Supersymmetry and Supergravity — Problem Sheet 2

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These problems refer to material covered in Lectures 1 through 10. They are due by Saturday before the class on week 5 by 11 am. Links to submit:
TA A. Boido: https://cloud.maths.ox.ac.uk/index.php/s/WP8kazik5pNZjmi
TA J. McGovern: https://cloud.maths.ox.ac.uk/index.php/s/oBKgZcaE9F4bw3z

1 Checking some super Jacobi identities

Consider the $\mathcal{N}$-extended SUSY algebra in 4d Minkowski spacetime. Use the following (anti)commutators,

$$[J_{\mu\nu}, Q^I{}_\alpha] = -i\,(\sigma_{\mu\nu})_\alpha{}^\beta\,Q^I{}_\beta\ , \qquad [J_{\mu\nu}, \overline{Q}_I{}^{\dot\alpha}] = -i\,(\overline{\sigma}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,\overline{Q}_I{}^{\dot\beta}\ , \tag{1}$$

$$\{Q^I{}_\alpha, Q^J{}_\beta\} = \epsilon_{\alpha\beta}\,Z^{IJ}\ , \qquad \{\overline{Q}_{I\dot\alpha}, \overline{Q}_{J\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\,\overline{Z}_{IJ}\ , \tag{2}$$

to verify by direct computation the super Jacobi identities JQQ and $J\overline{QQ}$.

2 Massless supermultiplets

Recall that the SUSY algebra written for a massless particle with momentum $p^\mu = (E, 0, 0, E)$ can be cast in the form

$$\{a^I, a^\dagger_J\} = \delta^I{}_J\ , \qquad \{a^I, a^J\} = 0\ , \qquad \{a^\dagger_I, a^\dagger_J\} = 0\ , \tag{3}$$

with $I = 1, \ldots, \mathcal{N}$. We start from a Clifford vacuum $|\Omega;\lambda\rangle$ of helicity λ annihilated by the a^I's, and we act with the $a^\dagger_I$'s in all possible ways. The $a^\dagger_I$'s increase helicity by 1/2.

2.a Consider $\mathcal{N} = 2$ and perform the redefinition

$$a^{I=1} = \tfrac{1}{\sqrt2}\,(b^1 + i\,b^2)\ , \qquad a^{I=2} = \tfrac{1}{\sqrt2}\,(b^3 + i\,b^4)\ , \qquad a^\dagger{}_{I=1} = \tfrac{1}{\sqrt2}\,(b^1 - i\,b^2)\ , \qquad a^\dagger{}_{I=2} = \tfrac{1}{\sqrt2}\,(b^3 - i\,b^4)\ . \tag{4}$$

Write the algebra in terms of b^r, $r = 1, 2, 3, 4$. Which algebra is found? What are the Hermiticity properties of b^r? What is the manifest symmetry of the algebra written with b^r? Explain how to generalize these observations to arbitrary $\mathcal{N} \geq 2$.

2.b Consider a multiplet of massless $\mathcal{N}$-exteded SUSY that is obtained starting from a Clifford vacuum of helicity λ. (This might not be CPT symmetric on its own, bet let us ignore this now.) What is the maximal helicity in the multiplet? Give an explicit formula for the number of states of helicity $\lambda + \frac{1}{2}p$ as a function of p and $\mathcal{N}$. Use the formula to verify that we always have an equal number of bosonic and fermionic degrees of freedom in the multiplet.

2.c Let us consider $\mathcal{N} = 3$. Find the allowed values for the helicity λ of the Clifford vacuum, such that the helicities of all particles in the multiplet are in the range $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. For the allowed helicities of the Clifford vacuum, enumerate the states in the multiplet and their degeneracies and describe how they transform under the $SU(3)$ R-symmetry. Compare your findings to the content of the massless vector multiplet of 4d $\mathcal{N}=4$ supersymmetry.

2.d Let us consider $\mathcal{N}=8$. Argue that there is only one massless multiplet that has helicities in the range $\{-2,-\frac{3}{2},-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2},2\}$. Enumerate its states and their degeneracies and describe how they transform under the $SU(8)$ R-symmetry. This multiplet is the gravity multiplet of maximal supergravity in 4d.

2.e Let us consider $\mathcal{N}=7$. Find the allowed values for the helicity λ of the Clifford vacuum, such that the helicities of all particles in the multiplet are in the range $\{-2,-\frac{3}{2},-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2},2\}$. For the allowed helicities of the Clifford vacuum, enumerate the states in the multiplet and their degeneracies and describe how they transform under the $SU(7)$ R-symmetry. Compare your findings with the gravity multiplet of 4d $\mathcal{N}=8$.

3 A closer look at $\mathcal{N}=2$ hypermultiplets

If we consider $\mathcal{N}=2$ and start from a Clifford vacuum of helicity $\lambda=-\frac{1}{2}$, we get the states

$$\begin{array}{ll} \lambda=-\frac{1}{2} & |\Omega\rangle \\ \lambda=0 & a^\dagger|\Omega\rangle \\ \lambda=\frac{1}{2} & a^\dagger a^\dagger|\Omega\rangle \end{array} \tag{5}$$

Even though the above multiplet has a symmetric pattern of helicities, it cannot be CPT self-conjugate. Let's see why.

3.a The helicity-0 states $a_I^\dagger|\Omega\rangle$ are associated to scalar fields q_I that carry an index I. This is an index of $SU(2)_R$. The transformation of q_I is $q_I'=U_I{}^J\,h_J$, $U_I{}^J\in SU(2)_R$. Consider the epsilon symbol ϵ_{IJ}, ϵ^{IJ} with the convention $\epsilon_{12}=1$, $\epsilon^{12}=1$. Define the object

$$(q^c)_I=\epsilon_{IJ}\,(q_J)^*\ . \tag{6}$$

Show that it has the same transformation law as q_I. Since q_I and $(q^c)_I$ transform in the same way, a reality condition of the form $(q^c)_I=q_I$ is covariant under $SU(2)_R$. It is not a consistent reality conditon, however: prove that $((q^c)^c)_I=-q_I$ This is why a single copy of (5) cannot be CPT self-conjugate. (In maths language, we say that the fundamental representation of $SU(2)$ is pseudo-real, as opposed to strictly real.)

Since (5) cannot be self-conjugate, the helicity $-\frac{1}{2}$ state is associated to a Weyl spinor $\overline{\widetilde{\psi}}_{\dot\alpha}$ and the helicity $+\frac{1}{2}$ state is associated to a Weyl spinor ψ_α, where $\widetilde{\psi}_\alpha\neq\psi_\alpha$. In total, we find two complex scalars q_I and two independent Weyl fermions ψ_α, $\widetilde{\psi}_\alpha$. The correspondence between 1-particle states and fields can be summarized as:

$$\begin{array}{lllllll} \overline{\widetilde{\psi}}_{\dot\alpha} & : & \lambda=-\frac{1}{2} & |\Omega\rangle & \overline{\psi}_{\dot\alpha} & : & \lambda=-\frac{1}{2} \quad |\Omega'\rangle \\ q_I & : & \lambda=0 & a^\dagger|\Omega\rangle & (q^c)_I & : & \lambda=0 \quad a^\dagger|\Omega'\rangle \\ \psi_\alpha & : & \lambda=\frac{1}{2} & a^\dagger a^\dagger|\Omega\rangle & \widetilde{\psi}_\alpha & : & \lambda=\frac{1}{2} \quad a^\dagger a^\dagger|\Omega'\rangle \end{array} \tag{7}$$

The multiplet $(q_I, \psi_\alpha, \widetilde{\psi}_\alpha)$ is sometimes referred to as a full hypermultiplet.

3.b Assign $U(1)_R$ charge $+1$ to $a_I^\dagger$ and charge $R[q_I]$ to q_I. Determine the $U(1)_R$ charges of the Clifford vacua $|\Omega\rangle$ and $|\Omega'\rangle$ and all the fields in (7).

3.c Suppose q_I transforms in a representation $\mathbf{R}$ of a flavor or gauge group G (which commutes with the supercharges). In which representation of G do the other fields in (7) transform?

3.d Recall that $USp(M)$ with M even is the group of matrices $V_i{}^j \in U(M)$ that satisfy $V_i{}^j\, V_k{}^\ell\, \omega^{ik} = \omega^{j\ell}$ where $\omega^{ij} := \left(\begin{smallmatrix} 0 & \mathbb{I}_{M/2} \\ -\mathbb{I}_{M/2} & 0 \end{smallmatrix} \right) =: \omega_{ij}$. Suppose q_I transforms in the fundamental of a $USp(M)$ flavor or gauge group, i.e. $q'_{Ii} = V_i{}^j\, q_{Ij}$ under the action of $V_i{}^j \in USp(M)$. Define

$$(q^c)_{Ii} = \epsilon_{IJ}\, \omega_{ij}\, (q_{Jj})^* \ . \tag{8}$$

Show that $(q^c)_{Ii}$ transforms in the same way as q_{Ii} under $SU(2)_R \times USp(M)$. Prove that $((q^c)^c)_{Ii} = q_{Ii}$. The reality condition $(q^c)_{Ii} = q_{Ii}$ is therefore consistent. (A hypermultiplet that transforms in a pseudo-real representation of a flavor or gauge group and that satisfies a reality condition of the form (8) is called a half-hypermultiplet.)

3.e Write $q_{I=1} = Q$ and $q_{I=2} = \widetilde{Q}^*$, where Q and $\widetilde{Q}$ are complex scalars. Consider the $\mathcal{N} = 1$ SUSY subalgebra of the full $\mathcal{N} = 2$ algebra generated by the supercharges with $I = 2$. Re-organize the states and fields in (7) in multiplets of this $\mathcal{N} = 1$ subalgebra. If q_I transforms in a representation $\mathbf{R}$ of a flavor or gauge group, in which representation do the $\mathcal{N} = 1$ multiplets transform?

A structure similar to (5) is found when we study the massless vector multiplet of 4d $\mathcal{N} = 4$ and the gravity multiplet of 4d $\mathcal{N} = 8$. In those cases, however, the multiplet is indeed equal to its CPT conjugate. Why? The answer has to do with a fact about representations of $SU(2M)$.

3.f Let us consider $SU(2M)$ and the representation $\wedge^M(\text{fund})$, i.e. the antisymmetrized tensor product of M copies of the fundamental representation. In terms of indices $I = 1, \dots, 2M$, an object in this representation has structure $q_{I_1 \dots I_M} = q_{[I_1 \dots I_M]}$. Define the operation

$$(q^c)_{I_1 \dots I_M} = \frac{1}{M!} \epsilon_{I_1 \dots I_M J_1 \dots J_M}\, (q_{J_1 \dots J_M})^* \ . \tag{9}$$

Prove that q^c transforms under $SU(2M)$ in the same was as q. Compute $(q^c)^c$ for generic M. Use your result to justify why the $\mathcal{N} = 4$ vector multiplet and the $\mathcal{N} = 8$ gravity multiplet can be their own CPT conjugates.

Hint: Our conventions for the epsilon tensors $\epsilon_{I_1 \dots I_{2M}}$, $\epsilon^{I_1 \dots I_{2M}}$ are $\epsilon_{1 \dots 2M} = +1 = \epsilon^{1 \dots 2M}$. The following identity holds,

$$\epsilon_{I_1 \dots I_p J_1 \dots J_q}\, \epsilon^{I_1 \dots I_p K_1 \dots K_q} = p!\, q!\, \delta^{[K_1}_{[J_1} \dots \delta^{K_q]}_{J_q]} \ , \qquad p + q = 2M \ . \tag{10}$$

4 SUSY invariant actions

In Lecture 8 we have considered the kinetic action

$$S_{\text{kin}} = \int d^4x \left[-\partial^\mu \overline{X}\, \partial_\mu X + i\, \partial_\mu \overline{\psi}\, \overline{\sigma}^\mu\, \psi + \overline{F}\, F \right] \tag{11}$$

and its variation under

$$\delta X = \sqrt{2}\, \xi\, \psi \ , \qquad \delta \overline{X} = \sqrt{2}\, \overline{\xi}\, \overline{\psi} \ , \tag{12}$$

$$\delta \psi_\alpha = i\, \sqrt{2}\, (\sigma^\mu\, \overline{\xi})_\alpha\, \partial_\mu X + \sqrt{2}\, F\, \xi_\alpha \ , \qquad \delta \overline{\psi}_{\dot\alpha} = -i\, \sqrt{2}\, (\xi\, \sigma^\mu)_{\dot\alpha}\, \partial_\mu \overline{X} + \sqrt{2}\, \overline{F}\, \overline{\xi}_{\dot\alpha} \ , \tag{13}$$

$$\delta F = i\, \sqrt{2}\, \overline{\xi}\, \overline{\sigma}^\mu\, \partial_\mu \psi \ , \qquad \delta \overline{F} = -i\, \sqrt{2}\, \partial_\mu \overline{\psi}\, \overline{\sigma}^\mu\, \xi \ . \tag{14}$$

In the computation, we have treated the SUSY parameters as if they were arbitrary functions of spacetime. We have neglected total derivatives in spacetime. We have verified that

$$\delta S_{\text{kin}} = \int d^4x\, (\partial_\mu \xi^\alpha\, J^\mu{}_\alpha + \partial_\mu \overline{\xi}_{\dot\alpha}\, \overline{J}^{\mu\dot\alpha}) \ , \quad J^\mu_\alpha = \sqrt{2}\, (\sigma^\nu\, \overline{\sigma}^\mu\, \psi)_\alpha\, \partial_\nu \overline{X} \ , \quad \overline{J}^{\mu\dot\alpha} = \sqrt{2}\, (\overline{\sigma}^\nu\, \sigma^\mu\, \overline{\psi})^{\dot\alpha}\, \partial_\nu X \ . \tag{15}$$

The quantities J^μ_α, $\overline{J}^{\mu\dot\alpha}$ are the components of the SUSY current.

Repeat the same steps for the actions

$$S_m = \int d^4x \left[m\, X\, F - \frac{1}{2}\, m\, \psi\, \psi \right] + \text{h.c.} \ , \qquad S_g = \int d^4x \left[g\, X^2\, F - g\, X\, \psi\, \psi \right] + \text{h.c.} \ , \tag{16}$$

where m, g are complex constants. Verify that δS_m, δS_g have the same form as δS_{kin} and read off the contributions of δS_m, δS_g to the SUSY current J^μ_α. You should get

$$J^\mu_\alpha = -i\, \sqrt{2}\, \frac{\partial \overline{W}}{\partial \overline{X}}\, (\sigma^\mu\, \overline{\psi})_\alpha \ , \qquad \overline{J}^{\mu\dot\alpha} = -i\, \sqrt{2}\, \frac{\partial W}{\partial X}\, (\overline{\sigma}^\mu\, \psi)^{\dot\alpha} \ , \tag{17}$$

where

$$W(X) = \frac{1}{2}\, m\, X^2 + \frac{1}{3}\, g\, X^2 \ , \qquad \overline{W}(\overline{X}) = \frac{1}{2}\, \overline{m}\, \overline{X}^2 + \frac{1}{3}\, \overline{g}\, \overline{X}^3 \ . \tag{18}$$

Hint: $(\psi\, \psi)\, \psi_\alpha \equiv 0$.