Supersymmetry and Supergravity — Problem Sheet 1

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These problems refer to material covered in Lectures 1 through 8. They are due by Saturday before the class on week 3 by 11 am. Links to submit:
TA A. Boido: https://cloud.maths.ox.ac.uk/index.php/s/WP8kazik5pNZjmi
TA J. McGovern: https://cloud.maths.ox.ac.uk/index.php/s/oBKgZcaE9F4bw3z

1 Poincaré symmetry

1.a Let us parametrize an element of the Poincaré group as $g(\Lambda, a)$, where $\Lambda^\mu{}_\nu$ is a matrix in the Lorentz group $SO(1,3)$ (i.e. $\Lambda^\mu{}_\nu \, \Lambda^\rho{}_\sigma \, \eta_{\mu\rho} = \eta_{\nu\sigma}$) and a^μ is a constant 4-vector. Let $g(\Lambda, a)$ act on the coordinates x^μ from the left according to

$$g(\Lambda, a) \cdot x^\mu = x'^\mu := \Lambda^\mu{}_\nu \, x^\nu + a^\mu \ . \tag{1}$$

Verify the composition law

$$g(\Lambda_2, a_2) \, g(\Lambda_1, a_1) = g(\Lambda_2 \, \Lambda_1, \Lambda_2 \, a_1 + a_2) \ . \tag{2}$$

This law shows that the Poincaré group is isomorphic to the semidirect product $SO(1,3) \ltimes \mathbb{R}^4$. Use (2) to show

$$g(\Lambda, a)^{-1} = g(\Lambda^{-1}, -\Lambda^{-1} \, a) \ . \tag{3}$$

1.b Let $U(\Lambda, a)$ be the unitary operator in the Hilbert space of the QFT that implements the transformation $g(\Lambda, a)$. We demand the composition law

$$U(\Lambda_2, a_2) \, U(\Lambda_1, a_1) = U(\Lambda_2 \, \Lambda_1, \Lambda_2 \, a_1 + a_2) \ . \tag{4}$$

For an infinitesimal transformation, the vector a^μ is infinitesimal and $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \lambda^\mu{}_\nu$ where $\lambda_{\mu\nu} = \eta_{\mu\rho} \, \lambda^\rho{}_\nu$ is antisymmetric in $\mu\nu$. Let us write the unitary operator associated to such an infinitesimal transformation as

$$U(\mathbb{I} + \lambda, a) = \mathbb{I} + \tfrac{i}{2} \, \lambda^{\mu\nu} \, J_{\mu\nu} - i \, a^\mu \, P_\mu \ , \tag{5}$$

where $J_{\mu\nu}$, P_μ are Hermitian operators. Use the composition law (5) to derive the commutation relations of the Poincaré generators:

$$\begin{aligned} [J_{\mu\nu}, J_{\rho\sigma}] &= i \, \eta_{\mu\rho} \, J_{\nu\sigma} - i \, \eta_{\nu\rho} \, J_{\mu\sigma} - i \, \eta_{\mu\sigma} \, J_{\nu\rho} + i \, \eta_{\nu\sigma} \, J_{\mu\rho} \ , \\ [J_{\mu\nu}, P_\rho] &= i \, \eta_{\mu\rho} \, P_\nu - i \, \eta_{\nu\rho} \, P_\mu \ , \qquad [P_\mu, P_\nu] = 0 \ . \end{aligned} \tag{6}$$

Hint: Recall (3) and compute $U(\Lambda_2, a_2) \, U(\Lambda_1, a_1) \, U(\Lambda_2, a_2)^{-1}$. Specialize to infinitesimal (Λ_1, a_1). Finally, specialize to infinitesimal (Λ_2, a_2).

1.c By definition, we say that a local operator $\mathcal{O}(x)$ transforms as a scalar under the action of the Poincaré group if

$$\mathcal{O}'(x') = \mathcal{O}(x) \qquad \text{where} \qquad \mathcal{O}'(x) := U(\Lambda,a)^{-1}\,\mathcal{O}(x)\,U(\Lambda,a) \quad \text{and} \quad x'^{\mu} := \Lambda^{\mu}{}_{\nu}\,x^{\nu} + a^{\mu} \ . \tag{7}$$

Specialize this definition to an infinitesimal transformation and compute the commutators $[P_\mu, \mathcal{O}(x)]$ and $[J_{\mu\nu}, \mathcal{O}(x)]$. Write them in the form

$$[P_\mu, \mathcal{O}(x)] = -\mathbf{P}_\mu \mathcal{O}(x) \ , \qquad [J_{\mu\nu}, \mathcal{O}(x)] = -\mathbf{J}_{\mu\nu}\mathcal{O}(x) \ , \tag{8}$$

and identify the differential operators $\mathbf{P}_\mu$, $\mathbf{J}_{\mu\nu}$. Verify that these differential operators satisfy the same commutation relations (6) as the abstract generators of the Poincaré algebra.

1.d Suppose $\mathcal{O}(x)$ is a scalar operator satisfying (7). Define $\mathcal{O}^\mu = \eta^{\mu\nu}\partial_\nu \mathcal{O}$. Verify the transformation law

$$U(\Lambda,a)^{-1}\,\mathcal{O}^{\mu}(x')\,U(\Lambda,a) = \Lambda^{\mu}{}_{\nu}\,\mathcal{O}^{\nu}(x) \quad \text{where} \quad x'^{\mu} := \Lambda^{\mu}{}_{\nu}\,x^{\nu} + a^{\mu} \ . \tag{9}$$

1.e Let us consider a local operator that carries an index $\mathcal{O}^{\mathcal{A}}(x)$ and transforms as a

$$U(\Lambda,a)^{-1}\,\mathcal{O}^{\mathcal{A}}(x')\,U(\Lambda,a) = M(\Lambda)^{\mathcal{A}}{}_{\mathcal{B}}\,\mathcal{O}^{\mathcal{B}}(x) \quad \text{where} \quad x'^{\mu} := \Lambda^{\mu}{}_{\nu}\,x^{\nu} + a^{\mu} \ . \tag{10}$$

Show that consistency with (4) requires the matrices $M(\Lambda)^{\mathcal{A}}{}_{\mathcal{B}}$ to form a representation of the Lorentz group, i.e.

$$M(\Lambda_1)^{\mathcal{A}}{}_{\mathcal{B}}\,M(\Lambda_2)^{\mathcal{B}}{}_{\mathcal{C}} = M(\Lambda_1\,\Lambda_2)^{\mathcal{A}}{}_{\mathcal{C}} \ . \tag{11}$$

1.f Let us write

$$M(\mathbb{I}+\lambda)^{\mathcal{A}}{}_{\mathcal{B}} = \delta^{\mathcal{A}}{}_{\mathcal{B}} + \tfrac{i}{2}\,\lambda^{\mu\nu}\,(S_{\mu\nu})^{\mathcal{A}}{}_{\mathcal{B}} \tag{12}$$

for an infinitesimal transformation, where $(S_{\mu\nu})^{\mathcal{A}}{}_{\mathcal{B}}$ are the Lorentz generators in the representation with indices $\mathcal{A}$, $\mathcal{B}$. Use (10) to compute the commutators $[P_\mu, \mathcal{O}^{\mathcal{A}}(x)]$ and $[J_{\mu\nu}, \mathcal{O}^{\mathcal{A}}(x)]$.

2 Clifford algebra and Lorentz generators

2.a Use the abstract Clifford algebra anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\,\eta^{\mu\nu}\,\mathbb{I}$ to verify that the objects

$$S_{\mu\nu} = -\tfrac{i}{4}\,(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu) \tag{13}$$

satisfy the same commutation relations as the $J_{\mu\nu}$'s in (6).

2.b Verify that the gamma matrices are invariant tensors of the Lorentz algebra, i.e.

$$0 = [S_{\mu\nu}, \gamma^\rho] + (S^{\rm vec}_{\mu\nu})^{\rho}{}_{\sigma}\,\gamma^\sigma \ , \tag{14}$$

where the generators in the vector representations are defined by the relation $\lambda^{\rho}{}_{\sigma} = \frac{i}{2}\,\lambda^{\mu\nu}\,(S^{\rm vec}_{\mu\nu})^{\rho}{}_{\sigma}$.

3 The homomorphism $SL(2,\mathbb{C}) \to SO(1,3)$

Our conventions for the σ matrices are

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \;, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;, \tag{15}$$

as well as

$$\overline{\sigma}^0 = \sigma^0 \;, \qquad \overline{\sigma}^1 = -\sigma^1 \;, \qquad \overline{\sigma}^2 = -\sigma^2 \;, \qquad \overline{\sigma}^3 = -\sigma^3 \;. \tag{16}$$

Notice that all these σ matrices are Hermitian. The index structure on σ^μ is $(\sigma^\mu)_{\alpha\dot\beta}$, and the index structure on $\overline{\sigma}^\mu$ is $(\overline{\sigma}^\mu)^{\dot\alpha\beta}$. The epsilon symbols $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$, $\epsilon_{\dot\alpha\dot\beta}$, $\epsilon^{\dot\alpha\dot\beta}$ satisfy $\epsilon_{12} = -1$, $\epsilon^{12} = +1$, both for dotted and undotted indices. We also define

$$\sigma^{\mu\nu} = \tfrac{1}{4} \left(\sigma^\mu \, \overline{\sigma}^\nu - \sigma^\nu \, \overline{\sigma}^\mu \right) \;, \qquad \overline{\sigma}^{\mu\nu} = \tfrac{1}{4} \left(\overline{\sigma}^\mu \, \sigma^\nu - \overline{\sigma}^\nu \, \sigma^\mu \right) \;. \tag{17}$$

Here are some useful identities:

$$\mathrm{Tr}(\sigma^\mu \, \overline{\sigma}^\nu) = -2 \, \eta^{\mu\nu} \;, \qquad (\sigma^\mu)_{\alpha\dot\beta} \, (\overline{\sigma}_\mu)^{\dot\gamma\delta} = -2 \, \delta_\alpha^\delta \, \delta_{\dot\beta}^{\dot\gamma} \;, \qquad (\overline{\sigma}^\mu)^{\dot\alpha\beta} = \epsilon^{\dot\alpha\dot\gamma} \, \epsilon^{\beta\delta} \, (\sigma^\mu)_{\delta\dot\gamma} \;, \tag{18}$$

$$\sigma^\mu \, \overline{\sigma}^\nu + \sigma^\nu \, \overline{\sigma}^\mu = -2 \, \eta^{\mu\nu} \, \mathbb{I}_2 \;, \qquad \overline{\sigma}^\mu \, \sigma^\nu + \overline{\sigma}^\nu \, \sigma^\mu = -2 \, \eta^{\mu\nu} \, \mathbb{I}_2 \tag{19}$$

$$\mathrm{Tr}(\sigma^{\mu\nu} \, \sigma^{\rho\sigma}) = -\tfrac{1}{2} \left(\eta^{\mu\rho} \, \eta^{\nu\sigma} - \eta^{\mu\sigma} \, \eta^{\nu\rho} \right) - \tfrac{i}{2} \, \epsilon^{\mu\nu\rho\sigma} \;, \qquad \epsilon^{0123} = +1 \;. \tag{20}$$

3.a We consider the following linear map that sends vectors x^μ in Minkowski space to Hermitian 2×2 matrices,

$$x^\mu \mapsto \sigma_\mu \, x^\mu \;. \tag{21}$$

Determine the inverse of this map, i.e. solve the equation $M = \sigma_\mu x^\mu$ for x^μ.

3.b The group $SL(2,\mathbb{C})$ acts on the vector space of Hermitian 2×2 matrices via

$$M' = A \, M \, A^\dagger \qquad \text{or equiv.} \qquad M'_{\alpha\dot\beta} = A_\alpha{}^\gamma \, M_{\gamma\dot\delta} \, \overline{A}_{\dot\beta}{}^{\dot\delta} \qquad \text{where} \qquad A \in SL(2,\mathbb{C}) \;. \tag{22}$$

In our notation $\overline{A}_{\dot\alpha}{}^{\dot\beta}$ are the entries of the complex conjugate of the matrix A with entries $A_\alpha{}^\beta$. Write $M = \sigma_\mu \, x^\mu$, $M' = \sigma_\mu \, x'^\mu$ and compute x'^μ in terms of x^μ and A. Write the result in the form $x'^\mu = \Lambda(A)^\mu{}_\nu \, x^\nu$ and identify the expression for the matrix $\Lambda(A)^\mu{}_\nu$ in terms of A.

3.c Use the expression $\Lambda(A)^\mu{}_\nu$ to verify by direct computation that the following composition law holds,

$$\Lambda(A_1)^\mu{}_\nu \, \Lambda(A_2)^\nu{}_\rho = \Lambda(A_1 A_2)^\mu{}_\rho \;, \tag{23}$$

and that $\Lambda(A)^\mu{}_\nu$ is a Lorentz transformation, i.e.

$$\Lambda(A)^\mu{}_\nu \, \Lambda(A)^\rho{}_\sigma \, \eta_{\mu\rho} = \eta_{\nu\sigma} \;. \tag{24}$$

These relations show that the map $SL(2,\mathbb{C}) \ni A \mapsto \Lambda(A)^\mu{}_\nu \in SO(1,3)$ is a group homomorphism.

3.d Compute the kernel of the homomorphism, i.e. the set of matrices A for which $\Lambda(A)^\mu{}_\nu = \delta^\mu{}_\nu$.

3.e Let us consider a matrix $A \in SL(2,\mathbb{C})$ that is infinitesimally close to the identity, $A = \mathbb{I} + \delta A$. This matrix is mapped to an infinitesimal Lorentz transformation of the form $\Lambda(A)^\mu{}_\nu = \delta^\mu{}_\nu + \lambda^\mu{}_\nu$. Find the explicit expression for $\lambda^\mu{}_\nu$ in terms of δA and verify $\lambda_{\mu\nu} = \lambda_{[\mu\nu]}$. The map $\delta A \mapsto \lambda^\mu{}_\nu$ is a Lie algebra homomorphism from $\mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{so}(1,3)$.

3.f Show that the Lie algebra homomorphism $\delta A \mapsto \lambda^\mu{}_\nu$ can be inverted (and thus is actually an isomorphism $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(1,3)$) by finding the explicit expression of δA in terms of $\lambda^\mu{}_\nu$. Hint: any complex traceless 2×2 matrix Y can be written uniquely as $Y = y^{\mu\nu}\,\sigma_{\mu\nu}$ for suitable real coefficients $y^{\mu\nu} = y^{[\mu\nu]}$.

4 Spinor bilinears

All 2-component spinors we consider are anticommuting, i.e. Grassmann-odd. Our raising/lowering conventions are $\psi_\alpha = \epsilon_{\alpha\beta}\,\psi^\beta$, $\psi^\alpha = \epsilon^{\alpha\beta}\,\psi_\beta$. When a pair of contracted α indices is implicit, it is understood as ${}^\alpha{}_\alpha$. The raising/lowering conventions for dotted indices are the same, but when a pair of contracted $\dot\alpha$ indices is implicit, it is understood as ${}_{\dot\alpha}{}^{\dot\alpha}$. Complex conjugation interchanges the order of a product: for any objects a, b, we have $(ab)^* = b^*\,a^*$. The complex conjugate of spinors satisfies $(\psi_\alpha)^* = \overline{\psi}_{\dot\alpha}$. The identities recorded in problem 3 can be useful.

4.a Verify the following "flip identities" for spinor bilinears:

$$\chi\,\psi = \psi\,\chi\ , \qquad \chi\,\sigma^\mu\,\overline{\psi} = -\overline{\psi}\,\overline{\sigma}^\mu\,\chi\ , \qquad \chi\,\sigma^\mu\,\overline{\sigma}^\nu\,\psi = \psi\,\sigma^\nu\,\overline{\sigma}^\mu\,\chi\ . \tag{25}$$

4.b Verify the following reality properties of spinor bilinears:

$$(\chi\,\psi)^* = \overline{\psi}\,\overline{\chi}\ , \qquad (\chi\,\sigma^\mu\,\overline{\psi})^* = \psi\,\sigma^\mu\,\overline{\chi}\ , \qquad (\chi\,\sigma^\mu\,\overline{\sigma}^\nu\,\psi)^* = \overline{\psi}\,\overline{\sigma}^\nu\,\sigma^\mu\,\overline{\chi}\ . \tag{26}$$

4.c Verify the following "Fierz identities":

$$(\chi_1\,\chi_2)\,\chi_{3\alpha} + (\chi_1\,\chi_3)\,\chi_{2\alpha} + (\chi_2\,\chi_3)\,\chi_{1\alpha} = 0\ , \qquad (\psi\,\phi)\,\overline{\chi}_{\dot\alpha} = -\tfrac{1}{2}\,(\phi\,\sigma^\mu\,\overline{\chi})\,(\psi\,\sigma_\mu)_{\dot\alpha}\ , \tag{27}$$

$$(\theta\,\sigma^\mu\,\overline{\theta})\,(\theta\,\sigma^\nu\,\overline{\theta}) = -\tfrac{1}{2}\,\eta^{\mu\nu}\,(\theta\,\theta)\,(\overline{\theta}\,\overline{\theta})\ . \tag{28}$$