta}_{\dot{\alpha}}\,\bar{\chi}^{\dot{\alpha}}(x) \\
& + \frac{1}{2}\,i\,(\theta\,\theta)\,M(x) - \frac{1}{2}\,i\,(\bar{\theta}\,\bar{\theta})\,\overline{M}(x) - (\theta\,\sigma^{\mu}\,\bar{\theta})\,v_{\mu}(x) \\
& + i\,(\theta\,\theta)\,\bar{\theta}_{\dot{\alpha}}\left[\bar{\lambda}^{\dot{\alpha}}(x) + \frac{1}{2}\,i\,(\bar{\sigma}^{\mu})^{\dot{\alpha}\beta}\,\partial_{\mu}\chi_{\beta}(x)\right] - i\,(\bar{\theta}\,\bar{\theta})\,\theta^{\alpha}\left[\lambda_{\alpha}(x) + \frac{1}{2}\,i\,(\sigma^{\mu})_{\alpha\dot{\beta}}\,\partial_{\mu}\bar{\chi}^{\dot{\beta}}(x)\right] \\
& + \frac{1}{2}\,(\theta\,\theta)\,(\bar{\theta}\,\bar{\theta})\left[D(x) + \frac{1}{2}\,\partial^{\mu}\,\partial_{\mu}C(x)\right]
\end{aligned}$$

specialized to:

$$C = X\,, \quad v_{\mu} = -\,i\,\partial_{\mu}X\;, \quad \chi_{\alpha} = -\,i\sqrt{2}\,\psi_{\alpha}\;, \quad M = -\,2\,i\,F\;,$$
$$\bar{\chi} = \lambda = \bar{\lambda} = \overline{M} = D = 0$$

# Chiral superfield expansion

$$\delta C = i\,\xi\,\chi - i\,\bar{\xi}\,\bar{\chi}\ ,$$

$$\delta\chi_\alpha = \chi_\alpha\,M + (\sigma^\mu\,\bar{\xi})_\alpha\,(\partial_\mu C + i\,v_\mu)\ ,$$

$$\delta\bar{\chi}_{\dot{\alpha}} = \bar{\xi}_{\dot{\alpha}}\,\overline{M} + (\xi\,\sigma^\mu)_{\dot{\alpha}}\,(\partial_\mu C - i\,v_\mu)\ ,$$

$$\delta M = 2\,i\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\chi + 2\,\bar{\xi}\,\bar{\lambda}\ ,$$

$$\delta\overline{M} = 2\,i\,\xi\,\sigma^\mu\,\partial_\mu\bar{\chi} + 2\,\xi\,\lambda\ ,$$

$$\delta v_\mu = i\,\bar{\xi}\,\sigma_\mu\,\bar{\lambda} + i\,\bar{\xi}\,\bar{\sigma}_\mu\,\lambda + \xi\,\partial_\mu\chi + \bar{\xi}\,\partial_\mu\bar{\chi}$$

$$\delta\lambda_\alpha = i\,\xi_\alpha\,D + 2\,(\sigma^{\mu\nu}\,\xi)_\alpha\,\partial_\mu v_\nu\ ,$$

$$\delta\bar{\lambda}_{\dot{\alpha}} = -i\,\bar{\xi}_{\dot{\alpha}}\,D - 2\,(\bar{\xi}\,\bar{\sigma}^{\mu\nu})_{\dot{\alpha}}\,\partial_\mu v_\nu\ ,$$

$$\delta D = -\xi\,\sigma^\mu\,\partial_\mu\bar{\lambda} + \bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\lambda$$

$$C = X\,,\quad v_\mu = -\,i\,\partial_\mu X$$

$$\chi_\alpha = -\,i\sqrt{2}\,\psi_\alpha \qquad M = -\,2\,i\,F$$

$$\bar{\chi} = \lambda = \bar{\lambda} = \overline{M} = D = 0$$

- We have identified a way to reduce the original SUSY reps on the full set of component fields

$$\delta X = \sqrt{2}\,\xi\,\psi$$

$$\delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu\,\xi)_\alpha\,\partial_\mu X + \sqrt{2}\,F\,\xi_\alpha$$

$$\delta F = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi$$

- If we impose the condition $\mathcal{S}^* = \mathcal{S}$, we find that $\mathcal{S}(x,\theta,\bar{\theta}) = \text{const}$

# F-type supersymmetric actions

The superspace analysis confirms that $F$ in a chiral superfield $\overline{D}_{\dot{\alpha}}\Phi = 0$ transforms as a total spacetime derivative.

Chiral superfields give us a new way to generate supersymmetric actions:

$$\int d^4x\, \Phi(x, \theta, \overline{\theta})\Big|_{\theta\theta}$$

Since a non-trivial chiral superfield is complex, in order to get a real action we have to add the hermitian conjugate by hand.

# F-type supersymmetric actions

We can write an F-term action as an integral over the slice at $\bar{\theta}^{\dot{\alpha}} = 0$

$$\int d^4x\, \Phi(x,\theta,\bar{\theta})\Big|_{\theta\theta} = \int d^4x\, d^2\theta\, \Phi(x,\theta,\bar{\theta})\Big|_{\bar{\theta}=0}$$

Equivalently, we integrate over the entire superspace with a delta function

$$\int d^4x\, \Phi(x,\theta,\bar{\theta})\Big|_{\theta\theta} = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \delta^{(2)}(\bar{\theta})\, \Phi(x,\theta,\bar{\theta})$$

For a single Grassman variable $\eta$ we have $\delta(\eta) = \eta$ because

$$\int d\eta\, \delta(\eta)\, f(\eta) = f(0) \quad \text{for any} \quad f(\eta) = f_0 + \eta\, f_1$$

In a similar way

$$d^2\bar{\theta} = \frac{1}{2}\, d\bar{\theta}^2\, d\bar{\theta}^1\,, \qquad \delta^{(2)}(\bar{\theta}) = 2\,\delta(\bar{\theta}^1)\,\delta(\bar{\theta}^2) = 2\,\bar{\theta}^1\,\bar{\theta}^2 = \bar{\theta}\,\bar{\theta}$$

# F-type supersymmetric actions

It is instructive to write the same quantity in the chiral superspace coords

$$y^\mu = x^\mu + i\,\theta\,\sigma^\mu\,\overline{\theta} \quad , \qquad \vartheta = \theta \ , \qquad \overline{\vartheta} = \overline{\theta}$$

One can check that the superJacobian of the coord change has superdet equal to 1, so

$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\, \delta^{(2)}(\overline{\theta})\, \Phi(x,\theta,\overline{\theta}) = \int d^4y\, d^2\vartheta\, d^2\overline{\vartheta}\, \delta^{(2)}(\overline{\vartheta})\, \Phi(y,\vartheta)$$

We can now verify manifest SUSY in two equivalent ways:

1. We know that $\Phi'(y',\vartheta') = \Phi(y,\vartheta)$ and also that $d^4y'\, d^2\vartheta'\, d^2\overline{\vartheta}' = d^4y\, d^2\vartheta\, d^2\overline{\vartheta}$ (we have checked SUSY invariance of the measure in the original coords, but it is valid in the new coords as well). As a result we can write

$$\int d^4y'\, d^2\vartheta'\, d^2\overline{\vartheta}'\, \delta^{(2)}(\overline{\vartheta}')\, \Phi'(y',\vartheta') = \int d^4y\, d^2\vartheta\, d^2\overline{\vartheta}\, \delta^{(2)}(\overline{\vartheta}-\overline{\xi})\, \Phi(y,\vartheta) = \int d^4y\, d^2\vartheta\, d^2\overline{\vartheta}\, \delta^{(2)}(\overline{\vartheta})\, \Phi(y,\vartheta)$$

where we have used translational invariance of the Berezin integral in the $\overline{\vartheta}$'s

# F-type supersymmetric actions

2. Using $-i\,\mathbf{Q}_\alpha = \partial/\partial\vartheta^\alpha$ we have

$$-i\int d^4y\,d^2\vartheta\,d^2\overline{\vartheta}\,\delta^{(2)}(\overline{\vartheta})\,\mathbf{Q}_\alpha\,\Phi(y,\vartheta) = \int d^4y\,d^2\vartheta\,d^2\overline{\vartheta}\;\tfrac{\partial}{\partial\vartheta^\alpha}\big[\delta^{(2)}(\overline{\vartheta})\,\Phi(y,\vartheta)\big]$$

while using $-i\overline{\mathbf{Q}}_{\dot\alpha} = -\,\partial/\partial\overline{\vartheta}^{\dot\alpha} + 2\,i\,(\vartheta\,\sigma^\mu)_{\dot\alpha}\,\partial/\partial y^\mu = \overline{D}_{\dot\alpha} + 2\,i\,(\vartheta\,\sigma^\mu)_{\dot\alpha}\,\partial/\partial y^\mu$ we have

$$-i\int d^4y\,d^2\vartheta\,d^2\overline{\vartheta}\,\delta^{(2)}(\overline{\vartheta})\,\overline{\mathbf{Q}}_{\dot\alpha}\,\Phi(y,\vartheta) = \int d^4y\,d^2\vartheta\,d^2\overline{\vartheta}\,\delta^{(2)}(\overline{\vartheta})\,\overline{D}_{\dot\alpha}\Phi + 2\,i\int d^4y\,d^2\vartheta\,d^2\overline{\vartheta}\;\tfrac{\partial}{\partial y^\mu}\big[\delta^{(2)}(\overline{\vartheta})\,(\vartheta\,\sigma^\mu)_{\dot\alpha}\,\Phi\big]$$

where the first term is zero because $\Phi$ is chiral.

# F-type actions as D-type actions

- Observation: $\overline{D}_{\dot\alpha}\,\overline{D}_{\dot\beta}\,\overline{D}_{\dot\gamma}(\ldots) = 0$ and $D_\alpha\, D_\beta\, D_\gamma(\ldots) = 0$ because these quantities are totally antisymmetric in 3 spinor indices.
- If $\mathcal{S}(x,\theta,\overline{\theta})$ is any generic superfield, then $\Phi = \overline{D}\,\overline{D}\mathcal{S}$ is automatically chiral. Conversely, given a chiral superfield $\overline{D}_{\dot\alpha}\Phi = 0$, it can always be written as $\Phi = \overline{D}\,\overline{D}\mathcal{S}$ for some superfield $\mathcal{S}$
- We suppose $\Phi = \overline{D}\,\overline{D}\mathcal{S}$ and we use the identities $\delta^{(2)}(\overline{\theta}) = \overline{\theta}\,\overline{\theta}$ , $\overline{D}\overline{D}\,(\overline{\theta}\overline{\theta}) = -\,4$ to write
$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\, \delta^2(\overline{\theta})\,\Phi = \int d^4x\, d^2\theta\, d^2\overline{\theta}\,(\overline{\theta}\overline{\theta})\,\overline{D}\overline{D}\mathcal{S} = -\,4\int d^4x\, d^2\theta\, d^2\overline{\theta}\,\mathcal{S}$$
- With this trick we can convert F-type functionals onto D-type functionals
- We get yet another argument for manifest SUSY of the integral $\displaystyle\int d^4x\, d^2\theta\, d^2\overline{\theta}\,\delta^2(\overline{\theta})\,\Phi$

# Supersymmetry and supergravity

Lecture 17

# Gauge invariance in superspace

- Recall that a real superfield is a Grassmann-even superfield with no spinor indices and satisfying $V(x,\theta,\bar\theta) = V(x,\theta,\bar\theta)^*$. Its expansion is

$$V(x,\theta,\bar\theta) = C(x) + i\,\theta^\alpha\chi_\alpha(x) - i\,\bar\theta_{\dot\alpha}\,\bar\chi^{\dot\alpha}(x)$$
$$+\frac{1}{2}\,i\,(\theta\,\theta)\,M(x) - \frac{1}{2}\,i\,(\bar\theta\,\bar\theta)\,\overline{M}(x) - (\theta\,\sigma^\mu\,\bar\theta)\,v_\mu(x)$$
$$+\,i\,(\theta\,\theta)\,\bar\theta_{\dot\alpha}\left[\bar\lambda^{\dot\alpha}(x) + \frac{1}{2}\,i\,(\bar\sigma^\mu)^{\dot\alpha\beta}\,\partial_\mu\chi_\beta(x)\right] - i\,(\bar\theta\,\bar\theta)\,\theta^\alpha\left[\lambda_\alpha(x) + \frac{1}{2}\,i\,(\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu\bar\chi^{\dot\beta}(x)\right]$$
$$+\frac{1}{2}\,(\theta\,\theta)\,(\bar\theta\,\bar\theta)\left[D(x) + \frac{1}{2}\,\partial^\mu\,\partial_\mu C(x)\right]$$

with the reality conditions

$$C^* = C\ ,\quad (\chi_\alpha)^* = \bar\chi_{\dot\alpha}\ ,\quad M^* = \overline{M}\,,\quad (v_\mu)^* = v_\mu\,,\quad (\lambda_\alpha)^* = \bar\lambda_{\dot\alpha}\ ,\ D^* = D$$

- This superfield is also known as **vector superfield** because of the component field $v_\mu$, which we want to interpret as a gauge field, $v_\mu \equiv A_\mu$
- In the simplest case the gauge group is $U(1)$. We need the superspace analog of $\delta_{\text{gauge}} A_\mu = \partial_\mu \Lambda$

# Gauge invariance in superspace

The correct recipe is as follows:

- promote the gauge parameter to be a chiral superfield $\Lambda(x, \theta, \overline{\theta}), \quad \overline{D}_{\dot{\alpha}}\Lambda = 0$
- define the gauge transformation of a $U(1)$ vector supermultiplet as

$$V \to V + \frac{i}{2}\left(\Lambda - \overline{\Lambda}\right)$$

If we write $\Lambda(y, \vartheta) = X_\Lambda(y) + \sqrt{2}\,\vartheta\,\psi_\Lambda(y) + \vartheta\,\vartheta\,F_\Lambda(y)$, this gauge transformation is the same as

$$\begin{aligned}
&C \to C - \mathrm{Im}\,X_\Lambda \\
&\chi \to \chi + \frac{1}{\sqrt{2}}\,\psi_\Lambda \\
&M \to M + F_\Lambda \\
&A_\mu \to A_\mu + \partial_\mu \mathrm{Re}\,X_\Lambda \\
&\lambda \to \lambda \\
&D \to D
\end{aligned}$$

- We get the expected shift of $A_\mu$ by a derivative
- $\mathrm{Re}\,X_\Lambda$ is the the standard gauge parameter
- $\lambda$ and $D$ are gauge invariant
- The other fields are shifted by a gauge transformation

# Wess-Zumino gauge

Let us compare gauge variations and SUSY variations

(recall $v_\mu \equiv A_\mu$)

$$C \to C - \mathrm{Im}\, X_\Lambda\,, \quad \chi \to \chi + \frac{1}{\sqrt{2}}\psi_\Lambda$$

$$M \to M + F_\Lambda\ , \qquad A_\mu \to A_\mu + \partial_\mu \mathrm{Re}\, X_\Lambda$$

$$\lambda \to \lambda\ \ , \qquad\qquad D \to D$$

Since $\mathrm{Im}\Lambda_X$, $\psi_\Lambda$, $F_\Lambda$ are arbitrary, we can always use a gauge transformation to impose

"**Wess-Zumino gauge**": $\quad C = 0, \quad \chi = 0, \quad M = 0$

- Notice that we still have an arbitrary $\mathrm{Re}\, X_\Lambda$ and gauge transf. for $A_\mu$
- These conditions are <u>not</u> preserved under a SUSY variation!

$$\begin{aligned}
\delta C &= i\,\xi\,\chi - i\,\bar\xi\,\bar\chi\ ,\\
\delta\chi_\alpha &= \chi_\alpha\, M + (\sigma^\mu\,\bar\xi)_\alpha\,(\partial_\mu C + i\,v_\mu)\ ,\\
\delta\bar\chi_{\dot\alpha} &= \bar\xi_{\dot\alpha}\,\overline{M} + (\xi\,\sigma^\mu)_{\dot\alpha}\,(\partial_\mu C - i\,v_\mu)\ ,\\
\delta M &= 2\,i\,\bar\xi\,\bar\sigma^\mu\,\partial_\mu\chi + 2\,\bar\xi\,\bar\lambda\ ,\\
\delta\overline{M} &= 2\,i\,\xi\,\sigma^\mu\,\partial_\mu\bar\chi + 2\,\xi\,\lambda\ ,\\
\delta v_\mu &= i\,\xi\,\sigma_\mu\,\bar\lambda + i\,\bar\xi\,\bar\sigma_\mu\,\lambda + \xi\,\partial_\mu\chi + \bar\xi\,\partial_\mu\bar\chi\\
\delta\lambda_\alpha &= i\,\xi_\alpha\,D + 2\,(\sigma^{\mu\nu}\,\xi)_\alpha\,\partial_\mu v_\nu\ ,\\
\delta\bar\lambda_{\dot\alpha} &= -i\,\bar\xi_{\dot\alpha}\,D - 2\,(\bar\xi\,\bar\sigma^{\mu\nu})_{\dot\alpha}\,\partial_\mu v_\nu\ ,\\
\delta D &= -\xi\,\sigma^\mu\,\partial_\mu\bar\lambda + \bar\xi\,\bar\sigma^\mu\,\partial_\mu\lambda
\end{aligned}$$

# Wess-Zumino gauge

Even though it is non-supersymmetric, the WZ gauge can be very useful because it leads to a superfield $V$ which is nilpotent:

$$V = -\,\theta\,\sigma^\mu\,\bar\theta\,A_\mu(x) + i\,(\theta\,\theta)\,\bar\theta\,\bar\lambda(x) - i\,(\bar\theta\,\bar\theta)\,\theta\,\lambda(x) + \frac{1}{2}\,(\theta\,\theta)(\bar\theta\,\bar\theta)\,D(x)$$

$$V\,V = -\frac{1}{2}\,(\theta\,\theta)(\bar\theta\,\bar\theta)\,A_\mu\,A^\mu$$

$$V\,V\,V \equiv 0$$

# Compensating gauge transformation

Suppose we start in WZ gauge

$C = 0, \quad \chi = 0, \quad M = 0$

If we perform a SUSY variation, we then have to perform a <u>compensating gauge transformation</u> to restore the WZ gauge: $V \to V + \frac{i}{2}(\Lambda - \overline{\Lambda})$ where $\Lambda$ has component fields

$$X_\Lambda = 0$$

$$\psi_{\Lambda\alpha} = -\,i\sqrt{2}\,A_\mu\,(\sigma^\mu\,\bar{\xi})_\alpha$$

$$F_\Lambda = -\,2\,\bar{\xi}\,\bar{\lambda}$$

The variations for $A_\mu$, $\lambda$, $D$ are those of an off-shell vector multiplet as discussed earlier

$$\delta C = 0\ ,$$

$$\delta\chi_\alpha = i\,(\sigma^\mu\,\bar{\xi})_\alpha\,A_\mu\ ,$$

$$\delta\overline{\chi}_{\dot\alpha} = -i\,(\xi\,\sigma^\mu)_{\dot\alpha}\,A_\mu\ ,$$

$$\delta M = 2\,\bar{\xi}\,\bar{\lambda}\ ,$$

$$\delta\overline{M} = 2\,\xi\,\lambda\ ,$$

$$\boxed{\begin{aligned}\delta A_\mu &= i\,\xi\,\sigma_\mu\,\bar{\lambda} + i\,\bar{\xi}\,\bar{\sigma}_\mu\,\lambda\ ,\\ \delta\lambda_\alpha &= i\,\xi_\alpha\,D + (\sigma^{\mu\nu}\,\xi)_\alpha\,F_{\mu\nu}\ ,\\ \delta\bar{\lambda}_{\dot\alpha} &= -i\,\bar{\xi}_{\dot\alpha}\,D - (\bar{\xi}\,\bar{\sigma}^{\mu\nu})_{\dot\alpha}\,F_{\mu\nu}\ ,\\ \delta D &= -\xi\,\sigma^\mu\,\partial_\mu\bar{\lambda} + \bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\lambda\end{aligned}}$$

$$C \to C - \mathrm{Im}\,X_\Lambda\ ,\quad \chi \to \chi + \frac{1}{\sqrt{2}}\psi_\Lambda$$

$$M \to M + F_\Lambda \quad A_\mu \to A_\mu + \partial_\mu \mathrm{Re}\,X_\Lambda$$

# SUSY variations and gauge covariance

- The SUSY variations realized geometrically in superspace always anticommute to an ordinary derivative. For example, the SUSY variation of the gauge field is
$$\delta A_\mu = i\,\bar{\xi}\,\sigma_\mu\,\bar{\lambda} + i\,\bar{\xi}\,\bar{\sigma}_\mu\,\lambda + \xi\,\partial_\mu\chi + \bar{\xi}\,\partial_\mu\bar{\chi}$$
and because of the $\chi$ terms it satisfies
$$\delta_1\delta_2 A_\mu - \delta_2\delta_1 A_\mu = -\,2\,i\,(\xi_1\,\sigma^\nu\,\bar{\xi}_2 - \xi_2\,\sigma^\nu\,\bar{\xi}_1)\,\partial_\mu A_\nu$$
(even if we start with $\chi = 0$ we pick up terms from the iterated variation)

- When we use the WZ gauge and the compensating gauge transformation we obtain gauge-covariant SUSY variations $\widetilde{\delta}$ which close to the gauge-cov. “completion” of translations. Eg:
$$\widetilde{\delta} A_\mu = i\,\bar{\xi}\,\sigma_\mu\,\bar{\lambda} + i\,\bar{\xi}\,\bar{\sigma}_\mu\,\lambda \quad , \qquad \widetilde{\delta}_1\widetilde{\delta}_2 A_\mu - \widetilde{\delta}_2\widetilde{\delta}_1 A_\mu = -\,2\,i\,(\xi_1\,\sigma^\nu\,\bar{\xi}_2 - \xi_2\,\sigma^\nu\,\bar{\xi}_1)\,F_{\mu\nu}$$

- The variation $\widetilde{\delta}$ is the one we encountered studying SUSY gauge theories before superspace. It was simply denoted $\delta$

# Field strength superfield

- Given a vector superfield $V$ one defines

$$\mathscr{W}_\alpha = -\frac{1}{4}\,\overline{D}\overline{D}D_\alpha V \quad \text{and its complex conj.} \quad \overline{\mathscr{W}}_{\dot\alpha} = -\frac{1}{4}\,DD\overline{D}_{\dot\alpha}V$$

- Since $V$ is bosonic and we act with an odd number of SUSY cov. der's, the superfields $\mathscr{W}_\alpha$, $\overline{\mathscr{W}}_{\dot\alpha}$ are fermionic
- Recall $\overline{D}_{\dot\alpha}\overline{D}_{\dot\beta}\overline{D}_{\dot\gamma}(\ldots) = 0$. We see that $\mathscr{W}_\alpha$ is chiral, and similarly $\overline{\mathscr{W}}_{\dot\alpha}$ is antichiral
- They are also gauge invariant: under $V \to V + i/2\,(\Lambda - \overline{\Lambda})$ we have

$$\mathscr{W}_\alpha \to \mathscr{W}_\alpha - \frac{i}{8}\,\overline{D}\overline{D}D_\alpha\Lambda + \frac{i}{8}\,\overline{D}\overline{D}D_\alpha\overline{\Lambda}$$

The second term goes away because $\overline{\Phi}$ is antichiral. For the first term we first use

$$\overline{D}_{\dot\beta}\overline{D}^{\dot\beta}D_\alpha\Lambda = \overline{D}_{\dot\beta}\{\overline{D}^{\dot\beta}, D_\alpha\}\Lambda - \overline{D}_{\dot\beta}D_\alpha\overline{D}^{\dot\beta}\Lambda = \overline{D}_{\dot\beta}\{\overline{D}^{\dot\beta}, D_\alpha\}\Lambda = \{\overline{D}^{\dot\beta}, D_\alpha\}\overline{D}_{\dot\beta}\Lambda$$

where we used that the anticommutator gives a $P_\mu$, which commutes with $\overline{D}_{\dot\beta}$

# Field strength superfield

- Since $\mathscr{W}_\alpha$ is a chiral superfield it is convenient to expand it in the $(y, \vartheta)$ coordinates:

$$\mathscr{W}_\alpha = -\,i\,\lambda_\alpha(y) + \left[\delta_\alpha{}^\beta\, D(y) - i\,(\sigma^{\mu\nu})_\alpha{}^\beta\, F_{\mu\nu}(y)\right] \vartheta_\beta + (\vartheta\,\vartheta)\,(\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu \overline{\lambda}^{\dot\beta}(y)$$

- As usual we have $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- This can be computed in any gauge, for example the WZ gauge
- The expression for $\overline{\mathscr{W}}_{\dot\alpha}$ in terms of the natural coords in antichiral superspace is similar
- Notice that both $\mathscr{W}_\alpha$ and $\overline{\mathscr{W}}_{\dot\alpha}$ are built with the same component fields. Indeed they obey a constraint in superspace:

$$D^\alpha \mathscr{W}_\alpha = \overline{D}_{\dot\alpha} \overline{\mathscr{W}}^{\dot\alpha}$$

# U(1) charged matter in superspace

- We have defined the $U(1)$ gauge transformation of the vector superfield as $V \to V + i/2\,(\Lambda - \overline{\Lambda})$. What about a field that has a charge $q$? We set

$$\Phi \to e^{-iq\Lambda}\,\Phi$$

- This gauge transformation is compatible with $\Phi$ being chiral
- An infinitesimal variation is $\delta_{\text{gauge}}\Phi = -\,i\,q\,\Lambda\,\Phi$. In component fields:

$$\delta_{\text{gauge}}X_\Phi = -\,i\,q\,X_\Phi\,X_\Lambda\ , \qquad \delta_{\text{gauge}}\psi_\Phi = -\,i\,q\,(\psi_\Phi\,X_\Lambda + X_\Phi\,\psi_\Lambda)$$

$$\delta_{\text{gauge}}F_\Phi = -\,i\,q\,(F_\Phi\,X_\Lambda + X_\Phi\,F_\Lambda - \psi_X\,\psi_\Lambda)$$

- Recall $\delta_{\text{gauge}}A_\mu = \partial_\mu(\mathrm{Re}\,X_\Lambda)$. We see that $\mathrm{Re}\,X_\Lambda$ is identified with the parameter of a standard gauge transformation.
- The standard parameter $\mathrm{Re}\,X_\Lambda$ is naturally combined with $\mathrm{Im}\,X_\Lambda$ when acting on the matter superfield $\Phi$. We have a natural action of the complexified gauge group $\mathbb{C}^\times$ on matter chiral superfields.

# Gauge-covariant SUSY variations

- Recall that if we have a $U(1)$ vector superfield in WZ gauge and we act with a SUSY variation in superspace, we need a compensating gauge transformation to restore the WZ gauge

$$X_\Lambda = 0 \ , \qquad \psi_{\Lambda\alpha} = -\, i \sqrt{2}\, A_\mu\, (\sigma^\mu\, \bar{\xi})_\alpha\, , \qquad F_\Lambda = -\, 2\, \bar{\xi}\, \bar{\lambda}$$

- The "non-covariant" SUSY variations of the matter field $\Phi$ are those defined by supertranslations. We combine them with the compensating gauge transformation

$$\delta_{\text{n.c.}} X_\Phi = \sqrt{2}\, \xi\, \psi_\Phi \qquad\qquad \delta_{\text{comp.gauge}}\, X_\Phi = 0$$

$$\delta_{\text{n.c.}} \psi_{\Phi\alpha} = i \sqrt{2}\, (\sigma^\mu\, \bar{\xi})_\alpha\, \partial_\mu X_\Phi + \sqrt{2}\, \xi_\alpha\, F_\Phi \qquad\qquad \delta_{\text{comp.gauge}}\, \psi_{\Phi\alpha} = -\sqrt{2}\, q\, X_\Phi\, A_\mu\, (\sigma^\mu\, \bar{\xi})_\alpha$$

$$\delta_{\text{n.c.}} F_\Phi = i \sqrt{2}\, \bar{\xi}\, \bar{\sigma}^\mu\, \partial_\mu \psi_\Phi \qquad\qquad \delta_{\text{comp.gauge}}\, F_\Phi = 2\, i\, q\, X_\Phi\, (\bar{\xi}\, \bar{\lambda}) - \sqrt{2}\, q\, (\bar{\xi}\, \bar{\sigma}^\mu\, \psi_X)\, A_\mu$$

- The total variations are the gauge-cov. SUSY variations

$$\widetilde{\delta} X_\Phi = \sqrt{2}\, \xi\, \psi_\Phi \ , \qquad \widetilde{\delta} \psi_\Phi = i \sqrt{2}\, (\sigma^\mu\, \bar{\xi})_\alpha\, (\partial_\mu + i\, q\, A_\mu) X_\Phi + \sqrt{2}\, \xi_\alpha\, F_\Phi$$

$$\widetilde{\delta} F_\Phi = i \sqrt{2}\, \bar{\xi}\, \bar{\sigma}^\mu\, (\partial_\mu + i\, q\, A_\mu)\, \psi_\Phi + 2\, i\, q\, X_\Phi\, (\bar{\xi}\, \bar{\lambda})$$

# Gauge-covariant SUSY variations

$$\widetilde{\delta} X_\Phi = \sqrt{2}\, \xi\, \psi_\Phi \quad , \qquad\qquad \widetilde{\delta} \psi_\Phi = i \sqrt{2}\, (\sigma^\mu\, \bar{\xi})_\alpha\, (\partial_\mu + i\, q\, A_\mu) X_\Phi + \sqrt{2}\, \xi_\alpha\, F_\Phi$$

$$\widetilde{\delta} F_\Phi = i \sqrt{2}\, \bar{\xi}\, \bar{\sigma}^\mu\, (\partial_\mu + i\, q\, A_\mu)\, \psi_\Phi + 2\, i\, q\, X_\Phi\, (\bar{\xi}\, \bar{\lambda})$$

We have recovered the gauge-cov. SUSY variation of a charged chiral multiplet off-shell. Notice:

- ordinary derivatives are replaced by gauge cov. derivatives
- we find the extra term in the variation of $F_\Phi$ of the form $(\bar{\xi}\, \bar{\lambda})\, X_\Phi$

Recall that the gauge cov. SUSY variations close on the gauge cov. version of $P_\mu$. In this example

$$\widetilde{\delta}_1 \widetilde{\delta}_2 X_\Phi - \widetilde{\delta}_2 \widetilde{\delta}_1 X_\Phi = -\, 2\, i\, (\xi_1\, \sigma^\nu\, \bar{\xi}_2 - \xi_2\, \sigma^\nu\, \bar{\xi}_1)\, (\partial_\mu + i\, q\, A_\mu)\, X_\Phi \qquad \text{etc.}$$

NB: The gauge-cov variation was simply denoted $\delta$ in the previous lectures, before we introduced superspace

# Gauge transformations of $\overline{\Phi}$ and $\overline{\Phi}\, e^{2qV}$

- Taking the complex conjugate of $\Phi \to e^{-iq\Lambda}\,\Phi$ we get
$$\overline{\Phi} \to e^{iq\overline{\Lambda}}\,\overline{\Phi}$$
- If $\Phi$ were an ordinary field charged under a $U(1)$ gauge group, $\Phi$ and its complex conjugate would transform in opposite ways, because the gauge parameter would be real
- The gauge parameter is now a chiral superfield $\overline{D}_{\dot{\alpha}}\Lambda = 0$ and we know that if we demand $\Lambda = \overline{\Lambda}$ then $\Lambda$ is a constant in superspace
- In order to get an object that transforms in the opposite way as $\Phi$ we have to combine $\overline{\Phi}$ and the vector superfield $V$. Recall: $V \to V + \frac{i}{2}\,(\Lambda - \overline{\Lambda})$. Then
$$\overline{\Phi}\, e^{2qV} \to e^{iq\Lambda}\,(\overline{\Phi}\, e^{2qV})$$
- If we take a function $K(z, \overline{z})$ of a complex variable $z$ that in invariant under $(z, \overline{z}) \to (e^{-iq\Lambda_0}\, z, e^{iq\Lambda_0}\overline{z})$ for a <u>constant</u> $\Lambda_0$, then the quantity $K(\Phi, \overline{\Phi}\, e^{2qV})$ is a superfield that is invariant under gauge transformations in superspace

# Non-Abelian gauge invariance

- We consider a non-Abelian gauge group $G$; we use $a, b = 1, \dots, \dim G$ for adjoint indices
- Let us pick a collection of vector superfields $V^a$ and let us choose a reference representation $\mathbf{R}$ of $G$. The generators of $G$ in this rep are denoted

$$(t_a^{\mathbf{R}})^i{}_j \quad , \quad i, j = 1, \dots, \dim \mathbf{R}$$

- We take them to be hermitian and normalized so that

$$\mathrm{tr}_{\mathbf{R}}(t_a\, t_b) = T(\mathbf{R})\, \delta_{ab}$$

- We can construct the matrix-valued superfield

$$V_{\mathbf{R}} := V^a\, t_a^{\mathbf{R}}$$

- In a similar way we use a collection of chiral superfields $\Lambda^a$ to define

$$\Lambda_{\mathbf{R}} := \Lambda^a\, t_a^{\mathbf{R}} \qquad \text{so that} \qquad \Lambda_{\mathbf{R}}^\dagger = \overline{\Lambda}^a\, t_a^{\mathbf{R}}$$

# Non-Abelian matter chiral superfields

- We want to write the transformation law for a chiral superfield $\Phi$ in the representation $\mathbf{R}$ of the gauge group. The natural expression to consider is

$$\Phi' = e^{-i\Lambda_{\mathbf{R}}}\,\Phi \qquad \text{or more explicitly} \qquad \Phi'^{\,i} = \exp(-i\Lambda^a\, t_a^{\mathbf{R}})^i{}_j\,\Phi^j$$

- The expression for an infinitesimal gauge transformation is simple

$$\delta_{\text{gauge}}\Phi^i = -\,i\,\Lambda^a\,(t_a^{\mathbf{R}})^i{}_j\,\Phi^j$$

- We interpret $\mathrm{Re}\,X_{\Lambda^a}$ as the standard gauge parameter. It is naturally accompanied by $\mathrm{Im}\,X_{\Lambda^a}$ leading to an action of the complexified gauge group $G^{\mathbb{C}}$ on $\Phi$

# The transformation law for $\Phi^\dagger$

- Taking the $\dagger$ of the matrix equation $\Phi' = e^{-i\Lambda_{\mathbf{R}}}\,\Phi$ we obtain

$$\Phi^{\dagger\,\prime} = \Phi^\dagger\, e^{i\Lambda^\dagger_{\mathbf{R}}} \quad \text{or more explicitly} \quad \overline{\Phi}'_i = \overline{\Phi}_j\, \exp(i\,\overline{\Lambda}^a\, t_a^{\mathbf{R}})^j{}_i$$

- As in the Abelian case, $\Phi$ and $\Phi^\dagger$ do not transform in opposite (i.e. contragradient) ways, because the gauge parameter is not a real quantity
- It is natural to use a collection of vector superfields $V^a$ with an adjoint index and to define their transformation law to be

$$e^{2\,V'_{\mathbf{R}}} = e^{-i\,\Lambda^\dagger_{\mathbf{R}}}\, e^{2\,V_{\mathbf{R}}}\, e^{i\,\Lambda_{\mathbf{R}}} \qquad \text{recall that} \quad V_{\mathbf{R}} = V^a\, t_a^{\mathbf{R}}$$

- In this way we have

$$(\Phi^\dagger\, e^{2V_{\mathbf{R}}})' = (\Phi^\dagger\, e^{2V_{\mathbf{R}}})\, e^{i\Lambda_{\mathbf{R}}}$$

which does indeed transform in the opposite way as $\Phi$

# Transformation law for vector superfields

- We have picked a representation $\mathbf{R}$ and we have demanded
$$e^{2\,V'_{\mathbf{R}}} = e^{-i\,\Lambda^{\dagger}_{\mathbf{R}}}\,e^{2\,V_{\mathbf{R}}}\,e^{i\,\Lambda_{\mathbf{R}}}$$
- This relation makes sense because it yields the same $V'^{a}$ irrespectively of the reference representation $\mathbf{R}$ we choose. This is because on the RHS we have to use the BCH formula, and we encounter the commutators
$$[t^{\mathbf{R}}_a, t^{\mathbf{R}}_b] = i f_{ab}{}^{c}\, t^{\mathbf{R}}_c$$
which have the same functional form in any representation $\mathbf{R}$. In the end, $V'^{a}$ can be expressed as an infinite sum in which all terms are built with $V^a$, $\Lambda^a, \overline{\Lambda}^a, f_{ab}{}^{c}$

# Transformation law for vector superfields

$$e^{2\,V'_{\mathbf{R}}} = e^{-i\,\Lambda^\dagger_{\mathbf{R}}}\,e^{2\,V_{\mathbf{R}}}\,e^{i\,\Lambda_{\mathbf{R}}}$$

We can be more explicit if we work at linear order in $\Lambda^a$ and $\overline{\Lambda}^a$. To compute $V'^a$ we need the following form of the BCH formula

$$\log(e^A\,e^B) = A + \left[\int_0^1 dt\,\psi(e^{\mathrm{ad}_A}\,e^{t\,\mathrm{ad}_B})\right]B \quad , \qquad \psi(x) := \frac{\log x}{1 - x^{-1}}$$

Using $\log(e^B\,e^A) = -\log(e^{-A}\,e^{-B})$ we also have a similar formula with the roles of $A$ and $B$ exchanged. These formulas are useful in extracting expressions that are exact in $A$ and linear in $B$.

<u>Fun fact</u>: $\psi(x)$ is closely related to the generating function for Bernoulli numbers

$$\frac{u}{e^u - 1} = \frac{u}{2}\left(\coth\frac{u}{2} - 1\right) = \sum_{n=0}^{\infty}\frac{B_n\,u^n}{n!}$$

# Transformation law for vector superfields

One finds

$$2\,\delta V_{\mathbf{R}} = i\,\mathrm{ad}_{V_{\mathbf{R}}}(\Lambda_{\mathbf{R}} + \Lambda_{\mathbf{R}}^{\dagger}) + i\,\mathrm{ad}_{V_{\mathbf{R}}}\,\coth \mathrm{ad}_{V_{\mathbf{R}}}(\Lambda_{\mathbf{R}} - \Lambda_{\mathbf{R}}^{\dagger})$$

To get some intuition, let us expand the RHS in powers of $V$ (notice that the expansion of $\coth x$ near 0 starts with a $1/x$ pole)

$$2\,\delta V_{\mathbf{R}} = i\,(\Lambda_{\mathbf{R}} - \Lambda_{\mathbf{R}}^{\dagger}) + i\,[V_{\mathbf{R}}, \Lambda_{\mathbf{R}} + \Lambda_{\mathbf{R}}^{\dagger}] + \frac{i}{3}[V_{\mathbf{R}}, [V_{\mathbf{R}}, \Lambda_{\mathbf{R}} - \Lambda_{\mathbf{R}}^{\dagger}]] + \mathcal{O}(V_{\mathbf{R}}^3)$$

We can translate this matrix equation into a relation for $\delta V^a$,

$$2\,\delta V^a = i\,(\Lambda^a - \overline{\Lambda}^a) - f_{bc}{}^{a}\,V^b\,(\Lambda^c + \overline{\Lambda}^c) - \frac{i}{3}\,f_{bc}{}^{a} f_{de}{}^{c}\,V^b\,V^d\,(\Lambda^e - \overline{\Lambda}^e) + \ldots$$

We get an expression in $V^a$, $\Lambda^a$, $\overline{\Lambda}^a$, $f_{ab}{}^c$, as promised. Notice that we do not have to raise/lower the adjoint indices in these equations.

# Non-Abelian WZ gauge

$$2\,\delta V^a = i\,(\Lambda^a - \overline{\Lambda}^a) - f_{bc}{}^a\, V^b\,(\Lambda^c + \overline{\Lambda}^c) - \frac{i}{3}\, f_{bc}{}^a f_{de}{}^c\, V^b\, V^d\,(\Lambda^e - \overline{\Lambda}^e) + \dots$$

The first term in the variation has the same structure as in the Abelian case. It implies that it is still possible to choose the WZ gauge for non-Abelian vector superfields:

$$C^a = 0, \quad \chi^a = 0, \quad M^a = 0$$

In this gauge

$$V^a = -\,\theta\,\sigma^\mu\,\overline{\theta}\,A^a_\mu(x) + i\,(\theta\,\theta)\,\overline{\theta}\,\overline{\lambda}^a(x) - i\,(\overline{\theta}\,\overline{\theta})\,\theta\,\lambda^a(x) + \frac{1}{2}\,(\theta\,\theta)(\overline{\theta}\,\overline{\theta})\,D^a(x)$$

and in particular $V^a\,V^b\,V^c \equiv 0$. This means that we can compute the gauge variation away from WZ gauge exactly:

$$2\,\delta V^a\Big|_{\text{from WZ gauge}} = i\,(\Lambda^a - \overline{\Lambda}^a) - f_{bc}{}^a\, V^b\,(\Lambda^c + \overline{\Lambda}^c) - \frac{i}{3}\, f_{bc}{}^a f_{de}{}^c\, V^b\, V^d\,(\Lambda^e - \overline{\Lambda}^e)$$

# Non-Abelian gauge-cov SUSY variations

$$2\,\delta V^a\Big|_{\text{from WZ gauge}} = i\,(\Lambda^a - \overline{\Lambda}^a) - f_{bc}{}^a\, V^b\,(\Lambda^c + \overline{\Lambda}^c) - \frac{i}{3}\, f_{bc}{}^a f_{de}{}^c\, V^b\, V^d\,(\Lambda^e - \overline{\Lambda}^e)$$

With the above expression one can extract the SUSY variations of the component fields (assuming the basepoint is in WZ gauge). Comparing with the SUSY variations away from the basepoint, one can extract the compensating gauge transformation that restores the WZ gauge after the SUSY variation. Combining the two transformations gives the gauge-covariant version of the SUSY variations we have seen earlier,

$$\delta A^a_\mu = i\,\overline{\xi}\,\overline{\sigma}_\mu\,\lambda^a + \mathrm{h.c.}$$

$$\delta\lambda^a_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\, F^a{}_{\mu\nu} + i\,D^a\,\xi_\alpha \qquad\qquad D_\mu\lambda^a = \partial_\mu\lambda^a - f_{bc}{}^a\, A^b_\mu\,\lambda^c$$

$$\delta D^a = \overline{\xi}\,\overline{\sigma}^\mu\, D_\mu\lambda^a + \mathrm{h.c.}$$

NB: we have reabsorbed the gauge coupling constant in $A^a_\mu$.

# Non-Abelian field strength superfields

- In analogy with the Abelian case, let us define

$$2\,\mathscr{W}_{\mathbf{R}\alpha} = -\frac{1}{4}\,\overline{D}\overline{D}e^{-2V_{\mathbf{R}}}\,D_\alpha e^{2V_{\mathbf{R}}} \qquad \text{where} \quad \mathscr{W}_{\mathbf{R}\alpha} := \mathscr{W}^a_\alpha\, t^{\mathbf{R}}_a$$

- This definition makes sense because $\mathscr{W}^a_\alpha$ does not depend on the representation $\mathbf{R}$. This follows from the fact that $e^{-2V_{\mathbf{R}}}\,D_\alpha e^{2V_{\mathbf{R}}}$ can be computed with a version of the BCH formula,

$$e^{-A}\,de^A = dA - \frac{1}{2!}\,[A, dA] + \frac{1}{3!}\,[A,[A,dA]] + \dots$$

- Since we only encounter Lie brackets, the result is independent of the representation $\mathbf{R}$. For example, in WZ gauge one has

$$e^{-2V_{\mathbf{R}}}\,D_\alpha e^{2V_{\mathbf{R}}} = 2\,D_\alpha V_{\mathbf{R}} - \frac{1}{2}\,[2\,V_{\mathbf{R}}, 2\,D_\alpha V_{\mathbf{R}}] = 2\left(D_\alpha V^a - i f_{bc}{}^a\,V^b\,D_\alpha V^c\right) t^{\mathbf{R}}_a$$

which implies the formula

$$\text{WZ gauge:} \qquad \mathscr{W}^a_\alpha = -\frac{1}{4}\,\overline{D}\,\overline{D}\left(D_\alpha V^a - i f_{bc}{}^a\,V^b\,D_\alpha V^c\right)$$

# Non-Abelian field strength superfields

$$2\,\mathscr{W}_{\mathbf{R}\alpha} = -\frac{1}{4}\,\overline{D}\overline{D}e^{-2V_{\mathbf{R}}}\,D_\alpha e^{2V_{\mathbf{R}}} \qquad \text{where} \quad \mathscr{W}_{\mathbf{R}\alpha} := \mathscr{W}^a_\alpha\, t_a^{\mathbf{R}}$$

- Just like its Abelian counterpart, $\mathscr{W}^a_\alpha$ is automatically a chiral superfield because it is $\overline{D}\,\overline{D}$ of something
- In the Abelian case $\mathscr{W}_\alpha$ is gauge-invariant. In the non-Abelian case one can show that

$$e^{2\,V'_{\mathbf{R}}} = e^{-i\,\Lambda^\dagger_{\mathbf{R}}}\,e^{2\,V_{\mathbf{R}}}\,e^{i\,\Lambda_{\mathbf{R}}} \quad \text{implies} \quad {\mathscr{W}_{\mathbf{R}\alpha}}' = e^{-i\,\Lambda_{\mathbf{R}}}\,\mathscr{W}_{\mathbf{R}\alpha}\,e^{i\,\Lambda_{\mathbf{R}}}$$

- Notice that the transformation law preserves the fact that $\mathscr{W}^a_\alpha$ is chiral
- Using the BCH formula $e^A\,B\,e^{-A} = e^{\mathrm{ad}_A}B$ one writes $\mathscr{W}^{a\,\prime}_\alpha$ in terms of $\mathscr{W}^a_\alpha$, $\Lambda^a$, ${f_{bc}}^a$ and verifies that it does not depend on the representation $\mathbf{R}$
- For an infinitesimal transf.: $\delta\mathscr{W}_{\mathbf{R}\alpha} = -\,i\,[\Lambda_{\mathbf{R}}, \mathscr{W}_{\mathbf{R}\alpha}]\,, \quad \delta\mathscr{W}^a_\alpha = {f_{bc}}^a\,\Lambda^b\,\mathscr{W}^c_\alpha$

# Non-Abelian field strength superfields

$$2\,\mathscr{W}_{\mathbf{R}\alpha} = -\frac{1}{4}\,\overline{D}\overline{D}e^{-2V_{\mathbf{R}}}\,D_\alpha e^{2V_{\mathbf{R}}} \qquad \text{where} \quad \mathscr{W}_{\mathbf{R}\alpha} := \mathscr{W}^a_\alpha\, t^{\mathbf{R}}_a$$

- The expression of $\mathscr{W}^a_\alpha$ is easier in WZ gauge, where it is the natural non-Abelian generalization of the gauge-invariant $\mathscr{W}_\alpha$ of the Abelian case

$$\text{WZ gauge:} \quad \mathscr{W}^a_\alpha = -\,i\,\lambda^a_\alpha(y) + \left[\delta_\alpha{}^\beta\,D^a(y) - i\,(\sigma^{\mu\nu})_\alpha{}^\beta\,F^a_{\mu\nu}(y)\right]\vartheta_\beta + (\vartheta\,\vartheta)\,(\sigma^\mu)_{\alpha\dot\beta}\,D_\mu\overline{\lambda}^{a\dot\beta}(y)$$

- We have introduced

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{bc}{}^a\,A^b_\mu\,A^c_\nu \quad , \qquad D_\mu\overline{\lambda}^a = \partial_\mu\overline{\lambda}^a - f_{bc}{}^a\,A^b_\mu\,\overline{\lambda}^c$$

- Similar remarks apply to the hermitian conjugate $\overline{\mathscr{W}}^a_{\dot\alpha}$ which is antichiral
- There is a non-Abelian analog of the constraint $D^\alpha\mathscr{W}_\alpha = \overline{D}_{\dot\alpha}\overline{\mathscr{W}}^{\dot\alpha}$ but to write it down one needs a suitable gauge covariant generalization of $D_\alpha$, $\overline{D}_{\dot\alpha}$. We will not pursue this further

# A remark on notation

- We are using the conventions of Wess-Bagger, up to a different normalization for $V$ and $\mathscr{W}$ (both in the Abelian and non-Abelian cases)

$$V_{\rm WB} = 2\, V_{\rm here} \qquad \text{and} \qquad (W_\alpha)_{\rm WB} = 2\, (\mathscr{W}_\alpha)_{\rm here}$$

- This factor of 2 in needed in order to get the gauge-cov derivatives

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{bc}{}^a\, A^b_\mu\, A^c_\nu \quad , \qquad D_\mu \bar\lambda^a = \partial_\mu \bar\lambda^a - f_{bc}{}^a\, A^b_\mu\, \bar\lambda^c$$

- Cfr exercise (7) of Chapter VII of Wess-Bagger. Notice that we choose to work with gauge-cov derivatives that do not contain explicitly the gauge coupling $g$. If desired one can make the rescaling $A^a_\mu \to g\, A^a_\mu$, $F^a_{\mu\nu} \to g\, F^a_{\mu\nu}$, $\lambda^a \to g\, \lambda^a$,$D^a \to g\, D^a$

# Supersymmetry and supergravity

Lecture 18

# Elementary and composite superfields

- Our goal is to use the superspace formalism to construct SUSY invariant actions for off-shell chiral multiplets and vector multiplets
- Our fundamental superfields are a collection of chiral fields $\Phi^i$ and of vector superfields $V^a$
- We have to construct composite superfields out of $\Phi^i$, $V^a$ and integrate them in superspace to get invariant actions
- Recall that we have two types of contributions:
    - type-D terms: full superspace integrals of a real superfield

$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\, V_{\text{composite}}$$

    - type-F terms: half superspace integrals of a chiral superfield (plus h.c.)

$$\int d^4x\, d^2\theta\, \Phi_{\text{composite}} + \mathrm{h.c.}$$

# Elementary and composite superfields

Here some useful facts to keep in mind when constructing composite superfields:

- Any linear combination of superfields with constant coefficients is a superfield
- Any linear combination of chiral superfields with constant coefficients is a chiral superfield
- The product of two superfields is a superfield
- The product of two chiral superfields is a chiral superfield
- The product of two real superfields is real superfield
- If we act with $\partial/\partial x^\mu$, $D_\alpha$ or $\overline{D}_{\dot\alpha}$ on a superfield we get a superfield

# Elementary and composite superfields

- As a first task, let us discuss how to write a renormalizable QFT of chiral superfields and vector superfields. We will generalize to non-renormalizable models later
- Let us start considering a model with chiral superfields only and no gauge invariance

# Canonical kinetic terms for chiral superfields

- We start with the chiral superfields $\overline{D}_{\dot\alpha}\Phi^i = 0$ with expansions

$$\Phi^i = X^i(y) + \sqrt{2}\,\vartheta\,\psi^i(y) + \vartheta\,\vartheta\,F^i(y)$$

- Their complex conjugates are antichiral: $D_\alpha \overline{\Phi}^{\bar\imath} = 0$
- The object $h_{j\bar\imath}\,\overline{\Phi}^{\bar\imath}\,\Phi^j$ is a real superfield as soon as the constant matrix $h$ is hermitian. Its component expansion can be worked out starting from the component expansion of $\Phi^i$. When the dust settles, one finds

$$\overline{\Phi}^{\bar\imath}\,\Phi^j = \ldots + \theta\theta\overline{\theta}\overline{\theta}\left[\overline{F}^{\bar\imath}\,F^j + \frac{1}{4}\,\overline{X}^{\bar\imath}\,\partial^\mu\partial_\mu X^j + \frac{1}{4}\,\partial^\mu\partial_\mu \overline{X}^{\bar\imath}\,X^j - \frac{1}{2}\,\partial_\mu \overline{X}^{\bar\imath}\,\partial^\mu X^j \right.$$
$$\left. + \frac{i}{2}\,\partial_\mu \overline{\psi}^{\bar\imath}\,\overline{\sigma}^\mu\,\psi^j - \frac{i}{2}\,\overline{\psi}^{\bar\imath}\,\overline{\sigma}^\mu\,\partial_\mu\psi^j\right]$$

# Canonical kinetic terms for chiral superfields

$$\overline{\Phi}^{\bar\imath}\,\Phi^j = \ldots + \theta\theta\overline{\theta}\overline{\theta}\left[\overline{F}^{\bar\imath}\,F^j + \frac{1}{4}\,\overline{X}^{\bar\imath}\,\partial^\mu\partial_\mu X^j + \frac{1}{4}\,\partial^\mu\partial_\mu\overline{X}^{\bar\imath}\,X^j - \frac{1}{2}\,\partial_\mu\overline{X}^{\bar\imath}\,\partial^\mu X^j \right.$$
$$\left. + \frac{i}{2}\,\partial_\mu\overline{\psi}^{\bar\imath}\,\overline{\sigma}^\mu\,\psi^j - \frac{i}{2}\,\overline{\psi}^{\bar\imath}\,\overline{\sigma}^\mu\,\partial_\mu\psi^j\right]$$

- We see that the real superfield $h_{j\bar\imath}\,\overline{\Phi}^{\bar\imath}\,\Phi^j$ can be used to write down the canonical kinetic terms for the chiral multiplets.
- In order to have non-degenerate kinetic terms with the correct sign, the constant matrix $h$ has to be positive definite. After a unitary field redefinition we can set $h_{i\bar\jmath} = \delta_{i\bar\jmath}$ without loss of generality
- It is customary to use the constant tensor $\delta_{j\bar\imath}$ to convert upper barred indices into lower unbarred indices, and write

$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\;\overline{\Phi}_i\,\Phi^i = \int d^4x\left[\overline{F}_i\,F^i - \partial_\mu\overline{X}_i\,\partial^\mu X^i + i\,\partial_\mu\overline{\psi}_i\,\overline{\sigma}^\mu\,\psi^i\right]$$

# Interaction terms for chiral superfields

- The monomials $\Phi^i\,\Phi^j$, $\Phi^i\,\Phi^j\,\Phi^k$, $\Phi^i\,\Phi^j\,\Phi^k\,\Phi^\ell$, etc. are all chiral superfields. In fact, any function that depends on $\Phi^i$ but not $\overline{\Phi}_i$ or derivatives of $\Phi^i$ is again a chiral superfield. This can be seen formally by thinking of the arbitrary function as a series expansion in $\Phi^i$.
- We can thus consider an arbitrary holomorphic function $W(\Phi^i)$
- This is the superspace origin of the **superpotential** for chiral multiplets
- Once $W(\Phi^i)$ is chosen, we can use it to build an F-type term

$$\int d^4x\, d^2\theta\, W(\Phi) + \mathrm{h.c.}$$

# Interaction terms for chiral superfields

- If we want a renormalizable model, $W$ should be a polynomial of degree at most 3

$$W = E_i\,\Phi^i + \frac{1}{2}\,m_{ij}\,\Phi^i\,\Phi^j + \frac{1}{3}\,g_{ijk}\,\Phi^i\,\Phi^j\,\Phi^k$$

- Our task is to find the component expansion of the monomials $\Phi^i\,\Phi^j$ and $\Phi^i\,\Phi^j\,\Phi^k$. This is most conveniently done in the $(y,\vartheta)$ coords:

$$\begin{aligned}\Phi^i\,\Phi^j &= X^i(y)\,X^j(y) + \sqrt{2}\,\vartheta\,[\psi^i(y)\,X^j(y) + \psi^j(y)\,X^i(y)]\\ &\quad + \vartheta\,\vartheta\,[X^i(y)\,F^j(y) + X^j(y)\,F^i(y) - \psi^i(y)\,\psi^j(y)]\end{aligned}$$

$$\begin{aligned}\Phi^i\,\Phi^j\,\Phi^k &= X^i(y)\,X^j(y)\,X^k(y) + \sqrt{2}\,\vartheta\,[\psi^i(y)\,X^j(y)\,X^k(y) + \psi^j(y)\,X^i(y)\,X^k(y) + \psi^k(y)\,X^i(y)\,X^j(y)]\\ &\quad + \vartheta\,\vartheta\,[F^i(y)\,X^j(y)\,X^k(y) + F^j(y)\,X^i(y)\,X^k(y) + F^k(y)\,X^i(y)\,X^j(y)\\ &\qquad - \psi^i(y)\,\psi^j(y)\,X^k(y) - \psi^i(y)\,\psi^k(y)\,X^j(y) - \psi^j(y)\,\psi^k(y)\,X^i(y)]\end{aligned}$$

- A useful identity: $(\vartheta\,\chi_1)\,(\vartheta\,\chi_2) = -\frac{1}{2}\,(\vartheta\,\vartheta)\,(\chi_1\,\chi_2)$ which follows from the Fierz id $(\chi_1\,\chi_2)\,\chi_{3\alpha} + \ldots = 0$

# Interaction terms for chiral superfields

- We collect the $\vartheta\vartheta$ components to arrive at

$$\int d^4x\, d^2\theta\, W = \int d^4x \left[ (E_i + m_{ij}\, X^i + g_{ijk}\, X^j\, X^k)\, F^j - \frac{1}{2}(m_{ij} + 2\, g_{ijk}\, X^k)\, \psi^i\, \psi^j \right]$$

- We have used the fact that $m_{ij}$, $g_{ijk}$ are symmetric in their indices
- The same expression is written more suggestively as

$$\int d^4x\, d^2\theta\, W = \int d^4x \left[ W_i\, F^j - \frac{1}{2}\, W_{ij}\, \psi^i\, \psi^j \right] \quad , \quad W_i = \frac{\partial W}{\partial X^i} \quad , \quad W_{ij} = \frac{\partial^2 W}{\partial X^i\, \partial X^j}$$

- We have used superspace to derive the superpotential terms we have discussed in previous lectures

# Maxwell term for Abelian vector superfields

- Next, let us use superspace to write a kinetic term for an Abelian vector superfield. Recall the gauge variation of $V$ and the definition of the field strength superfield

$$V' = V + \frac{i}{2}\left(\Lambda - \overline{\Lambda}\right) \quad , \qquad \mathscr{W}_\alpha = -\frac{1}{4}\,\overline{D}\,\overline{D} D_\alpha V$$

- For Abelian gauge fields $\mathscr{W}_\alpha$ is chiral and gauge-invariant. Its expansion is

$$\mathscr{W}_\alpha = -\,i\,\lambda_\alpha(y) + \left[\delta_\alpha{}^\beta\, D(y) - i\,(\sigma^{\mu\nu})_\alpha{}^\beta\, F_{\mu\nu}(y)\right]\vartheta_\beta + (\vartheta\,\vartheta)\,(\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu\overline{\lambda}^{\dot\beta}(y)$$

- The composite object $\mathscr{W}^\alpha\mathscr{W}_\alpha$ is a chiral superfield and a Lorentz scalar and is thus a good candidate to build an F-type term. Indeed, one finds

$$\mathscr{W}^\alpha\mathscr{W}_\alpha = \ldots + \vartheta\,\vartheta\left[-\,2\,i\,\lambda\,\sigma^\mu\,\partial_\mu\overline{\lambda} - \frac{1}{2}\,F^{\mu\nu}\,F_{\mu\nu} + D^2 + \frac{i}{4}\,\epsilon_{\mu\nu\rho\sigma}\,F^{\mu\nu}\,F^{\rho\sigma}\right]$$

- The F-term built from $\mathscr{W}^\alpha\mathscr{W}_\alpha$ is the desired SUSY completion of the Maxwell term:

$$\int d^4x\, d^2\theta\,\frac{1}{4}\,\mathscr{W}^\alpha\mathscr{W}_\alpha + \mathrm{h.c.} = \int d^4x\left[-\,i\,\lambda\,\sigma^\mu\,\partial_\mu\overline{\lambda} - \frac{1}{4}\,F^{\mu\nu}\,F_{\mu\nu} + \frac{1}{2}\,D^2\right]$$

- NB: the $\epsilon FF$ term is a total derivative. It does not affect the EOMs

# FI term for Abelian vector superfield

- A vector superfield $V$ is itself a real superfield. What happens if we try to build a D-type term out of it? Let us go back to the expression for $V$ before WZ gauge fixing:

$$
\begin{aligned}
V(x,\theta,\overline{\theta}) = C(x) &+ i\,\theta^\alpha\,\chi_\alpha(x) - i\,\overline{\theta}_{\dot\alpha}\,\overline{\chi}^{\dot\alpha}(x) \\
&+\tfrac{1}{2}\,i\,(\theta\,\theta)\,M(x) - \tfrac{1}{2}\,i\,(\overline{\theta}\,\overline{\theta})\,\overline{M}(x) - (\theta\,\sigma^\mu\,\overline{\theta})\,v_\mu(x) \\
&+ i\,(\theta\,\theta)\,\overline{\theta}_{\dot\alpha}\left[\overline{\lambda}^{\dot\alpha}(x) + \tfrac{1}{2}\,i\,(\overline{\sigma}^\mu)^{\dot\alpha\beta}\,\partial_\mu\chi_\beta(x)\right] - i\,(\overline{\theta}\,\overline{\theta})\,\theta^\alpha\left[\lambda_\alpha(x) + \tfrac{1}{2}\,i\,(\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu\overline{\chi}^{\dot\beta}(x)\right] \\
&+\tfrac{1}{2}\,(\theta\,\theta)\,(\overline{\theta}\,\overline{\theta})\left[D(x) + \tfrac{1}{2}\,\partial^\mu\,\partial_\mu C(x)\right]
\end{aligned}
$$

- The above expression gives us

$$
\int d^4x\,d^2\theta\,d^2\overline{\theta}\,V = \int d^4x\,\left[\tfrac{1}{2}\,D + \tfrac{1}{4}\,\partial^\mu\partial_\mu C\right] = \tfrac{1}{2}\int d^4x\,D
$$

- While $V$ is not gauge invariant, the component field $D(x)$ is gauge invariant for a $U(1)$ gauge field
- This is the superspace origin of the Fayet-Iliopoulos term

# Kinetic terms for matter charged under U(1)

- As a first example of a system with gauge interactions, let us consider an elementary chiral superfield of charge $q$,

$$V' = V + \frac{i}{2}(\Lambda - \overline{\Lambda}) \quad , \qquad \Phi' = e^{-i\,q\,\Lambda}\,\Phi \quad , \qquad \overline{\Phi}' = e^{i\,q\,\overline{\Lambda}}\,\overline{\Phi}$$

- Since $\Lambda$ is complex, $\Phi$ and $\overline{\Phi}$ do not transform in opposite ways, but we know how to fix this:

$$(\overline{\Phi}\,e^{2qV})' = e^{iq\Lambda}\,(\overline{\Phi}\,e^{2qV})$$

- The monomial $\overline{\Phi}\,e^{2qV}\,\Phi$ is a gauge-invariant real superfield. It contains the appropriate gauge-cov version of the canonical kinetic term for $\Phi$

# Kinetic terms for matter charged under U(1)

- Indeed, one can compute the $\theta\theta\bar{\theta}\bar{\theta}$ component of $\overline{\Phi}\, e^{2qV}\,\Phi$ by direct expansion (for convenience, in WZ gauge)

$$\overline{\Phi}\, e^{2qV}\,\Phi = \ldots + \theta\,\theta\,\bar{\theta}\,\bar{\theta}\,\Big[\overline{F}\,F + X\,\partial^\mu\partial_\mu\overline{X} + i\,\partial_\mu\overline{\psi}\,\overline{\sigma}^\mu\,\psi$$

$$+q\,A_\mu(\overline{\psi}\,\overline{\sigma}^\mu\,\psi + i\,\overline{X}\,\partial_\mu X - i\,\partial_\mu\overline{X}\,X) - q^2\,A_\mu\,A^\mu\,\overline{X}\,X$$

$$-i\sqrt{2}\,q\,(X\,\bar{\lambda}\psi - \overline{X}\,\lambda\,\psi) + q\,D\,\overline{X}\,X\Big]$$

- This result does not look gauge covariant. After some partial integrations in $x$-space, however, we can write

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\;\overline{\Phi}\, e^{2qV}\,\Phi = \int d^4x\Big[\overline{F}\,F - D^\mu\overline{X}\,D_\mu X + i\,D_\mu\overline{\psi}\,\overline{\sigma}^\mu\,\psi - i\sqrt{2}\,q\,(X\,\bar{\lambda}\psi - \overline{X}\,\lambda\,\psi) + q\,D\,\overline{X}\,X\Big]$$

$$D_\mu X = \partial_\mu X + i\,q\,A_\mu\,X\ , \qquad D_\mu\psi = \partial_\mu X + i\,q\,A_\mu\,\psi$$

# Kinetic terms for matter charged under U(1)

$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\, \overline{X}\, e^{2qV}\, X = \int d^4x \Big[ \overline{F}\, F - D^\mu \overline{\Phi}\, D_\mu \Phi + i\, D_\mu \overline{\psi}\, \overline{\sigma}^\mu\, \psi - i\sqrt{2}\, q\, (X\, \overline{\lambda}\, \psi - \overline{X}\, \lambda\, \psi) + q\, D\, \overline{X}\, X \Big]$$

$$D_\mu X = \partial_\mu X + i\, q\, A_\mu\, X \ , \qquad D_\mu \psi = \partial_\mu X + i\, q\, A_\mu\, \psi$$

- We have found the desired gauge-cov. kinetic terms
- We have also automatically generated the interaction terms

$$-i\sqrt{2}\, q\, (X\, \overline{\lambda}\, \psi - \overline{X}\, \lambda\, \psi) + q\, D\, \overline{X}\, X$$

- Those were first encountered without derivation in previous lectures

# Non-Abelian gauge theory

- We can now turn to non-Abelian renormalizable models. The elementary fields are a vector superfield $V^a$ in the adjoint rep of the gauge group $G$ and a collection of chiral superfields in a representation $\mathbf{R}$. We take $G$ to be a simple non-Abelian group. The generalization to several simple factors and $U(1)$ factors is straightforward
- The generators $t_a^{\mathbf{R}}$ are hermitian and satisfy

$$[t_a^{\mathbf{R}}, t_b^{\mathbf{R}}] = i f_{ab}{}^c\, t_c^{\mathbf{R}} \ , \qquad \mathrm{Tr}_{\mathbf{R}}\,(t_a\, t_b) = T(\mathbf{R})\,\delta_{ab}$$

# Short reminder

- Reminder of gauge transformations: $V_{\mathbf{R}} = V^a\, t_a^{\mathbf{R}},\ \Lambda_{\mathbf{R}} = \Lambda^a\, t_a^{\mathbf{R}},$

$$e^{2\,V'_{\mathbf{R}}} = e^{-i\,\Lambda^\dagger_{\mathbf{R}}}\, e^{2\,V_{\mathbf{R}}}\, e^{i\,\Lambda_{\mathbf{R}}} \quad , \qquad \Phi' = e^{-i\Lambda_{\mathbf{R}}}\,\Phi \quad ,$$

$$(\Phi^\dagger\, e^{2V_{\mathbf{R}}})' = (\Phi^\dagger\, e^{2V_{\mathbf{R}}})\, e^{i\Lambda_{\mathbf{R}}}$$

- Non-Abelian field strength superfield

$$2\,\mathscr{W}_{\mathbf{R}\alpha} = -\frac{1}{4}\,\overline{D}\overline{D}e^{-2V_{\mathbf{R}}}\, D_\alpha e^{2V_{\mathbf{R}}} \quad , \qquad \mathscr{W}_{\mathbf{R}\alpha} := \mathscr{W}^a_\alpha\, t_a^{\mathbf{R}}$$

- Its gauge transformation:

$$\mathscr{W}_{\mathbf{R}\alpha}{}' = e^{-i\,\Lambda_{\mathbf{R}}}\,\mathscr{W}_{\mathbf{R}\alpha}\, e^{i\,\Lambda_{\mathbf{R}}}$$

# YM term and SUSY completion

$$\Phi' = e^{-i\Lambda_{\mathbf{R}}}\,\Phi \ \ , (\Phi^\dagger\, e^{2V_{\mathbf{R}}})' = (\Phi^\dagger\, e^{2V_{\mathbf{R}}})\, e^{i\Lambda_{\mathbf{R}}} \ \ , \ \mathscr{W}_{\mathbf{R}\alpha}{}' = e^{-i\,\Lambda_{\mathbf{R}}}\,\mathscr{W}_{\mathbf{R}\alpha}\, e^{i\,\Lambda_{\mathbf{R}}}$$

The quantity $\mathrm{Tr}_{\mathbf{R}}\,(\mathscr{W}^\alpha\,\mathscr{W}_\alpha)$ is a gauge-invariant chiral superfield. It is convenient to compute it in WZ gauge. In the end, one finds

$$\mathrm{Tr}_{\mathbf{R}}\,(\mathscr{W}^\alpha\,\mathscr{W}_\alpha) = \ldots + \vartheta\,\vartheta\,\mathrm{Tr}_{\mathbf{R}}\left[-2\,i\,\lambda\,\sigma^\mu\,D_\mu\bar{\lambda} - \tfrac{1}{2}\,F^{\mu\nu}\,F_{\mu\nu} + D^2 + \tfrac{i}{4}\,\epsilon_{\mu\nu\rho\sigma}\,F^{\mu\nu}\,F^{\rho\sigma}\right]$$

Let $\tau$ be any complex constant. We can construct the F-type term

$$\int d^4x\, d^2\theta\, \frac{-i\,\tau}{16\pi T(\mathbf{R})}\,\mathrm{Tr}_{\mathbf{R}}\,(\mathscr{W}^\alpha\,\mathscr{W}_\alpha) + \mathrm{h.c.} \quad , \qquad\qquad \tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2}$$

This F-term yields the Lagrangian

$$\mathscr{L} = \frac{1}{T(\mathbf{R})}\mathrm{Tr}_{\mathbf{R}}\left[-\frac{1}{4\,g^2}\,F^{\mu\nu}\,F_{\mu\nu} + \frac{1}{2\,g^2}\,D^2 - \frac{1}{g^2}\,i\,\lambda\,\sigma^\mu\,D_\mu\bar{\lambda} + \frac{1}{64\pi^2}\,\theta\,\epsilon_{\mu\nu\rho\sigma}\,F^{\mu\nu}\,F^{\rho\sigma}\right]$$

# YM term and SUSY completion

$$\int d^4x\, d^2\theta\, \frac{-i\,\tau}{16\pi T(\mathbf{R})}\, \mathrm{Tr}_{\mathbf{R}}\,(\mathscr{W}^\alpha \mathscr{W}_\alpha) + \mathrm{h.c.} \quad , \qquad\qquad \tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2}$$

$$\mathscr{L} = \frac{1}{T(\mathbf{R})}\mathrm{Tr}_{\mathbf{R}}\left[ -\frac{1}{4\,g^2}\, F^{\mu\nu}\, F_{\mu\nu} + \frac{1}{2\,g^2}\, D^2 - \frac{1}{g^2}\, i\,\lambda\,\sigma^\mu\, D_\mu \bar{\lambda} + \frac{1}{64\pi^2}\,\theta\,\epsilon_{\mu\nu\rho\sigma}\, F^{\mu\nu}\, F^{\rho\sigma}\right]$$

Remarks:

- The trace $\mathrm{Tr}_{\mathbf{R}}\,(t_a\, t_b) = T(\mathbf{R})\,\delta_{ab}$ cancels the prefactor $T(\mathbf{R})$ and yields $\mathscr{L} = -\frac{1}{4g^2}\,\delta_{ab} F^a_{\mu\nu}\, F^b_{\mu\nu} + \ldots$. One can achieve canonical normalization with the rescaling $A^a_\mu \to g\, A^a_\mu$, $F^a_\mu \to g\, F^a_\mu$ (and similarly for $\lambda$ and $D$). The presentation with $\tau$ as an overall prefactor is better suited for a non-perturbative analysis.
- The $\theta\epsilon FF$ term is a total derivative but has important non-perturbative effects related to instantons (cfr to $\theta$ angle in QCD). The real parameter $\theta$ is identified modulo $2\pi$
- Each factor in the gauge group can have a different $\tau$

# Gauge-cov. kinetic terms for matter

$$\Phi' = e^{-i\Lambda_{\mathbf{R}}}\,\Phi \quad , \qquad (\Phi^\dagger\, e^{2V_{\mathbf{R}}})' = (\Phi^\dagger\, e^{2V_{\mathbf{R}}})\, e^{i\Lambda_{\mathbf{R}}} \quad , \qquad {\mathscr{W}_{\mathbf{R}\alpha}}' = e^{-i\,\Lambda_{\mathbf{R}}}\,\mathscr{W}_{\mathbf{R}\alpha}\, e^{i\,\Lambda_{\mathbf{R}}}$$

- Gauge-cov kinetic term for matter fields

$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\,\Phi^\dagger\, e^{2qV}\,\Phi \quad , \qquad \Phi^\dagger\, e^{2qV}\,\Phi = \overline{\Phi}_i\,(e^{2qV^a\, t_a^{\mathbf{R}}})^i{}_j\,\Phi^j$$

- Since this is gauge invariant, it can be computed in WZ gauge for convenience. This D-type action yields the gauge-cov. kinetic terms for matter fields, plus additional interactions

$$\mathscr{L}_{\rm kin} = -\,D^\mu \overline{X}_i\, D_\mu X^i + i\, D_\mu \overline{\psi}_i\, \overline{\sigma}^\mu\, \psi^i + \overline{F}_i\, F^i$$

$$\mathscr{L}_{\rm coupl} = i\sqrt{2}\,\left[\overline{X}_i\,(t_a)^i{}_j\,\psi^j\,\lambda^a - \overline{\lambda}^a\,\overline{\psi}_i\,(t_a)^i{}_j\,X^j\right] + D^a\,\overline{X}_i\,(t_a)^i{}_j\,X^j$$

- NB: compared to previous lectures, we have reabsorbed $g$ in the vector multiplet fields

# Superpotential interactions

- They work exactly as in the case without gauge symmetry

$$\int d^4x\, d^2\theta\, W(\Phi) + \mathrm{h.c.}$$

- For this term to be allowed, however, $W$ must be gauge invariant
- NB: if $W$ is invariant under a rigid transformation $\Phi^{i\prime} = (e^{-i\Lambda_0^a t_a^{\mathbf{R}}})^i{}_j\,\Phi^j$ where $\Lambda_0^a$ are real and constant, then $W$ is automatically invariant under a complexified rigid transformation where the $\Lambda_0^a$ are complex and constants. It is also invariant under a superspace gauge transformations $\Phi' = e^{-i\Lambda_{\mathbf{R}}}\,\Phi$, where now $\Lambda$ is a full chiral superfield. This is because $W$ contains $\Phi$ but no $\Phi^\dagger$ or derivatives

# Supersymmetry and supergravity

Lecture 19

# Non-renormalizable SUSY models

- The superspace formalism allows us to construct non-renormalizable SUSY models for chiral and vector superfields
- A non-renormalizable model should be regarded as a low-energy effective theory that is valid up to a cut-off scale
- A natural way to organize terms in a low-energy effective action is by derivative counting
- The leading-order terms are those with at most two derivatives

# Models with chiral superfields

- We have seen that a renormalizable SUSY model of chiral superfields has action

$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \delta_{j\bar{\imath}} \overline{\Phi}^{\bar{\imath}}\, \Phi^j + \left[ \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \delta^{(2)}(\bar{\theta})\, W(\Phi) + \mathrm{h.c.} \right]$$

  where $W$ is at most cubic

- The most general SUSY model at 2-derivative level is given by

$$S = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, K(\Phi, \overline{\Phi}) + \left[ \int d^4x\, d^2\theta\, d^2\bar{\theta}\, \delta^{(2)}(\bar{\theta})\, W(\Phi) + \mathrm{h.c.} \right]$$

- It is specified by two functions:
  - an arbitrary holomorphic superpotential $W(\Phi)$
  - a real **Kähler potential** $K(\Phi, \overline{\Phi})$

# Superpotential terms

- We have verified explicitly that, when $W$ is a cubic polynomial, the Lagrangian can be written as

$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\, \delta^{(2)}(\overline{\theta})\, W = \int d^4x \left[\partial_i W\, F^j - \frac{1}{2}\, \partial_i \partial_j W\, \psi^i\, \psi^j\right]$$

- We use the notation

$$\partial_i = \frac{\partial}{\partial X^i} \quad , \qquad \partial_{\bar{\imath}} = \frac{\partial}{\partial \overline{X}^{\bar{\imath}}}$$

- This relation remains valid when $W(\Phi)$ is an arbitrary holomorphic function of $\Phi^i$

# Kähler potential terms

The component field expansion of the quantity

$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\, K(\Phi, \overline{\Phi})$$

can be found by working monomial by monomial. In the end, one finds an expression in terms of derivatives of $K(X, \overline{X})$:

$$\begin{aligned} K(\Phi, \overline{\Phi}) = \cdots + \theta\,\theta\,\overline{\theta}\,\overline{\theta}\, \Big[ & \partial_i\, \partial_{\bar{\jmath}} K\, F^i\, \overline{F}^{\bar{\jmath}} - \partial_i\, \partial_{\bar{\jmath}} K\, \partial_\mu X^i\, \partial^\mu \overline{X}^{\bar{\jmath}} - i\, \partial_i\, \partial_{\bar{\jmath}} K\, \overline{\psi}^{\bar{\jmath}}\, \overline{\sigma}^\mu\, \partial_\mu \psi^i \\ & - \tfrac{1}{2}\, \partial_i\, \partial_{\bar{\jmath}}\, \partial_{\bar{k}} K\, F^i\, \overline{\psi}^{\bar{\jmath}}\, \overline{\psi}^{\bar{k}} - \tfrac{1}{2}\, \partial_{\bar{\imath}}\, \partial_j\, \partial_k K\, \overline{F}^{\bar{\imath}}\, \psi^j\, \psi^k \\ & - i\, \partial_i\, \partial_j\, \partial_{\bar{k}} K\, \overline{\psi}^{\bar{k}}\, \overline{\sigma}^\mu\, \psi^i\, \partial_\mu X^j + \tfrac{1}{4}\, \partial_i\, \partial_j\, \partial_{\bar{k}}\, \partial_{\bar{\ell}} K\, (\psi^i\, \psi^j)\, (\overline{\psi}^{\bar{k}}\, \overline{\psi}^{\bar{\ell}}) \Big] \end{aligned}$$

# Kinetic terms and non-linear sigma-models

- The terms in the action that come from the Käher potential have a geometric interpretation
- Let us first consider a simpler situation: no SUSY, and a collection $\phi^M$ of real scalar fields
- A non-linear sigma model is a theory defined by an action of the form

$$S = -\frac{1}{2}\int d^4x\, G_{MN}(\phi)\, \partial^\mu \phi^M\, \partial_\mu \phi^N$$

- The quantity $G_{MN}(\phi)$ is symmetric in its $MN$ indices. In order to have a well-behaved theory, it should be positive-definite

# Kinetic terms and non-linear sigma-models

$$S = -\frac{1}{2}\int d^4x\, G_{MN}(\phi)\, \partial^\mu \phi^M\, \partial_\mu \phi^N$$

- Geometric interpretation:
  - the scalar fields $\phi^M$ are local coordinates on a target space $\mathscr{M}$
  - the target space is equipped with a Riemannian metric $G_{MN}$
- The canonical kinetic terms are recovered if we choose $\mathscr{M}$ = flat space, we identify $\phi^M$ with Cartesian coordinates, and we choose the metric $G_{MN} = \delta_{MN}$. This is the only option if we want a renormalizable model
- In non-renormalizable models we can consider any target space $\mathscr{M}$ and any metric $G_{MN}$

# Kähler manifolds from SUSY

- The superspace object $K(\Phi, \overline{\Phi})$ gives among other terms
$$\mathscr{L} = -\frac{1}{2}\,\partial_i \partial_{\bar{j}} K\, \partial^\mu \overline{X}^{\bar{j}}\, \partial_\mu X^i + \ldots$$
- We interpret the fields $X^i$ as complex coordinates on a target space $\mathscr{M}$
- On the physics side, we are free to perform any field redefinition of the form $X^{i'} = X^{i'}(X)$, as long as $X^{i'}(X)$ is a holomorphic function (to preserve the fact that the $X$'s should be chiral superfields)
- On the maths side, we say that the manifold $\mathscr{M}$ can be covered by local complex coordinates with holomorphic transition functions between patches. This condition defines a complex manifold

# Kähler manifolds from SUSY

- The metric on $\mathscr{M}$ that we read off from the action is $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$. Notice that it does not have $ij$ or $\bar{\imath}\bar{\jmath}$ components. Such a metric is usually referred to as hermitian
- The metric $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$ is written locally as the derivative of a real function. Such a metric is called a Kähler metric, and the function $K$ is known as Kähler potential (hence the name for the same object in superspace)
- Lesson: SUSY constrains the allowed target spaces $\mathscr{M}$ and the metric on them. We must have a complex manifold endowed with a Kähler metric
- NB: the canonical kinetic terms of a renormalizable model correspond to the choice

$$K(\Phi, \overline{\Phi}) = \delta_{i\bar{j}} \, \Phi^i \, \overline{\Phi}^{\bar{j}}$$

# Geometric interpretation of the fermions

- If the scalar fields $X^i$ are complex coordinates on the target space $\mathscr{M}$, what is the interpretation of the fermions $\psi^i$?
- As anticipated by their index structure, they are interpreted as tangent vector fields on $\mathscr{M}$
- This means that under a field redefinition (holom coord change) $X^{i'} = X^{i'}(X)$, the fermions transform as vectors

$$\psi^{i'} = \frac{\partial X^{i'}}{\partial X^j} \psi^j \quad , \qquad \bar{\psi}^{\bar{\imath}'} = \frac{\partial \bar{X}^{\bar{\imath}'}}{\partial \bar{X}^{\bar{\jmath}}} \bar{\psi}^{\bar{\jmath}}$$

# Geometric interpretation of the fermions

- The fact that $\psi^i$ transform as a vector under coord changes suggests that it is natural to replace $\partial_\mu \psi^i$ with a suitable covariant derivative, which is compatible with $\psi^{i'} = \dfrac{\partial X^{i'}}{\partial X^j}\psi^j$
- We can construct such covariant derivative using the Levi-Civita connection of the Kähler metric
- It turns out that the only non-zero components of the Levi-Civita connection are

$$\Gamma^i_{jk} = G^{i\bar{\ell}}\,\partial_j\,G_{k\bar{\ell}} = G^{i\bar{\ell}}\,\partial_j\partial_k\partial_{\bar{\ell}}K \qquad \text{and its c.c.}$$

- The covariant derivative of the fermions is

$$\mathscr{D}_\mu\psi^i = \partial_\mu\psi^i + \Gamma^i{}_{jk}\partial_\mu X^j\,\psi^k$$

# The full action with auxiliary fields

- The full Lagrangian, including the auxiliary fields, can be written by combining the terms that come from the Kähler potentials and those that come from the superpotential. This final result is still rather cumbersome:

$$
\begin{aligned}
\mathcal{L} = {} & G_{i\bar{\jmath}}\, F^i\, \overline{F}^{\bar{\jmath}} - G_{i\bar{\jmath}}\, \partial^\mu X^i\, \partial_\mu \overline{X}^{\bar{\jmath}} - i\, G_{i\bar{\jmath}}\, \overline{\psi}^{\bar{\jmath}}\, \overline{\sigma}^\mu\, \mathscr{D}_\mu \psi^i \\
& + \tfrac{1}{4}\, \partial_k \partial_{\bar{\ell}} G_{i\bar{\jmath}}\, (\psi^i\, \psi^k)\, (\overline{\psi}^{\bar{\jmath}}\, \overline{\psi}^{\bar{\ell}}) - \tfrac{1}{2}\, G_{i\bar{h}}\, \Gamma^{\bar{h}}_{\bar{\jmath}\bar{k}}\, F^i\, \overline{\psi}^{\bar{\jmath}}\, \overline{\psi}^{\bar{k}} - \tfrac{1}{2}\, G_{h\bar{\imath}}\, \Gamma^{h}_{jk}\, \overline{F}^{\bar{\imath}}\, \overline{\psi}^{\bar{\jmath}}\, \overline{\psi}^{\bar{k}} \\
& + \partial_i W\, F^i + \partial_{\bar{\imath}} \overline{W}\, \overline{F}^{\bar{\imath}} - \tfrac{1}{2}\, \partial_i \partial_j W\, \psi^i\, \psi^j - \tfrac{1}{2}\, \partial_{\bar{\imath}}\, \partial_{\bar{\jmath}}\, \overline{W}\, \overline{\psi}^{\bar{\imath}}\, \overline{\psi}^{\bar{\jmath}}
\end{aligned}
$$

- We have made progress in elucidating the geometric meaning of the action, but we still have Christoffel symbols and partial derivatives of $W$

# Integrating out the auxiliary fields

- The EOM for the auxiliary fields is

$$G_{i\bar{\jmath}}\, F^i - \frac{1}{2}\, G_{k\bar{\jmath}}\, \Gamma^k_{h\ell}\, \psi^h\, \psi^\ell + \partial_{\bar{\imath}} \overline{W} = 0 \quad \text{and its c.c.}$$

- Since the metric is non-degenerate, this can be solved for the $F$'s. Plugging back in the action we get a form that is fully covariant

$$\mathcal{L} = -G_{i\bar{\jmath}}\, \partial^\mu X^i\, \partial_\mu \overline{X}^{\bar{\jmath}} - i\, G_{i\bar{\jmath}}\, \overline{\psi}^{\bar{\jmath}}\, \overline{\sigma}^\mu\, \mathscr{D}_\mu \psi^i + \frac{1}{4}\, R_{i\bar{\jmath}k\bar{\ell}}\, (\psi^i\, \psi^k)\, (\overline{\psi}^{\bar{\jmath}}\, \overline{\psi}^{\bar{\ell}})$$
$$- \frac{1}{2}\, \mathscr{D}_i \mathscr{D}_j W\, \psi^i\, \psi^j - \frac{1}{2}\, \mathscr{D}_{\bar{\imath}}\, \mathscr{D}_{\bar{\jmath}} \overline{W}\, \overline{\psi}^{\bar{\imath}}\, \overline{\psi}^{\bar{\jmath}} - G^{i\bar{\jmath}}\, \mathscr{D}_i W\, \mathscr{D}_{\bar{\jmath}} \overline{W}$$

# Integrating out the auxiliary fields

$$\mathcal{L} = -G_{i\bar{\jmath}}\,\partial^\mu X^i\,\partial_\mu \overline{X}^{\bar{\jmath}} - i\,G_{i\bar{\jmath}}\,\overline{\psi}^{\bar{\jmath}}\,\overline{\sigma}^\mu\,\mathscr{D}_\mu \psi^i + \tfrac{1}{4}\,R_{i\bar{\jmath}k\bar{\ell}}\,(\psi^i\,\psi^k)\,(\overline{\psi}^{\bar{\jmath}}\,\overline{\psi}^{\bar{\ell}})$$
$$- \tfrac{1}{2}\,\mathscr{D}_i\mathscr{D}_j W\,\psi^i\,\psi^j - \tfrac{1}{2}\,\mathscr{D}_{\bar{\imath}}\,\mathscr{D}_{\bar{\jmath}}\,\overline{W}\,\overline{\psi}^{\bar{\imath}}\,\overline{\psi}^{\bar{\jmath}} - G^{i\bar{\jmath}}\,\mathscr{D}_i W\,\mathscr{D}_{\bar{\jmath}}\overline{W}$$

Comments:

- The quantities $R_{i\bar{\jmath}k\bar{\ell}}$ are the components of the Riemann tensor computed from $G_{i\bar{\jmath}}$
- The function $W$ is a scalar field on the target space $\mathscr{M}$. Its first derivative is already a covariant derivative: $\partial_i W = \mathscr{D}_i W$. Its second derivative is not covariant, because $\partial_i W = \mathscr{D}_i W$ has a contravariant index. We know how to fix this using the Christoffel symbols:

$$\mathscr{D}_i\mathscr{D}_j W = \partial_i\partial_j W - \Gamma^k_{ij}\,\partial_k W$$

# Scalar potential and SUSY vacua

$$\mathcal{L} = -G_{i\bar{\jmath}}\,\partial^\mu X^i\,\partial_\mu \overline{X}^{\bar{\jmath}} - i\,G_{i\bar{\jmath}}\,\overline{\psi}^{\bar{\jmath}}\,\overline{\sigma}^\mu\,\mathscr{D}_\mu\psi^i + \tfrac{1}{4}\,R_{i\bar{\jmath}k\bar{\ell}}\,(\psi^i\,\psi^k)\,(\overline{\psi}^{\bar{\jmath}}\,\overline{\psi}^{\bar{\ell}})$$
$$-\tfrac{1}{2}\,\mathscr{D}_i\mathscr{D}_j W\,\psi^i\,\psi^j - \tfrac{1}{2}\,\mathscr{D}_{\bar{\imath}}\,\mathscr{D}_{\bar{\jmath}}\,\overline{W}\,\overline{\psi}^{\bar{\imath}}\,\overline{\psi}^{\bar{\jmath}} - G^{i\bar{\jmath}}\,\mathscr{D}_i W\,\mathscr{D}_{\bar{\jmath}}\overline{W}$$

The scalar potential of this model is $V = G^{i\bar{\jmath}}\,\mathscr{D}_i W\,\mathscr{D}_{\bar{\jmath}}\overline{W}$. As usual in rigid SUSY theories, it is non-negative. SUSY is unbroken iff $V = 0$, which is the same as

$$\mathscr{D}_i W \equiv \partial_i W = 0$$

We can also see this from the SUSY variation of the fermions:

$$\delta\psi^i = i\sqrt{2}\,\sigma^\mu\,\overline{\xi}\,\partial_\mu X^i + \sqrt{2}\,\xi\,F^i\ , \qquad G_{i\bar{\jmath}}\,F^i - \frac{1}{2}\,G_{k\bar{\jmath}}\,\Gamma^k_{h\ell}\,\psi^h\,\psi^\ell + \partial_{\bar{\imath}}\overline{W} = 0$$

In the vacuum the fermions are zero and $\partial_{\bar{\imath}}\overline{W} = 0$, and thus the $F$'s are zero. Moreover the $X$'s are constant.

# Kähler transformations

- On the maths side, a Kahler transformation is any shift of the Kähler potential of the form
$$K(X,\overline{X}) \to K(X,\overline{X}) + f(X) + \bar{f}(\overline{X})$$
- Here $f(X)$ is any holomorphic function. This shift does not change the geometry, because the extra terms go away under $\partial_i \partial_{\bar{j}}$
- Superspace knows about this! Indeed, the superspace version of a Kahler transformation is
$$K(\Phi,\overline{\Phi}) \to K(\Phi,\overline{\Phi}) + f(\Phi) + \bar{f}(\overline{\Phi})$$
- The new terms vanish under $\int d^4x\, d^2\theta\, d^2\overline{\theta}$. To see this, we recall the $x$-expansion of a chiral superfield:

$$\begin{aligned} f(x,\theta,\overline{\theta}) = & X_f(x) + \sqrt{2}\,\theta^\alpha\,\psi_{f\alpha}(x) + (\theta\,\theta)\,F_f(x) \\ & + i\,\theta\,\sigma^\mu\,\overline{\theta}\,\partial_\mu X_f(x) + \frac{1}{4}\,(\theta\,\theta)\,(\overline{\theta}\,\overline{\theta})\,\partial^\mu\,\partial_\mu X_f - \frac{i}{\sqrt{2}}\,(\theta\,\theta)\,\partial_\mu \psi_f(x)\,\sigma^\mu\,\overline{\theta} \end{aligned}$$

The $\theta\theta\overline{\theta}\overline{\theta}$ component of $f(\Phi) + \bar{f}(\overline{\Phi})$ is a total derivative in $x$-space.

# R-symmetry

- Our discussion of R-symmetry in renormalizable models with chiral superfields extends immediately to general models
- The Kähler potential $K(\Phi, \overline{\Phi}) = \delta_{i\bar{j}}\, X^i\, \overline{X}^{\bar{j}}$ for renormalizable models is automatically invariant, but for a general Kähler potential we have to require $R[K] = 0$
- The problem is to ensure that the superpotential is compatible with R-symmetry. As we have seen, the condition is

$$R[W] = 2$$

- There is an easy superspace argument to see this. When we integrate $W$ in superspace $d^2\overline{\theta}$ cancels against $\delta^{(2)}(\overline{\theta})$ and we are left with $d^2\theta$. Recall $R[\theta] = 1$. The volume form in a Berezin integral transforms in the opposite way as a normal bosonic integral under coordinate transformations. Hence $R[d\theta] = -1$. This is way we must have $R[W] = 2$ to have an invariant action

# Remarks

- SUSY non-linear sigma-models exist in various dimensions and with various amounts of supersymmetry
- The larger the number of real supercharges, the more constrained is the geometry of the target space
- For example, with 4d $\mathscr{N}=2$ SUSY the target space is a specific kind of Kähler manifold, known as special-Kähler
- With 4d $\mathscr{N}=4$ SUSY the target space is so severely restricted that it can only be flat space with the flat metric

# Supersymmetry and supergravity

Lecture 20

# Non-renormalizable models with gauge fields

- We have seen that the standard YM term for a non-Abelian gauge field in superspace takes the form

$$\int d^4x\, d^2\theta\, \frac{-i\,\tau}{16\pi T(\mathbf{R})}\, \mathrm{Tr}_{\mathbf{R}}\,(\mathscr{W}^{\alpha}\mathscr{W}_{\alpha}) + \mathrm{h.c.} \quad , \qquad \tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2}$$

- The idea is to replace the complex constant $\tau$ with an arbitrary chiral superfield constructed from the elementary chiral matter fields $\Phi$. As usual, we must use a holomorphic function in order to get a chiral composite

# Non-renormalizable models with gauge fields

- This leads us to the notion of **holomorphic gauge coupling function** $f_{ab}(\Phi)$

$$\int d^4x\, d^2\theta\, f_{ab}(\Phi)\, \mathscr{W}^{a\alpha} \mathscr{W}^{b}_{\alpha} + \mathrm{h.c.}$$

- The indices $ab$ are adjoint indices. $f_{ab}(\Phi)$ is symmetric in $ab$
- Recall that $\mathscr{W}_{\mathbf{R}\alpha}{}' = e^{-i\Lambda_{\mathbf{R}}} \mathscr{W}_{\mathbf{R}\alpha} e^{i\Lambda_{\mathbf{R}}}$. For an infinitesimal gauge transformation

$$\delta \mathscr{W}_{\mathbf{R}\alpha} = -i\,[\Lambda_{\mathbf{R}}, \mathscr{W}_{\mathbf{R}\alpha}] \qquad \text{or} \qquad \delta \mathscr{W}^{a}_{\alpha} = f_{bc}{}^{a} \Lambda^{b} \mathscr{W}^{c}_{\alpha}$$

- The gauge coupling function must transform as dictated by its lower $ab$ indices:

$$\delta f_{ab} = -f_{ca}{}^{d} \Lambda^{c} f_{db} - f_{cb}{}^{d} \Lambda^{c} f_{ad}$$

# Non-renormalizable models with gauge fields

- We recover the canonical SYM term with the choice

$$f_{ab}(\Phi) = \frac{-i\,\tau}{16\pi T(\mathbf{R})}\,\mathrm{Tr}_{\mathbf{R}}(t_a t_b) = \frac{-i\,\tau}{16\pi}\,\delta_{ab}$$

- In this case the equation $\delta f_{ab} = -f_{ca}{}^{d}\,\Lambda^c f_{db} - f_{cb}{}^{d}\,\Lambda^c f_{ad}$ is true because both sides are 0 (The structure constants with their upper index lowered with the inverse of $\mathrm{Tr}_{\mathbf{R}}(t_a t_b)$ are totally antisymmetric.)
- A simple generalization is for example

$$f_{ab}(\Phi) = f(\Phi)\,\mathrm{Tr}_{\mathbf{R}}(t_a t_b)$$

  where $f(\Phi)$ is gauge-invariant
- If $ab$ label $U(1)$ factors in the gauge group (instead of generators of a simple non-Abelian factor), then $f_{ab}(\Phi)$ can be any gauge invariant holomorphic function, possibly with off-diagonal entries that induce "kinetic mixing" among the $U(1)$ gauge fields

# Some models with gauged chiral superfields

- In order to discuss the coupling between gauge fields and matter we have to understand the relation between the Kähler potential and gauge symmetries
- The general story is quite complicated so we start with a simpler scenario
- We fix the gauge transformation of the scalars to be the same standard linear variation that we have seen in renormalizable models

$$\delta_{\text{gauge}}\Phi = -\,i\,\Lambda^a\,t_a^{\mathbf{R}}\,\Phi \qquad \text{or using indices} \qquad \delta_{\text{gauge}}\Phi^i = -\,i\,\Lambda^a\,(t_a^{\mathbf{R}})^i{}_j\,\Phi^j$$

- "Linear" here means linear in $X$. Recall that $\Lambda^a$ is a chiral superfield
- We already know that the combination $X^\dagger\, e^{2V_{\mathbf{R}}}$ has a nice gauge variation that contains $\Lambda^a$ and not $\overline{\Lambda}^a$

$$\delta_{\text{gauge}}(\Phi^\dagger\, e^{2V_{\mathbf{R}}}) = i\,(\Phi^\dagger\, e^{2V_{\mathbf{R}}})\,\Lambda^a\,t_a^{\mathbf{R}}$$

# Some models with gauged chiral superfields

- Let us suppose that the Kähler potential of the model before introducing gauge fields in invariant under a rigid transformation $\delta\Phi^i = -\, i\, \Lambda_0^a\, (t_a^{\mathbf{R}})^i{}_j\, \Phi^j$ for real and constant params $\Lambda_0^a$
- With the replacement

$$K(\Phi, \Phi^\dagger) \to K(\Phi, \Phi^\dagger\, e^{2qV_{\mathbf{R}}})$$

  we are sure that the quantity $K(\Phi, \Phi^\dagger\, e^{2qV_{\mathbf{R}}})$ is a gauge-invariant real superfield. It gives the desired gauge-covariant completion of the couplings that we have seen in the ungauged sigma-model

- Example: imagine that $\Phi$ is in the fundamental representation of $U(N)$ and consider $K(\Phi, \Phi^\dagger) = \Phi^\dagger\Phi + \alpha\,(\Phi^\dagger\Phi)^2$. This is a non-renormalizable model of the kind we are considering

# Some models with gauged chiral superfields

<u>To summarize</u>: We take a collection of chiral superfields with linear gauge transformation $\delta_{\text{gauge}}\Phi = -i\,\Lambda^a\, t_a^{\mathbf{R}}\,\Phi$ and we couple them to vector superfields. The kinetic terms for the vectors are inside

$$\int d^4x\, d^2\theta\, f_{ab}(\Phi)\, \mathscr{W}^{a\alpha}\, \mathscr{W}^b_\alpha + \mathrm{h.c.}$$

where $f_{ab}(\Phi)$ is symmetric, holomorphic, and transforms under gauge transformations according to its $ab$ adjoint indices. The Kähler and superpotential terms for matter fields are

$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\, K(\Phi, \Phi^\dagger\, e^{2qV_{\mathbf{R}}}) + \left[\int d^4x\, d^2\theta\, W(\Phi) + \mathrm{h.c.}\right]$$

where we assume that $K(\Phi, \Phi^\dagger)$ and $W(\Phi)$ are invariant under the rigid variations $\delta\Phi = -i\,\Lambda_0^a\, t_a^{\mathbf{R}}\,\Phi$.

# Is this the most general story?

- The models in the previous slide are appealing because the gauge transformation of the matter superfields $\Phi$ is simple and we can write an action in superspace with small modifications from the ungauged case
- The general story is richer. We have seen that in a model with chiral superfields only, everything is covariant under arbitrary holomorphic field redefinitions $\Phi^{i'} = \Phi^{i'}(\Phi^i)$. This covariance is lost if we fix only allow for linear gauge transformations
- Rather than fixing the gauge transformations and demanding that the Kähler potential is invariant, the natural thing to do is to pick a Kähler potential, find the isometries of the Kähler metric, and determine the gauge variations accordingly
- The superspace treatment of this most general case is considerably more involved than the models we have considered so far (see e.g. Chapter XXIV of Wess-Bagger)
- Let us just state a few facts about these general models, without derivations. We abandon superspace and work in ordinary space, in component fields

# Geometric point of view on gauging

- The isometries of a generic Riemannian manifolds can be described using Killing vector fields, which preserve the metric
- A generic Killing field on a complex manifold with local coordinates $X^i$ has components $k^i(X,\overline{X})$ and their complex conjugates $k^{\bar{\imath}}(X,\overline{X})$
- On a Kähler manifold, the natural notion is that of a **holomorphic Killing vector**. A holomorphic Killing vector is a Killing vector that satisfies an extra condition: the components $k^i$ must be a function of $X$ but not $\overline{X}$.
- A holomorphic Killing vector generates a symmetry that does not mix the $X$'s and the $\overline{X}$'s

$$X^i \to X^i + k^i(X) \quad , \qquad \overline{X}^{\bar{\imath}} \to \overline{X}^{\bar{\imath}} + k^{\bar{\imath}}(\overline{X})$$

while at the same time preserving the Kähler metric

# Moment maps

- Let us take a holomorphic vector field $k^i(X)$ and let us demand that it is also a Killing vector. It turns out that this is equivalent to demanding that there exists locally a real function $\mathscr{P} = \mathscr{P}(X, \overline{X})$ such that
$$k^i(X) = -\,i\,G^{i\bar{\jmath}}\,\partial_{\bar{\jmath}}\mathscr{P}$$
- The function $\mathscr{P}$ is called moment map. It is only determined up by a shift by a real constant
- Since the LHS is holomorphic, we have a constraint on $\mathscr{P}$: $\partial_{\bar{k}}(G^{i\bar{\jmath}}\,\partial_{\bar{\jmath}}\mathscr{P}) = 0$
- Finding all local solutions to $\partial_{\bar{k}}(G^{i\bar{\jmath}}\,\partial_{\bar{\jmath}}\mathscr{P}) = 0$ for a given Kähler metric is equivalent to finding all local holomorphic Killing vectors (and is in general a hard task)

# Non-linear gauge transformations

- Recall that the Lie bracket of two Killing vector fields is also a Killing vector field. This is why (infinitesimal) isometries form a Lie algebra.
- It turns out that the Lie bracket of two holomorphic Killing vector fields is also a holomorphic Killing vector field, so (infinitesimal) holom. isom's also form a Lie algebra $\mathfrak{g}_{\text{hol.isom}}$
- To build a gauged model, we gauge a subalgebra $\mathfrak{g}_{\text{gauge}}$ of $\mathfrak{g}_{\text{hol.isom}}$. We use the label $a$ for the holom. killing vectors $k_a^i(X)$ that generate $\mathfrak{g}_{\text{gauge}}$. Thus $a$ in interpreted as an adjoint index in the model. The moment maps are labeled $\mathscr{P}_a(X, \overline{X})$

# Non-linear gauge transformations

- A vector field can be thought of as an infinitesimal displacement $\delta X^i$. In a sigma-model, $X^i$ is a complex scalar field (we work in ordinary space, not superspace). We identify the displacement $\delta X^i$ induced by holomorphic Killing vector as a gauge transformation

$$\delta_{\text{gauge}} X^i = \varepsilon^a \, k_a^i(X)$$

- Here $\varepsilon^a$ is a real $x$-dependent infinitesimal gauge parameter, carrying as usual an adjoint index $a$
- Notice that now $\delta_{\text{gauge}} X^i$ can be a non-linear function of the $X$'s

# A summary on the conditions on $\mathscr{P}$

- The real functions $\mathscr{P}_a(X, \overline{X})$ must solve $\partial_{\bar{k}}(G^{i\bar{j}}\,\partial_{\bar{j}}\mathscr{P}_a) = 0$, in such a way that $k_a^i(X) = -\,i\,G^{i\bar{j}}\,\partial_{\bar{j}}\mathscr{P}_a$ is indeed a holomorphic Killing vector
- At this stage we can freely shift $\mathscr{P}_a(X, \overline{X})$ by a real constant $p_a$
- In order to "take seriously" the label $a$ on $\mathscr{P}_a(X, \overline{X})$ as an adjoint index of the gauge group, the following "equivariance relation" has to hold

$$(k_a^i\,\partial_i + k_a^{\bar{\imath}}\,\partial_{\bar{\imath}})\mathscr{P}_b = f_{ab}{}^c\,\mathscr{P}_c$$

- It turns out that the equivariance relation fixes the ambiguity of shifts by $p_a$ if $a$ labels a generator of a simple non-Abelian factor in the gauge group. If instead $a$ labels a $U(1)$ factor, the ambiguity $\mathscr{P}_a(X, \overline{X}) \to \mathscr{P}_a(X, \overline{X}) + p_a$ remains
- This is the origin of FI terms in the geometry of Kähler manifolds

# The data of the most general gauged model

We can recap the data that defines the most general 2-derivative SUSY action for chiral multiplets and vector multiplets:

- The Kähler potential $K(X, \overline{X})$ (up to a Kähler transformation)
- The choice of subgroup of the holom. isom. group of the Kähler metric; this subgroup becomes the gauge group of the physical model. The real moment maps $\mathscr{P}_a(X, \overline{X})$ must be found for the generators of the gauge group, subject to the conditions summarized before
- The holomorphic superpotential $W(X)$, which must be invariant under the subgroup of the holom. isom. group that we are gauging
- The holomorphic gauge coupling function $f_{ab}(X)$, which is symmetric in $ab$, transforms according to these adjoint indices, and is also such that $\mathrm{Re} f_{ab}(X)$ is positive definite (to get well-behaved kinetic terms)

# The scalar potential of the most general model

- The full action of the most general SUSY model if quite involved. It can be found for example in Chapter 14 of the "Supergravity" book by Freedman and van Proeyen. Let us just record here the scalar potential. After eliminating the auxiliary fields, one finds

$$V = G^{i\bar{j}}\,\partial_i W\,\partial_{\bar{j}}\overline{W} + \frac{1}{2}\,(\mathrm{Re}f\,)^{-1ab}\,\mathscr{P}_a\,\mathscr{P}_b$$

- This is the sigma-model generalization of the result for renormalizable models. We still find a sum of "F-terms" $G^{i\bar{j}}\,\partial_i W\,\partial_{\bar{j}}\overline{W}$ and "D-terms" $(\mathrm{Re}f\,)^{-1ab}\,\mathscr{P}_a\,\mathscr{P}_b$

- As usual SUSY is unbroken iff $V = 0$ in the vacuum. Both $G^{i\bar{j}}$ and $(\mathrm{Re}f\,)^{-1ab}$ are non-degenerate (otherwise the model would have ill-behaved kinetic terms). We conclude that a SUSY vacuum must satisfy

$$\partial_i W = 0 \quad \text{and} \quad \mathscr{P}_a = 0 \quad \text{for all } i \text{ and } a$$

- Conversely, as soon as any of the $\partial_i W$ or $\mathscr{P}_a$ are non-zero in the vacuum, SUSY is spontaneously broken