Supersymmetry and supergravity

Lecture 21

SUSY and quantum corrections

- So far we have discussed the construction of classical Lagrangians that are invariant under 4d $\mathcal{N} = 1$ supersymmetry
- Let us now discuss some aspects of the quantum corrections in these models
- We begin with the simplest Wess-Zumino model

$$S = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, \Phi\, \overline{\Phi} + \left(\int d^4x\, d^2\theta\, W(\Phi) + \mathrm{h.c.} \right)$$

$$W = \frac{1}{2} m_0\, \Phi^2 + \frac{1}{3} g_0\, \Phi^3$$

- The 0 subscripts on the mass and coupling are a reminder that these are the bare mass and coupling that enter the bare Lagrangian

WZ model

$$S = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, \Phi\, \overline{\Phi} + \left(\int d^4x\, d^2\theta\, W(\Phi) + \mathrm{h.c.} \right), \quad W = \frac{1}{2}\, m_0\, \Phi^2 + \frac{1}{3}\, g_0\, \Phi^3$$

- In terms of component fields, the Lagrangian reads

$$\begin{aligned}\mathscr{L} = &- \partial^\mu \overline{X}\, \partial_\mu X + i\, \partial_\mu \overline{\psi}\, \overline{\sigma}^\mu\, \psi + \overline{F}\, F \\ &+ m_0\, X\, F + \overline{m}_0\, \overline{X}\, \overline{F} - \frac{1}{2}\, m_0\, \psi\, \psi - \frac{1}{2}\, \overline{m}_0\, \overline{\psi}\, \overline{\psi} \\ &+ g_0\, X^2\, F + \overline{g}_0\, \overline{X}^2\, \overline{F} - g_0\, X\, \psi\, \psi - \overline{g}_0\, \overline{X}\, \overline{\psi}\, \overline{\psi}\end{aligned}$$

- After eliminating the auxiliary fields we find

$$\begin{aligned}\mathscr{L} = &- \partial^\mu \overline{X}\, \partial_\mu X + i\, \partial_\mu \overline{\psi}\, \overline{\sigma}^\mu\, \psi - |\, m_0\,|^2\, \overline{X}\, X - \frac{1}{2}\, m_0\, \psi\, \psi - \frac{1}{2}\, \overline{m}_0\, \overline{\psi}\, \overline{\psi} \\ &- g_0\, X\, \psi\, \psi - \overline{g}_0\, \overline{X}\, \overline{\psi}\, \overline{\psi} - m_0\, \overline{g}_0\, X\, \overline{X}^2 - \overline{m}_0\, g_0\, \overline{X}\, X^2 - |\, g_0\,|^2\, \overline{X}^2\, X^2\end{aligned}$$

WZ model

$$\mathscr{L} = -\,\partial^\mu \overline{X}\,\partial_\mu X + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi - |\,m_0\,|^2\,\overline{X}\,X - \frac{1}{2}\,m_0\,\psi\,\psi - \frac{1}{2}\,\overline{m}_0\,\overline{\psi}\,\overline{\psi}$$
$$-\,g_0\,X\,\psi\,\psi - \overline{g}_0\,\overline{X}\,\overline{\psi}\,\overline{\psi} - m_0\,\overline{g}_0\,X\,\overline{X}^2 - \overline{m}_0\,g_0\,\overline{X}\,X^2 - |\,g_0\,|^2\,\overline{X}^2\,X^2$$

- This is a renormalizable QFT with Yukawa interactions and quartic scalar potential
- All divergences that we encounter in perturbation theory can be reabsorbed by a finite number of counterterms
- Let us analyze this model with methods that we learn in QFT, ignoring SUSY for the time being

WZ model

$$\mathscr{L} = -\,\partial^\mu \overline{X}\,\partial_\mu X + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi - |\,m_0|^2\,\overline{X}\,X - \frac{1}{2}\,m_0\,\psi\,\psi - \frac{1}{2}\,\overline{m}_0\,\overline{\psi}\,\overline{\psi}$$
$$-\,g_0\,X\,\psi\,\psi - \overline{g}_0\,\overline{X}\,\overline{\psi}\,\overline{\psi} - m_0\,\overline{g}_0\,X\,\overline{X}^2 - \overline{m}_0\,g_0\,\overline{X}\,X^2 - |\,g_0|^2\,\overline{X}^2\,X^2$$

- By a constant phase rotation of X and ψ we can set m_0 to be real and non-negative
- Let us parametrize g_0 as $g_0 = \sqrt{2}\,y_0\,e^{i\alpha}$ where y_0 is real. We can define

$$X = \frac{1}{\sqrt{2}}\,e^{-i\alpha}\,(A + i\,B)$$

where A, B are real scalar fields. The Lagrangian reads

$$\mathscr{L} = -\frac{1}{2}\,\partial^\mu A\,\partial_\mu A - \frac{1}{2}\,\partial^\mu B\,\partial_\mu B + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi - \frac{1}{2}\,m_0^2\,(A^2 + B^2) - \frac{1}{2}\,m_0\,\psi\,\psi - \frac{1}{2}\,m_0\,\overline{\psi}\,\overline{\psi}$$
$$-\,y_0\,A\,(\psi\,\psi + \overline{\psi}\,\overline{\psi}) - i\,y_0\,B\,(\psi\,\psi - \overline{\psi}\,\overline{\psi})$$
$$-\,m_0\,y_0\,A\,(A^2 + B^2) - \frac{1}{2}\,y_0^2\,(A^2 + B^2)^2$$

WZ model

$$\mathscr{L} = -\frac{1}{2}\,\partial^\mu A\,\partial_\mu A - \frac{1}{2}\,\partial^\mu B\,\partial_\mu B + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi - \frac{1}{2}\,m_0^2\,(A^2 + B^2) - \frac{1}{2}\,m_0\,\psi\,\psi - \frac{1}{2}\,m_0\,\overline{\psi}\,\overline{\psi}$$
$$- y_0\,A\,(\psi\,\psi + \overline{\psi}\,\overline{\psi}) - i\,y_0\,B\,(\psi\,\psi - \overline{\psi}\,\overline{\psi})$$
$$- m_0\,y_0\,A\,(A^2 + B^2) - \frac{1}{2}\,y_0^2\,(A^2 + B^2)^2$$

- In this language it is easier to verify that the WZ model admits parity as a symmetry. Parity acts on spacetime coordinates as $\mathscr{P} : (x^0, x^i) \to (x^0, -x^i)$ and is implemented by a unitary operator P with

$$\mathsf{P}^{-1}\,A(x)\,\mathsf{P} = A(\mathscr{P}x) \quad , \qquad \mathsf{P}^{-1}\,B(x)\,\mathsf{P} = -\,B(\mathscr{P}x)$$
$$\mathsf{P}^{-1}\,\psi_\alpha(x)\,\mathsf{P} = -\,i\,(\sigma^0)_{\alpha\dot\beta}\,\overline{\psi}^{\dot\beta}(\mathscr{P}x) \qquad\qquad (\sigma^0 = -\,\mathbb{I}_2)$$

- We can regard parity as an accidental symmetry of the WZ model with a single chiral superfield (for more details: Weinberg vol III, pages 82, 83)

Renormalization procedure

$$\mathscr{L} = -\frac{1}{2}\,\partial^\mu A\,\partial_\mu A - \frac{1}{2}\,\partial^\mu B\,\partial_\mu B + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi - \frac{1}{2}\,m_0^2\,(A^2 + B^2) - \frac{1}{2}\,m_0\,\psi\,\psi - \frac{1}{2}\,m_0\,\overline{\psi}\,\overline{\psi}$$
$$- y_0\,A\,(\psi\,\psi + \overline{\psi}\,\overline{\psi}) - i\,y_0\,B\,(\psi\,\psi - \overline{\psi}\,\overline{\psi})$$
$$- m_0\,y_0\,A\,(A^2 + B^2) - \frac{1}{2}\,y_0^2\,(A^2 + B^2)^2$$

- The original Lagrangian is written in terms of bare fields and bare masses/couplings
- It is convenient to rewrite it in terms of renormalized fields and physical masses/couplings
- We split $\mathscr{L}$ as $\mathscr{L} = \mathscr{L}_{\mathrm{r}} + \mathscr{L}_{\mathrm{ct}}$ into the Lagrangian written in terms of renormalized quantities, plus counterterms
- The counterterms can be adjusted order-by-order in perturbation theory to absorb all UV divergences from loop integrals

Renormalization procedure

- In the free theory ($g = 0$) the 2-pt function of A has a simple pole at $p^2 = -m_0^2$ with residue one. The VEV of A is zero.
- In the interacting theory:
  - the location of the pole is no longer the bare mass m_0^2 that enters the Lagrangian, but at the physical mass m_A^2 of particles of the A field
  - the residue at the pole is generically different from one
  - the VEV of the scalar A might get shifted
- We shift and rescale the bare field A by a positive constant Z_A and a constant v: $\quad A = Z_A^{1/2} A_\mathrm{r} + v$
- The renormalized field A_r has zero VEV and is such that its 2-pt function has unit residue at the physical mass m_A^2
- Similar remarks apply to B and the fermion: $\quad B = Z_B^{1/2} B_\mathrm{r} \;, \qquad \psi = Z_\psi^{1/2} \psi_\mathrm{r} \;, \qquad \overline{\psi} = Z_\psi^{1/2} \overline{\psi}_\mathrm{r}$
- NB: Lorentz invariance of the vacuum forbids a VEV for the fermion; parity forbids a VEV for B
- These Z factors are usually referred to as **wavefunction renormalization**

Renormalization procedure

$$\mathscr{L} = -\frac{1}{2}\,\partial^\mu A\,\partial_\mu A - \frac{1}{2}\,\partial^\mu B\,\partial_\mu B + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi - \frac{1}{2}\,m_0^2\,(A^2+B^2) - \frac{1}{2}\,m_0\,\psi\,\psi - \frac{1}{2}\,m_0\,\overline{\psi}\,\overline{\psi}$$
$$-\,y_0\,A\,(\psi\,\psi + \overline{\psi}\,\overline{\psi}) - i\,y_0\,B\,(\psi\,\psi - \overline{\psi}\,\overline{\psi})$$
$$-\,m_0\,y_0\,A\,(A^2+B^2) - \frac{1}{2}\,y_0^2\,(A^2+B^2)^2$$

We perform the replacements $A = Z_A^{1/2}\,A_\mathrm{r} + v, \qquad B = Z_B^{1/2}\,B_\mathrm{r}\ , \qquad \psi = Z_\psi^{1/2}\,\psi_\mathrm{r}\ , \qquad \overline{\psi} = Z_\psi^{1/2}\,\overline{\psi}_\mathrm{r}$

We get an ugly expression that can be parametrized as

$$\mathscr{L} = -\frac{1}{2}\,Z_A\,\partial^\mu \overline{A}_\mathrm{r}\,\partial_\mu A_\mathrm{r} - \frac{1}{2}\,Z_B\,\partial^\mu \overline{B}_\mathrm{r}\,\partial_\mu B_\mathrm{r} + i\,Z_\psi\,\partial_\mu \overline{\psi}_\mathrm{r}\,\overline{\sigma}^\mu\,\psi_\mathrm{r}$$
$$-\,Z_{m_A^2}\,m_A^2\,A_\mathrm{r}^2 - Z_{m_B^2}\,m_B^2\,B_\mathrm{r}^2 - \frac{1}{2}\,Z_{m_\psi}\,m_\psi\,(\psi_\mathrm{r}\,\psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\,\overline{\psi}_\mathrm{r})$$
$$-\,Z_{y_A}\,y_A\,A_\mathrm{r}\,(\psi_\mathrm{r}\,\psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\,\overline{\psi}_\mathrm{r}) - i\,Z_{y_B}\,y_B\,B_\mathrm{r}\,(\psi_\mathrm{r}\,\psi_\mathrm{r} - \overline{\psi}_\mathrm{r}\,\overline{\psi}_\mathrm{r})$$
$$-\,Z_{\lambda_{AAA}}\,\lambda_{AAA}\,A_\mathrm{r}^3 - Z_{\lambda_{ABB}}\,\lambda_{ABB}\,A_\mathrm{r}\,B_\mathrm{r}^2 - Y\,A_\mathrm{r}$$
$$-\,Z_{\lambda_{AAAA}}\,\lambda_{AAAA}A_\mathrm{r}^4 - Z_{\lambda_{BBBB}}\,\lambda_{BBBB}\,B_\mathrm{r}^4 - Z_{\lambda_{AABB}}\,\lambda_{AABB}\,A_\mathrm{r}^2\,B_\mathrm{r}^2 - V_0$$

For example

$$Z_{m_A^2}\,m_A^2 \equiv Z_A\,m_0^2 + 6\,m_0\,v\,y_0\,Z_A + 6\,v^2\,y_0^2\,Z_A\ , \qquad Z_{y_A}\,y_A \equiv Z_A^{1/2}\,Z_\psi\,y_0\ , \qquad V_0 \equiv \frac{1}{2}\,v^2\,(m_0 + v\,y_0)^2\ , \quad \text{etc}$$

Renormalization procedure

$$\begin{aligned}
\mathscr{L} = &-\frac{1}{2} Z_A\, \partial^\mu \overline{A}_\mathrm{r}\, \partial_\mu A_\mathrm{r} - \frac{1}{2} Z_B\, \partial^\mu \overline{B}_\mathrm{r}\, \partial_\mu B_\mathrm{r} + i\, Z_\psi\, \partial_\mu \overline{\psi}_\mathrm{r}\, \overline{\sigma}^\mu\, \psi_\mathrm{r} \\
&- Z_{m_A^2}\, m_A^2\, A_\mathrm{r}^2 - Z_{m_B^2}\, m_B^2\, B_\mathrm{r}^2 - \frac{1}{2} Z_{m_\psi}\, m_\psi\, (\psi_\mathrm{r}\,\psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\,\overline{\psi}_\mathrm{r}) \\
&- Z_{y_A}\, y_A\, A_\mathrm{r}\, (\psi_\mathrm{r}\,\psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\,\overline{\psi}_\mathrm{r}) - i\, Z_{y_B}\, y_B\, B_\mathrm{r}\, (\psi_\mathrm{r}\,\psi_\mathrm{r} - \overline{\psi}_\mathrm{r}\,\overline{\psi}_\mathrm{r}) \\
&- Z_{\lambda_{AAA}}\, \lambda_{AAA}\, A_\mathrm{r}^3 - Z_{\lambda_{ABB}}\, \lambda_{ABB}\, A_\mathrm{r}\, B_\mathrm{r}^2 - Y A_\mathrm{r} \\
&- Z_{\lambda_{AAAA}}\, \lambda_{AAAA} A_\mathrm{r}^4 - Z_{\lambda_{BBBB}}\, \lambda_{BBBB}\, B_\mathrm{r}^4 - Z_{\lambda_{AABB}}\, \lambda_{AABB}\, A_\mathrm{r}^2\, B_\mathrm{r}^2 - V_0
\end{aligned}$$

- We have generated all terms that are compatible with Lorentz symmetry, parity, renormalizability, with arbitrary coefficients
- Notice how the shift v in the scalar field A has generated a term linear in A_r (tadpole) and the constant term V_0

Renormalization procedure

$$
\begin{aligned}
\mathscr{L} = &-\frac{1}{2} Z_A\, \partial^\mu \overline{A}_\mathrm{r}\, \partial_\mu A_\mathrm{r} - \frac{1}{2} Z_B\, \partial^\mu \overline{B}_\mathrm{r}\, \partial_\mu B_\mathrm{r} + i\, Z_\psi\, \partial_\mu \overline{\psi}_\mathrm{r}\, \overline{\sigma}^\mu\, \psi_\mathrm{r} \\
&- Z_{m_A^2}\, m_A^2\, A_\mathrm{r}^2 - Z_{m_B^2}\, m_B^2\, B_\mathrm{r}^2 - \frac{1}{2} Z_{m_\psi}\, m_\psi\, (\psi_\mathrm{r}\, \psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\, \overline{\psi}_\mathrm{r}) \\
&- Z_{y_A}\, y_A\, A_\mathrm{r}\, (\psi_\mathrm{r}\, \psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\, \overline{\psi}_\mathrm{r}) - i\, Z_{y_B}\, y_B\, B_\mathrm{r}\, (\psi_\mathrm{r}\, \psi_\mathrm{r} - \overline{\psi}_\mathrm{r}\, \overline{\psi}_\mathrm{r}) \\
&- Z_{\lambda_{AAA}}\, \lambda_{AAA}\, A_\mathrm{r}^3 - Z_{\lambda_{ABB}}\, \lambda_{ABB}\, A_\mathrm{r}\, B_\mathrm{r}^2 - Y\, A_\mathrm{r} \\
&- Z_{\lambda_{AAAA}}\, \lambda_{AAAA} A_\mathrm{r}^4 - Z_{\lambda_{BBBB}}\, \lambda_{BBBB}\, B_\mathrm{r}^4 - Z_{\lambda_{AABB}}\, \lambda_{AABB}\, A_\mathrm{r}^2\, B_\mathrm{r}^2 - V_0
\end{aligned}
$$

- The 10 parameters $m_A^2, m_B^2, m_\psi, y_A, y_B$, $\lambda_{AAA}, \lambda_{ABB}, \lambda_{AAAA},\ \lambda_{BBBB}, \lambda_{AABB}$ encode the physical/measurable masses and couplings of the particles in the model
- They are defined by suitable renormalization condition. For instance, we can define m_A^2 as the location of the pole in the exact A_r 2-pt function; y_A can be defined as the value of the 3-point amplitude $A_\mathrm{r}\, \psi_\mathrm{r}\, \psi_\mathrm{r}$ at the limit where external momenta go to zero, etc…

Renormalization procedure

$$\begin{aligned}\mathscr{L} = &-\frac{1}{2}\,Z_A\,\partial^\mu \overline{A}_\mathrm{r}\,\partial_\mu A_\mathrm{r} - \frac{1}{2}\,Z_B\,\partial^\mu \overline{B}_\mathrm{r}\,\partial_\mu B_\mathrm{r} + i\,Z_\psi\,\partial_\mu \overline{\psi}_\mathrm{r}\,\overline{\sigma}^\mu\,\psi_\mathrm{r} \\ &- Z_{m_A^2}\,m_A^2\,A_\mathrm{r}^2 - Z_{m_B^2}\,m_B^2\,B_\mathrm{r}^2 - \frac{1}{2}\,Z_{m_\psi}\,m_\psi\,(\psi_\mathrm{r}\,\psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\,\overline{\psi}_\mathrm{r}) \\ &- Z_{y_A}\,y_A\,A_\mathrm{r}\,(\psi_\mathrm{r}\,\psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\,\overline{\psi}_\mathrm{r}) - i\,Z_{y_B}\,y_B\,B_\mathrm{r}\,(\psi_\mathrm{r}\,\psi_\mathrm{r} - \overline{\psi}_\mathrm{r}\,\overline{\psi}_\mathrm{r}) \\ &- Z_{\lambda_{AAA}}\,\lambda_{AAA}\,A_\mathrm{r}^3 - Z_{\lambda_{ABB}}\,\lambda_{ABB}\,A_\mathrm{r}\,B_\mathrm{r}^2 - Y\,A_\mathrm{r} \\ &- Z_{\lambda_{AAAA}}\,\lambda_{AAAA}A_\mathrm{r}^4 - Z_{\lambda_{BBBB}}\,\lambda_{BBBB}\,B_\mathrm{r}^4 - Z_{\lambda_{AABB}}\,\lambda_{AABB}\,A_\mathrm{r}^2\,B_\mathrm{r}^2 - V_0\end{aligned}$$

- At zeroth order in perturbation theory the physical couplings are given by

zeroth order: $m_A^2 = m_B^2 = m_\psi^2 = m_0^2\;, \qquad y_A = y_B = y_0$

$$\lambda_{AAA} = \lambda_{ABB} = m_0\,y_0\;, \qquad 2\,\lambda_{AAAA} = 2\,\lambda_{BBBB} = \lambda_{AABB} = y_0^2$$

Renormalization procedure

$$\begin{aligned}\mathscr{L} = &-\frac{1}{2} Z_A\, \partial^\mu \overline{A}_\mathrm{r}\, \partial_\mu A_\mathrm{r} - \frac{1}{2} Z_B\, \partial^\mu \overline{B}_\mathrm{r}\, \partial_\mu B_\mathrm{r} + i\, Z_\psi\, \partial_\mu \overline{\psi}_\mathrm{r}\, \overline{\sigma}^\mu\, \psi_\mathrm{r} \\ &- Z_{m_A^2}\, m_A^2\, A_\mathrm{r}^2 - Z_{m_B^2}\, m_B^2\, B_\mathrm{r}^2 - \frac{1}{2} Z_{m_\psi}\, m_\psi\, (\psi_\mathrm{r}\, \psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\, \overline{\psi}_\mathrm{r}) \\ &- Z_{y_A}\, y_A\, A_\mathrm{r}\, (\psi_\mathrm{r}\, \psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\, \overline{\psi}_\mathrm{r}) - i\, Z_{y_B}\, y_B\, B_\mathrm{r}\, (\psi_\mathrm{r}\, \psi_\mathrm{r} - \overline{\psi}_\mathrm{r}\, \overline{\psi}_\mathrm{r}) \\ &- Z_{\lambda_{AAA}}\, \lambda_{AAA}\, A_\mathrm{r}^3 - Z_{\lambda_{ABB}}\, \lambda_{ABB}\, A_\mathrm{r}\, B_\mathrm{r}^2 - Y A_\mathrm{r} \\ &- Z_{\lambda_{AAAA}}\, \lambda_{AAAA} A_\mathrm{r}^4 - Z_{\lambda_{BBBB}}\, \lambda_{BBBB}\, B_\mathrm{r}^4 - Z_{\lambda_{AABB}}\, \lambda_{AABB}\, A_\mathrm{r}^2\, B_\mathrm{r}^2 - V_0\end{aligned}$$

- The factors $Z_A, \ldots, Z_{\lambda_{AABB}}$ and Y encode the counterterms. Let us write Z_i to denote the Z factors collectively.
- At tree-level we have $Z_i = 1$, $Y = 0$. We write

$$Z_i = 1 + \delta Z_i \quad , \qquad Y = 0 + \delta Y$$

Renormalization procedure

$$Z_i = 1 + \delta Z_i \quad , \qquad Y = 0 + \delta Y$$

- The original Lagrangian splits as the sum of the Lagrangian written in terms of renormalized fields and physical masses/couplings, plus counterterms and a constant: $\mathscr{L} = \mathscr{L}_{\rm r} + \mathscr{L}_{\rm ct} - V_0$

$$\begin{aligned}
\mathscr{L}_{\rm r} = & -\tfrac{1}{2}\,\partial^\mu \overline{A}_{\rm r}\,\partial_\mu A_{\rm r} - \tfrac{1}{2}\,\partial^\mu \overline{B}_{\rm r}\,\partial_\mu B_{\rm r} + i\,\partial_\mu \overline{\psi}_{\rm r}\,\overline{\sigma}^\mu\,\psi_{\rm r} \\
& - m_A^2\,A_{\rm r}^2 - m_B^2\,B_{\rm r}^2 - \tfrac{1}{2}\,m_\psi\,(\psi_{\rm r}\psi_{\rm r} + \overline{\psi}_{\rm r}\overline{\psi}_{\rm r}) \\
& - y_A\,A_{\rm r}\,(\psi_{\rm r}\psi_{\rm r} + \overline{\psi}_{\rm r}\overline{\psi}_{\rm r}) - i\,y_B\,B_{\rm r}\,(\psi_{\rm r}\psi_{\rm r} - \overline{\psi}_{\rm r}\overline{\psi}_{\rm r}) \\
& - \lambda_{AAA}\,A_{\rm r}^3 - \lambda_{ABB}\,A_{\rm r}\,B_{\rm r}^2 \\
& - \lambda_{AAAA} A_{\rm r}^4 - \lambda_{BBBB}\,B_{\rm r}^4 - \lambda_{AABB}\,A_{\rm r}^2\,B_{\rm r}^2 \quad ; \qquad\qquad \mathscr{L}_{\rm ct} = \text{all terms with } \delta Z_i, \delta Y
\end{aligned}$$

- The counterterms are adjusted order-by-order in perturbation theory to preserve the renormalization conditions (δY is adjusted to cancel tadpoles of $A_{\rm r}$)
- In the process, all UV divergences from loop integrals are cancelled; physical observables are finite when expressed in terms of the physical masses/couplings

What about SUSY?

Some natural questions:

- Can the model be regularized and renormalized preserving SUSY?
- How does SUSY constrain the wavefunction renormalizations and the shift v in the scalar A? How does it constrain the Z factors for masses and couplings?
- Since the bare Lagrangian only had two parameters m_0, y_0, we have some relations among the renormalized couplings at zeroth order in perturbation theory. For instance

zeroth order: $$m_A^2 = m_B^2 = m_\psi^2 \quad , \quad y_A = y_B \quad ,$$

$$\lambda_{AAA} = \lambda_{ABB} = m_\psi \, y_A \, , \qquad 2\,\lambda_{AAAA} = 2\,\lambda_{BBBB} = \lambda_{AABB} = y_A^2$$

Is there a renormalization scheme in which these relations preserved beyond zeroth order in perturbation theory?

What about SUSY?

- Addressing these questions in the model without auxiliary fields is possible but a bit cumbersome. Recall that after integrating out the auxiliary fields,
  - the SUSY algebra only closes up to the EOMs
  - the SUSY variations are non-linear in the fields, because the on-shell value $F(\overline{X})$ (that enters $\delta\psi$) is a non-linear function of $\overline{X}$
- For these reasons, it is best to go back to the model before integrating out the auxiliary fields

$$
\begin{aligned}
\mathscr{L} = &- \partial^\mu \overline{X}\, \partial_\mu X + i\, \partial_\mu \overline{\psi}\, \overline{\sigma}^\mu\, \psi + \overline{F}\, F \\
&+ m_0\, X\, F + \overline{m}_0\, \overline{X}\, \overline{F} - \frac{1}{2}\, m_0\, \psi\, \psi - \frac{1}{2}\, \overline{m}_0\, \overline{\psi}\, \overline{\psi} \\
&+ g_0\, X^2\, F + \overline{g}_0\, \overline{X}^2\, \overline{F} - g_0\, X\, \psi\, \psi - \overline{g}_0\, \overline{X}\, \overline{\psi}\, \overline{\psi}
\end{aligned}
$$

The model keeping auxiliary fields

- As before, we can set m_0 real and non-negative without loss of generality
- To make parity symmetry manifest, we write

$$g_0 = \sqrt{2}\, y_0\, e^{i\alpha}\,, \qquad X = \frac{1}{\sqrt{2}}\, e^{-i\alpha}\,(A + i\,B)\,, \qquad F = \frac{1}{\sqrt{2}}\, e^{i\alpha}\,(\mathscr{F} - i\,\mathscr{G})$$

  with real scalar fields A, B, $\mathscr{F}$, $\mathscr{G}$

- The Lagrangian takes the form

$$\begin{aligned}\mathscr{L} = &-\frac{1}{2}\,\partial^\mu A\,\partial_\mu A - \frac{1}{2}\,\partial^\mu B\,\partial_\mu B + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi + \frac{1}{2}\,(\mathscr{F}^2 + \mathscr{G}^2)\\ &+ m_0\,(A\,\mathscr{F} + B\,\mathscr{G}) - \frac{1}{2}\,m_0\,(\psi\,\psi + \overline{\psi}\,\overline{\psi})\\ &- y_0\,A\,(\psi\,\psi + \overline{\psi}\,\overline{\psi}) - i\,y_0\,B\,(\psi\,\psi - \overline{\psi}\,\overline{\psi}) + y_0\,\mathscr{F}\,(A^2 - B^2) + 2\,y_0\,\mathscr{G}\,A\,B\end{aligned}$$

- Under parity, $\mathscr{F}$ transforms as A, while $\mathscr{G}$ transforms as B

The model keeping auxiliary fields

- We can now set up the renormalization procedure as we did in the model without auxiliary fields. When we re-write the same Lagrangian in terms of renormalized fields and couplings, it takes the form

$$
\begin{aligned}
\mathscr{L} = &-\frac{1}{2}\, Z_A\, \partial^\mu A_\mathrm{r}\, \partial_\mu A_\mathrm{r} - \frac{1}{2}\, Z_B\, \partial^\mu B_\mathrm{r}\, \partial_\mu B_\mathrm{r} + i\, Z_\psi\, \partial_\mu \overline{\psi}_\mathrm{r}\, \overline{\sigma}^\mu\, \psi_\mathrm{r} + \frac{1}{2}\, (Z_\mathscr{F}\, \mathscr{F}_\mathrm{r}^2 + Z_\mathscr{G}\, \mathscr{G}_\mathrm{r}^2) \\
&+ Z_{m_{A\mathscr{F}}}\, m_{A\mathscr{F}}\, A_\mathrm{r}\, \mathscr{F}_\mathrm{r} + Z_{m_{B\mathscr{G}}}\, m_{B\mathscr{G}}\, B_\mathrm{r}\, \mathscr{G}_\mathrm{r} - \frac{1}{2}\, Z_{m_\psi}\, m_\psi\, (\psi_\mathrm{r}\, \psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\, \overline{\psi}_\mathrm{r}) \\
&- Z_{y_A}\, y_A\, A_\mathrm{r}\, (\psi_\mathrm{r}\, \psi_\mathrm{r} + \overline{\psi}_\mathrm{r}\, \overline{\psi}_\mathrm{r}) - i\, Z_{y_B}\, y_B\, B_\mathrm{r}\, (\psi_\mathrm{r}\, \psi_\mathrm{r} - \overline{\psi}_\mathrm{r}\, \overline{\psi}_\mathrm{r}) \\
&+ Z_{\lambda_{\mathscr{F}AA}} \lambda_{\mathscr{F}AA}\, \mathscr{F}_\mathrm{r}\, A_\mathrm{r}^2 + Z_{\lambda_{\mathscr{F}BB}} \lambda_{\mathscr{F}AA}\, \mathscr{F}_\mathrm{r}\, B_\mathrm{r}^2 + Z_{\lambda_{\mathscr{G}AB}}\, \lambda_{\mathscr{G}AB}\, \mathscr{G}_\mathrm{r}\, A_\mathrm{r}\, B_\mathrm{r} \\
&- Z_{m_{AA}}\, m_{AA}\, A_\mathrm{r}^2 - Z_{m_{BB}}\, m_{BB}\, B_\mathrm{r}^2 + Y_A\, A + Y_\mathscr{F}\, \mathscr{F}_\mathrm{r} - V_0
\end{aligned}
$$

- NB: a priori, Lorentz and parity allow for a constant shift of both A and $\mathscr{F}$. These shifts are responsible for the appearance of the terms on the last line

Some results without derivation

$$
\begin{aligned}
\mathscr{L} = & -\frac{1}{2} Z_A\, \partial^\mu A_\mathrm{r}\, \partial_\mu A_\mathrm{r} - \frac{1}{2} Z_B\, \partial^\mu B_\mathrm{r}\, \partial_\mu B_\mathrm{r} + i\, Z_\psi\, \partial_\mu \overline{\psi}_\mathrm{r}\, \overline{\sigma}^\mu\, \psi_\mathrm{r} + \frac{1}{2} (Z_\mathscr{F}\, \mathscr{F}_\mathrm{r}^2 + Z_\mathscr{G}\, \mathscr{G}_\mathrm{r}^2) \\
& + Z_{m_{A\mathscr{F}}}\, m_{A\mathscr{F}}\, A_\mathrm{r}\, \mathscr{F}_\mathrm{r} + Z_{m_{B\mathscr{G}}}\, m_{B\mathscr{G}}\, B_\mathrm{r}\, \mathscr{G}_\mathrm{r} - \frac{1}{2} Z_{m_\psi}\, m_\psi\, (\psi_\mathrm{r} \psi_\mathrm{r} + \overline{\psi}_\mathrm{r} \overline{\psi}_\mathrm{r}) \\
& - Z_{y_A}\, y_A\, A_\mathrm{r}\, (\psi_\mathrm{r} \psi_\mathrm{r} + \overline{\psi}_\mathrm{r} \overline{\psi}_\mathrm{r}) - i\, Z_{y_B}\, y_B\, B_\mathrm{r}\, (\psi_\mathrm{r} \psi_\mathrm{r} - \overline{\psi}_\mathrm{r} \overline{\psi}_\mathrm{r}) \\
& + Z_{\lambda_{\mathscr{F}AA}} \lambda_{\mathscr{F}AA}\, \mathscr{F}_\mathrm{r}\, A_\mathrm{r}^2 + Z_{\lambda_{\mathscr{F}BB}} \lambda_{\mathscr{F}AA}\, \mathscr{F}_\mathrm{r}\, B_\mathrm{r}^2 + Z_{\lambda_{\mathscr{G}AB}}\, \lambda_{\mathscr{G}AB}\, \mathscr{G}_\mathrm{r}\, A_\mathrm{r}\, B_\mathrm{r} \\
& - Z_{m_{AA}}\, m_{AA}\, A_\mathrm{r}^2 - Z_{m_{BB}}\, m_{BB}\, B_\mathrm{r}^2 + Y_A\, A + Y_\mathscr{F}\, \mathscr{F}_\mathrm{r} - V_0
\end{aligned}
$$

1. The model can be regularized and renormalized preserving SUSY
2. SUSY forbids a constant shift of A and/or $\mathscr{F}$. In particular, the last line in the Lagrangian is not generated
3. All wavefunction renormalization factors are equal:

$$Z_A = Z_B = Z_\psi = Z_\mathscr{F} = Z_\mathscr{G} \equiv Z_\Phi$$

Some results without derivation

4. There is no need to introduce independent Z factors for mass terms and couplings: <u>wavefunction renormalization in the only source of renormalization</u>

$$\mathscr{L} = Z_\Phi\Big[-\frac{1}{2}\,\partial^\mu A_r\,\partial_\mu A_r - \frac{1}{2}\,\partial^\mu B_r\,\partial_\mu B_r + i\,\partial_\mu\overline{\psi}_r\,\overline{\sigma}^\mu\,\psi_r + \frac{1}{2}\,(\mathscr{F}_r^2 + \mathscr{G}_r^2)\Big]$$
$$+\, m_r\,(A_r\,\mathscr{F}_r + B_r\,\mathscr{G}_r) - \frac{1}{2}\,m_r\,(\psi_r\,\psi_r + \overline{\psi}_r\,\overline{\psi}_r)$$
$$-\, y_r\,A_r\,(\psi_r\,\psi_r + \overline{\psi}_r\,\overline{\psi}_r) - i\,y_r\,B_r\,(\psi_r\,\psi_r - \overline{\psi}_r\,\overline{\psi}_r)$$
$$+\, y_r\,\mathscr{F}_r\,(A_r^2 - B_r^2) + 2\,y_r\,\mathscr{G}_r\,A_r\,B_r$$

where we the relation between renormalized and bare couplings is

$$m_r = Z_\Phi\,m_0 \quad , \qquad y_r = Z_\Phi^{3/2}\,y_0$$

- Dramatic simplifications compared to non-SUSY models of the same kind!

Some results without derivation

- We can write both the bare Lagrangian and the Lagrangian in terms of renormalized fields and couplings using superspace, to emphasize that SUSY is manifest

$$S = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, \Phi\, \overline{\Phi} + \left[\int d^4x\, d^2\theta \left(\frac{1}{2} m_0\, \Phi^2 + \frac{1}{3} g_0\, \Phi^3\right) + \mathrm{h.c.}\right]$$

$$= \int d^4x\, d^2\theta\, d^2\overline{\theta}\, Z_\Phi\, \Phi_\mathrm{r}\, \overline{\Phi}_\mathrm{r} + \left[\int d^4x\, d^2\theta \left(\frac{1}{2} m_\mathrm{r}\, \Phi_\mathrm{r}^2 + \frac{1}{3} g_\mathrm{r}\, \Phi_\mathrm{r}^3\right) + \mathrm{h.c.}\right]$$

$$\Phi_\mathrm{r} = Z_\Phi^{-1/2}\, \Phi \quad , \quad m_\mathrm{r} = Z_\Phi\, m_0 \quad , \quad g_\mathrm{r} = Z_\Phi^{3/2}\, g_0$$

Non-renormalization theorem

The results that we have stated for the simple WZ model with one chiral superfield extend to all renormalizable models with an arbitrary number of chiral superfields:

- The model can be regularized and renormalized preserving SUSY
- The wavefunction renormalization factors are the same for all component fields in the same chiral superfield, at all orders in perturbation theory
- **Non-renormalization theorem**: the mass terms and couplings in the superpotential are not renormalized at any order in perturbation theory, except for wavefunction renormalization
- In particular:
  - If a term is absent in the classical superpotential, it is not generated at any order in perturbation theory (for example the linear term $W \supset E\,\Phi$ in the WZ model)
  - The zeroth order relations between the couplings are preserved at all orders in perturbation theory

How are these results derived?

There are various approaches to deriving the non-renormalization theorem

- In the simplest WZ model, one can verify it explicitly at one-loop by a brute force computation. One finds “miraculous” cancellations between bosonic and fermionic loops
- The simplest WZ model (formulated with auxiliary fields) is discussed in

  Iliopoulos, Zumino, “Broken supergauge symmetry and renormalization”

  http://cds.cern.ch/record/415096

  The authors give an argument based on “elementary” QFT methods (i.e. methods borrowed from non-SUSY QFTs)
- Special techniques for Feynman diagrams in superspace have been developed, which show the origin of the “miraculous” one-loop cancellations and demonstrate that these cancellations persist at all orders in perturbation theory
- The most elegant proof is based on **holomorphy** ideas, perfected by Seiberg in the ‘90s. We will see the power of holomorphy later

Some simple observations

- Let us assume that we have found a way to regularize the theory that is manifestly supersymmetric. (For models with chiral superfields, one can use the Pauli-Villars method, or introduce higher derivatives in a controlled way to change the propagators.)
- Since SUSY is manifest, we can perform the redefinition from bare fields to renormalized fields in superspace, and write

$$\Phi = Z_\Phi^{1/2}\,\Phi_\mathrm{r} + v$$

  where Z_Φ is a positive constant and v is a chiral superfield that encodes the potential shift in the VEV of Φ due to quantum corrections

- What components of v can be non-zero? To preserve Lorentz symmetry we can only give a VEV to the scalar components of the chiral superfield v, and these VEVs must be constant

$$v(x,\theta,\bar\theta) = X_v + \theta^2\,F_v$$

- If the constant F_v is non-zero, however, SUSY is broken: we can see it from the SUSY variation of the fermion component of v, $\delta\psi_{v\alpha} = i\sqrt{2}\,\sigma^\mu\,\bar\xi\,\partial_\mu X_v + \sqrt{2}\,\xi\,F_v = \sqrt{2}\,\xi\,F_v$

Some simple observations

- We conclude that $\Phi = Z_\Phi^{1/2}\,\Phi_{\rm r} + v$ where v does not depend on x, θ or $\overline{\theta}$
- We plug this in the bare Lagrangian and find

$$S = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, \Phi\, \overline{\Phi} + \left[\int d^4x\, d^2\theta \left(\tfrac{1}{2}\, m_0\, \Phi^2 + \tfrac{1}{3}\, g_0\, \Phi^3\right) + \text{h.c.}\right]$$

$$= \int d^4x\, d^2\theta\, d^2\overline{\theta}\, Z_\Phi\, \Phi_{\rm r}\, \overline{\Phi}_{\rm r} + \left[\int d^4x\, d^2\theta \left(Z_\Phi^{1/2}\, v\,(m_0 + g_0\, v)\, \Phi_{\rm r} + \tfrac{1}{2}\, Z_\Phi\,(m_0 + 2\, v\, g_0)\, \Phi_{\rm r}^2 + \tfrac{1}{3}\, Z_\Phi^{3/2}\, g_0\, \Phi_{\rm r}^3\right) + \text{h.c.}\right]$$

- NB: we can safely drop any constant from the superpotential, because the action only depends on its derivatives
- By renaming a few parameters, the above can always be written as

$$S = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, Z_\Phi\, \Phi_{\rm r}\, \overline{\Phi}_{\rm r} + \left[\int d^4x\, d^2\theta \left(Z_E\, E_{\rm r}\, \Phi_{\rm r} + \tfrac{1}{2}\, Z_m\, m_{\rm r}\, \Phi_{\rm r}^2 + \tfrac{1}{3}\, Z_g\, g_{\rm r}\, \Phi_{\rm r}^3\right) + \text{h.c.}\right]$$

where $E_{\rm r}$, $m_{\rm r}$, $g_{\rm r}$ are physical renormalized couplings and the Z factors are a priori arbitrary

Some simple observations

$$S = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, Z_\Phi\, \Phi_{\rm r}\, \overline{\Phi}_{\rm r} + \left[\left[\int d^4x\, d^2\theta \left(Z_E\, E_{\rm r}\, \Phi_{\rm r} + \frac{1}{2}\, Z_m\, m_{\rm r}\, \Phi_{\rm r}^2 + \frac{1}{3}\, Z_g\, g_{\rm r}\, \Phi_{\rm r}^3 \right) + {\rm h.c.} \right] \right.$$

- It is at this point that the hard part of the non-renormalization theorem kicks in. It guarantees that $E_r = 0$ (equiv $v = 0$) and it states that $Z_m = 1, Z_g = 1$ at all orders in perturbation theory, in such a way that

$$m_{\rm r} = Z_\Phi\, m_0 \quad , \qquad g_{\rm r} = Z_\Phi^{3/2}\, g_0$$

Supersymmetry and supergravity

Lecture 22

Holomorphy arguments

- Arguments based on holomorphy are a powerful and insightful way to “understand” the origin of the non-renormalization theorem
- To address holomorphy arguments, we need some preliminary material:
    - background superfields
    - R-symmetry
    - Wilsonian effective action VS 1PI effective action
- We recall the first two ingredients, then we consider the holomorphy arguments. We leave the comparison of effective actions at the end (it is a technical point)

Background fields and spurions

- A background field is a field that enters the Lagrangian, but that it is not integrated over in the path integral
- A background field does not have to satisfy any equation of motion
- NB: auxiliary fields do not have propagating degrees of freedom, but we do perform the path integral over them
- A constant parameter in the Lagrangian can be considered as a background scalar field that has a constant profile in spacetime

Background fields and spurions

- Thinking of parameters as background fields is useful because we can consider the action of a global symmetry on dynamical fields and background fields simultaneously
- For example: let us consider a model with free massless real scalars and an $O(N)$ symmetry

$$\mathscr{L} = -\frac{1}{2}\,\delta_{ij}\,\partial^\mu\phi^i\,\partial_\mu\phi^j\ , \qquad \phi^i \to R^i{}_j\,\phi^j$$

- Let us turn on a mass term

$$\mathscr{L} = -\frac{1}{2}\,\delta_{ij}\,\partial^\mu\phi^i\,\partial_\mu\phi^j - \frac{1}{2}\,m_{ij}\,\phi^i\,\phi^j$$

- If the matrix m_{ij} is generic, $O(N)$ is broken. However, we can regard m_{ij} as a background field, and let it transform under $O(N)$ together with the dynamical scalars ϕ^i:

$$\mathscr{L} = -\frac{1}{2}\,\delta_{ij}\,\partial^\mu\phi^i\,\partial_\mu\phi^j - \frac{1}{2}\,m_{ij}\,\phi^i\,\phi^j\ , \qquad \phi^i \to R^i{}_j\,\phi^j\ , \qquad m_{ij} \to m_{k\ell}\,(R^{-1})^k{}_i\,(R^{-1})^\ell{}_j$$

- In this way we are formally preserving $O(N)$ invariance
- A background field that transforms under a global symmetry is also known as a **spurion**

Background superfields

- In SUSY field theories it is convenient to regard coupling constants and mass parameters are background superfields
- In a renormalizable model with chiral superfields and superpotential

$$W = E_i\,\Phi^i + \frac{1}{2}\,m_{ij}\,\Phi^i\,\Phi^j + \frac{1}{3}\,g_{ijk}\,\Phi^i\,\Phi^j\,\Phi^k$$

  we interpret the mass parameters m_{ij} and the couplings E_i, g_{ijk} as **background chiral superfields**
- The only non-zero component of the background chiral superfields is their $\theta = \bar{\theta} = 0$ component X, which is a constant. The fermionic component ψ and the auxiliary components F are set to zero
- This preserves SUSY, because $\delta\psi = 0$, as can be seen from

$$\delta\psi = i\sqrt{2}\,\sigma^\mu\,\bar{\xi}\,\partial_\mu X + \sqrt{2}\,\xi\,F$$

R-symmetry

- Recall: an R-symmetry is a U(1) global symmetry that acts non-trivially on the supercharges. It follows that component fields in the same supermultiplet have different charges
- We have already discussed R-symmetry for chiral multiplets:

field/param	X^i	ψ^i_α	F^i	ξ
charge	$R[X^i]$	$R[X^i] - 1$	$R[X^i] - 2$	1

- The superpotential preserves R-symmetry if it has definite charge +2:

$$R[W] = 2$$

- Caveat: R-symmetry is a chiral symmetry (left-handed and right-handed spinors are rotated differently) and in some cases it is subject to quantum anomalies. This does not happen in models that only have chiral multiplets

The holomorphy argument

- Let us now consider the holomorphy argument for the Wess-Zumino model
- The starting point is the classical action

$$S = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, \Phi\, \overline{\Phi} + \left[\int d^4x\, d^2\theta\, W_{\mathrm{cl}} + \mathrm{h.c.} \right], \quad W_{\mathrm{cl}} = \frac{1}{2}\, m\, \Phi^2 + \frac{1}{3}\, g\, \Phi^3$$

- We regard m and g as background chiral superfields
- They can be considered as spurions that transform under a global $U(1)_A \times U(1)_R$ symmetry. Here $U(1)_A$ is a non-R-symmetry (i.e. all component fields in a supermultiplet have the same charge), while $U(1)_R$ is an R-symmetry

The holomorphy argument

$$W_{\rm cl} = \frac{1}{2}\, m\, \Phi^2 + \frac{1}{3}\, g\, \Phi^3$$

The table of charges is as follows:

	$U(1)_A$	$U(1)_R$
Φ	1	1
m	-2	0
g	-3	-1

Checks:

1. Both terms in $W_{\rm cl}$ have charge 0 under the non-R-symmetry $U(1)_A$

$$A[m\,\Phi^2] = A[m] + 2\,A[\Phi] = (-2) + 2(1) = 0\,, \quad A[g\,\Phi^3] = A[g] + 3\,A[\Phi] = (-3) + 3(1) = 0$$

2. Both terms in $W_{\rm cl}$ have charge 2 under the R-symmetry $U(1)_R$

$$R[m\,\Phi^2] = R[m] + 2\,A[\Phi] = (0) + 2(1) = 2, \qquad R[g\,\Phi^3] = R[g] + 3\,R[\Phi] = (-1) + 3(1) = 2$$

NB: The canonical kinetic term $\Phi\,\overline{\Phi}$ in the classical action is also $U(1)_A \times U(1)_R$ invariant

The holomorphy argument

- Let us now consider the Wilsonian effective action that we get at low energies if we integrate out the high-momentum modes of the chiral superfield Φ (we recall some facts on Wilsonian effective actions below)
- We assume that the theory can be regulated preserving SUSY
- We can write the effective action in superspace. Schematically

$$S_{\text{eff}} = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, K_{\text{eff}}(\Phi, m, g, \overline{\Phi}, \overline{m}, \overline{g}) + \left[\int d^4x\, d^2\theta\, W_{\text{eff}}(\Phi, m, g) + \mathrm{h.c.}\right]$$

- Remarks:

1. The effective action contains in general both renormalizable and non-renormalizable terms. We are displaying the terms that give at most 2 derivatives in spacetime. S_{eff} also contains infinitely many higher-derivative terms. We do not need them because we want to study W_{eff}.
2. The effective Kähler potential K_{eff} is a priori a generic real, non-holomorphic function of Φ, m, g. Symmetry under $U(1)_A \times U(1)_R$ tells us that $A[K_{\text{eff}}] = 0$ and $R[K_{\text{eff}}] = 0$
3. The effective superpotential W_{eff} is a priori a generic holomorphic function of Φ, m, g. Symmetry under $U(1)_A \times U(1)_R$ tells us that $A[W_{\text{eff}}] = 0$ and $R[W_{\text{eff}}] = 2$

The holomorphy argument

The effective superpotential W_{eff} is a priori a generic holomorphic function of Φ, m, g. Symmetry under $U(1)_A \times U(1)_R$ tells us that $A[W_{\text{eff}}] = 0$ and $R[W_{\text{eff}}] = 2$.

	$U(1)_A$	$U(1)_R$
Φ	1	1
m	-2	0
g	-3	-1

- What terms can possibly enter W_{eff}? Let us consider the quantity

$$\Phi^a\, m^b\, g^c$$

- We must demand $A[\Phi^a\, m^b\, g^c] = 0$ and $R[\Phi^a\, m^b\, g^c] = 2$
- The solution is $(a, b, c) = (n+2, 1-n, 1-n)$

$$\Phi^a\, m^b\, g^c = m\, \Phi^2\, (m^{-1}\, g\, \Phi)^n$$

- This shows that, if we include factor out $m\,\Phi^2$ from W_{eff}, what is left can be an arbitrary function of the combination $m^{-1}\, g\, \Phi$
- We conclude that the effective superpotential can be written as

$$W_{\text{eff}}(\Phi, m, g) = m\, \Phi^2 f(m^{-1}\, g\, \Phi) \quad \text{for some holomorphic function } f(z)$$

The holomorphy argument

$$W_{\text{eff}}(\Phi, m, g) = m\,\Phi^2 f(m^{-1}\,g\,\Phi) \quad \text{for some holomorphic function } f(z)$$

- The function $f(z)$ must be compatible with two special limits:

1. Weak-coupling limit $g \to 0$

   This is the limit we consider in perturbation theory. There cannot be any singularity as $g \to 0$, and therefore $f(z)$ can only contain non-negative powers of its arguments in its Laurent series

$$W_{\text{eff}} = \sum_{n \geq 0} a_n\, m^{1-n}\, g^n\, \Phi^{n+2}$$

2. Massless limit $m \to 0$

   The Wilsonian effective action is unambiguous even if the field is massless. There should be no singularity in the limit $m \to 0$. We learn that the only a_n coeffs that can be nonzero are a_0 and a_1:

$$W_{\text{eff}} = a_0\, m\, \Phi^2 + a_1\, g\, \Phi^3$$

The holomorphy argument

$$W_{\text{eff}} = a_0\, m\, \Phi^2 + a_1\, g\, \Phi^3$$

- But a_0, a_1 are numerical constants that do not depend on the background chiral superfields m, g. In order for W_{eff} to be compatible with W_{cl} we must have $a_0 = \frac{1}{2}$, $a_1 = \frac{1}{3}$

$$W_{\text{eff}} = \frac{1}{2}\, m\, \Phi^2 + \frac{1}{3}\, g\, \Phi^3$$

- This is exactly the same as the classical superpotential! No quantum corrections are generated, at all
- We have proven the non-renormalization theorem using holomorphy
- The holomorphy argument does not rely on perturbation theory. It is not clear, however, if the WZ model exists non-perturbatively, because it is not asymptotically free
- There is no clever argument about $K_{\text{eff}}(\Phi, m, g, \overline{\Phi}, \overline{m}, \overline{g})$. Indeed, wavefunction renormalization receives contributions at all orders in perturbation theory

1PI effective action

- Let us recall the steps in the definition of the 1PI effective action. To keep the notation simple, we consider a real scalar field ϕ, but the arguments apply to general fields
- We take the QFT of interest and we couple it to classical sources J to define the partition function

$$Z[J] = \int \mathscr{D}\phi \, e^{i\,S[\phi]+i\,\int d^4x\,\phi\,J}$$

- The partition function is the generating functional of the total n-pt functions, including contributions from Feynman diagrams that consist of several disconnected components

$$\langle 0|\,T\,\phi(x_1)\ldots\phi(x_n)\,|0\rangle = \frac{1}{Z[J]}\,\frac{\delta}{i\,\delta J(x_1)}\ldots\frac{\delta}{i\,\delta J(x_1)}Z[J]\bigg|_{J=0}$$

- Taking a log we get the functional $i\,W[J] = \log Z[J]$ that generates <u>connected</u> n-pt functions

$$\langle 0|\,T\,\phi(x_1)\ldots\phi(x_n)\,|0\rangle_{\rm conn} = \frac{\delta}{i\,\delta J(x_1)}\ldots\frac{\delta}{i\,\delta J(x_1)}i\,W[J]\bigg|_{J=0}$$

1PI effective action

- Next, we consider the 1-pt function of ϕ in the presence of a generic source J

$$\langle 0 | \phi(x) | 0 \rangle_J = \frac{\delta W[J]}{\delta J(x)}$$

- The quantity $\langle 0 | \phi(x) | 0 \rangle_J$ can be regarded as a classical field, so we write

$$\phi_{\text{cl}}(x) = \langle 0 | \phi(x) | 0 \rangle_J$$

- Let us perform a Legendre transform of $i\, W[J]$ to construct a functional of $\phi_{\text{cl}}(x)$, as follows:

$$\Gamma[\phi_{\text{cl}}] = W[J] - \int d^4x\, J\, \phi_{\text{cl}}$$

- On the RHS we regard J as a functional of $\phi_{\text{cl}}(x)$, obtained by inverting the relation $\phi_{\text{cl}}(x) = \frac{\delta W[J]}{\delta J(x)}$. One verifies from the definition that

$$J(x) = -\frac{\delta \Gamma[\phi_{\text{cl}}]}{\delta \phi_{\text{cl}}(x)}$$

Why “effective action”?

$$\Gamma[\phi_{\rm cl}] = W[J] - \int d^4x\, J\, \phi_{\rm cl} \quad , \quad \phi_{\rm cl}(x) = \frac{\delta W[J]}{\delta J(x)}, \qquad J(x) = -\frac{\delta \Gamma[\phi_{\rm cl}]}{\delta \phi_{\rm cl}(x)}$$

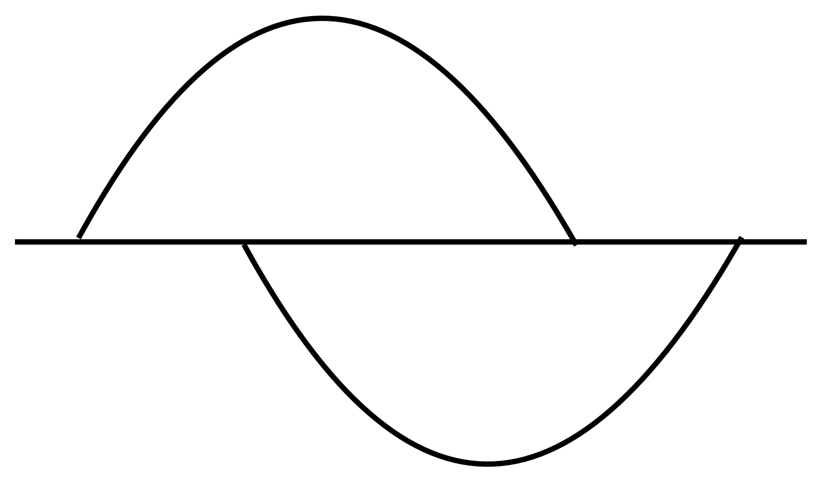
- The functional $\Gamma[\phi_{\rm cl}]$ is an <u>effective action</u>:
  - In the classical theory, a field configuration solves the EOMs iff $\dfrac{\delta S[\phi]}{\delta \phi(x)} = 0$
  - In the quantum theory, if we demand zero external source, $J = 0$, we find the condition $\dfrac{\delta \Gamma[\phi_{\rm cl}]}{\delta \phi_{\rm cl}(x)} = 0$
- One can prove that amplitudes in the quantum theory can be computed replacing the classical action $S[\phi]$ with the effective action $\Gamma[\phi_{\rm cl}]$ and retaining <u>only tree diagrams</u>
- $\Gamma[\phi_{\rm cl}]$ gives an effective tree-level description of all quantum corrections

Why “1PI”?

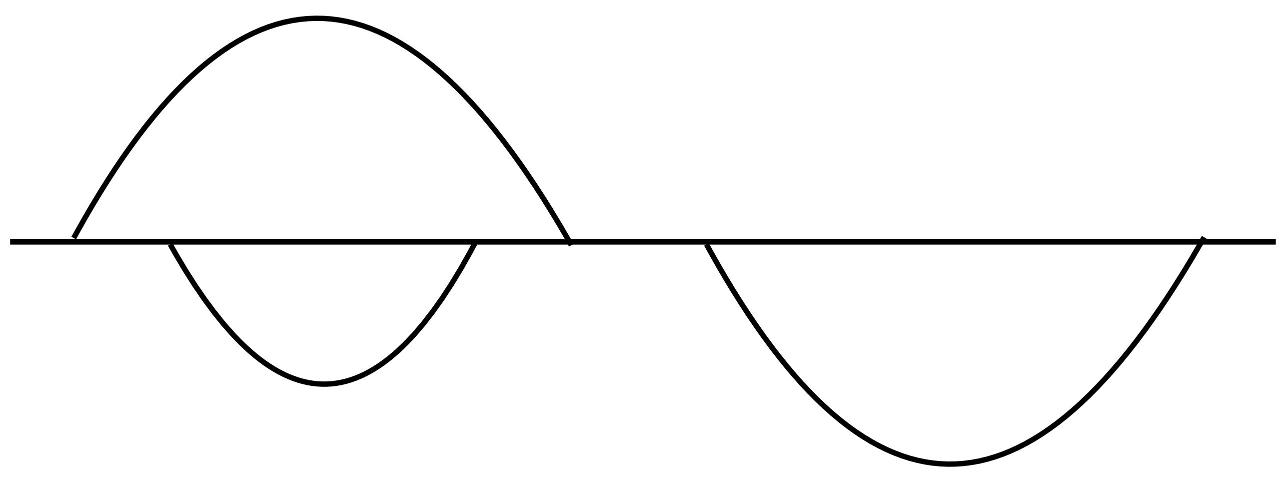
- If we write $\Gamma[\phi_{\rm cl}]$ in momentum space, we get schematically

$$\Gamma[\phi_{\rm cl}] = \int \frac{d^4p}{(2\pi)^4}\phi_{\rm cl}(-p)\boldsymbol{\Delta}(p)\,\phi_{\rm cl}(p) + \sum_{n=3}^{\infty}\int \frac{d^4p_1}{(2\pi)^4}\dots\frac{d^4p_n}{(2\pi)^4}\,\mathbf{V}_n(p_1,\dots,p_n)\,\phi_{\rm cl}(p_1)\dots\phi_{\rm cl}(p_n)$$

- The quantity $\boldsymbol{\Delta}$ is the exact propagator; $\mathbf{V}_n$ are the 1PI vertices. They are computed summing over connected and 1PI diagrams with n external legs
- 1PI = cannot be divided in two disconnected pieces by cutting a single line



is 1PI



is not 1PI

Caveats on the 1PI effective action

$$\Gamma[\phi_{\rm cl}] = \int \frac{d^4p}{(2\pi)^4}\phi_{\rm cl}(-p)\boldsymbol{\Delta}(p)\,\phi_{\rm cl}(p) + \sum_{n=3}^{\infty}\int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\,\mathbf{V}_n(p_1,\ldots,p_n)\,\phi_{\rm cl}(p_1)\ldots\phi_{\rm cl}(p_n)$$

- In general, the 1PI vertex functions $\mathbf{V}_n$ are not analytic in the momenta, which means that (undoing the Fourier transform) the 1PI effective action is non-local in spacetime
- If we consider a massive theory, one can expand in powers of p_i/m. The outcome is an infinite sum of local terms
- In a theory with massless particles, however, this is not possible, and non-locality remains (the vertex functions can have branch cuts that start at $p = 0$)
- In general, the 1PI effective action is sensitive to IR effects. This is because $\Gamma[\phi_{\rm cl}]$ is supposed to account for all loop corrections, coming from all momenta running in loops, including arbitrarily low momenta

Wilsonian effective action

- Wilsonian philosophy:
  - split quantum fields into low-momentum and high-momentum modes, with low and high determined with reference to some scale μ
  - perform the path integral on the high-momentum modes (i.e. “integrate them out”)
- Schematically (after Wick rotation to Euclidean signature)

$$\phi_{\mathrm{L}}(x) = \int_{|p|<\mu} \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \widetilde{\phi}(p) \ , \qquad \phi_{\mathrm{H}}(x) = \int_{|p|>\mu} \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \widetilde{\phi}(p)$$

$$e^{-S_\mu^{\mathrm{W}}[\phi_{\mathrm{L}}]} = \int \mathscr{D}\phi_{\mathrm{H}}\, e^{-S[\phi_L,\phi_H]}$$

Wilsonian effective action

- NB: We still have to perform the path integral over ϕ_{L}!
- To compute an observable, we use $S_\mu^{\mathrm{W}}[\phi_{\mathrm{L}}]$ to compute tree diagrams, <u>as well as loop diagrams</u>, where loop momenta are cutoff at μ
- In the end, we still perform the full path integral, but we do it in two steps: high momenta first, low momenta second

Wilsonian effective action

- $S_\mu^{\mathrm{W}}[\phi_{\mathrm{L}}]$ contains in general an infinite sum of local terms, with μ-dependent coefficients. For example, for a real scalar field with a $\phi \leftrightarrow -\phi$ symmetry, $S_\mu^{\mathrm{W}}[\phi_{\mathrm{L}}] = \int d^4x\, \mathscr{L}_\mu^{\mathrm{W}}$ where

$$\mathscr{L}_\mu^{\mathrm{W}} = -\frac{1}{2}\left[b_0(\mu) + \frac{b_2(\mu)}{\mu^2}\,\phi^2 + \frac{b_4(\mu)}{\mu^4}\,\phi^4 + \ldots\right]\partial^\mu\phi\,\partial_\mu\phi$$
$$+\left[a_0(\mu)\,\mu^4 + a_2(\mu)\,\mu^2\,\phi^2 + a_4(\mu)\,\phi^4 + \frac{a_6(\mu)}{\mu^2}\phi^6 + \ldots\right] + \ldots$$

- When we use $S_\mu^{\mathrm{W}}[\phi_{\mathrm{L}}]$ in loop diagrams, we get an extra μ dependence, because the loop momenta are cutoff at μ
- This extra μ dependence has to cancel against the explicit μ dependence of the couplings in $S_\mu^{\mathrm{W}}[\phi_{\mathrm{L}}]$
- This is because we are still performing the full path integral, and the scale μ that we use to separate low and high momenta is arbitrary. Physical quantities cannot depend on it

Holomorphy and the Wilsonian effective action

- The Wilsonian effective action does not suffer from the IR problems that one encounters in the 1PI effective action
- The Wilsonian effective action is always an (infinite) sum of local terms. In a SUSY theory locality is needed to distinguish F-terms and D-terms
- Formally, we can always convert a D-term as an F-term, but the price to pay is a non-local $\Box^{-1}$ (where $\Box = \partial^\mu \partial_\mu$)
- To see this, notice the following identity in superspace

$$\int d^4x\, d^2\theta\, d^2\overline{\theta}\, \Phi_1 (-\tfrac{1}{4}\, D^2) \Phi_2 = \int d^4x\, d^2\theta\, \Phi_1 \Box \Phi_2 \quad \text{for any chiral superfields } \Phi_1, \Phi_2$$

- If we set $\Phi_2 = \Box^{-1} \Phi_3$ we get an identity that converts a non-local D-term into a local F-term
- Because of its IR singularities in a massless theory, the 1PI action might develop non-local D-terms. They get converted into F-terms and ruin the non-renormalization theorem
- This can never happen with the Wilsonian effective action

Supersymmetry and supergravity

Lecture 23

Reminder on SUSY gauge theory actions

- The superspace action for a renormalizable model with vector and chiral superfields is

$$S = S_{\rm SYM} + S_K + S_W + S_{\rm FI}$$

$$S_{\rm SYM} = \int d^4x\, d^2\theta\, \frac{-i\,\tau}{16\pi T(\mathbf{R})}\, {\rm Tr}_{\mathbf{R}}\,(\mathscr{W}^{\alpha}\,\mathscr{W}_{\alpha}) + {\rm h\,.\,c\,.}\;, \qquad {\rm Tr}_{\mathbf{R}}\,(t_a\, t_b) = T(\mathbf{R})\,\delta_{ab}\;, \qquad \tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2}$$

$$S_K = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, \Phi^\dagger\, e^{2qV}\,\Phi \qquad S_W = \int d^4x\, d^2\theta\, W(\Phi) + {\rm h\,.\,c\,.} \qquad S_{\rm FI} = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, p\, V_{U(1)}$$

- Let us focus on $S_{\rm SYM}$ and S_K. In component fields:

$$\mathscr{L}_{\rm SYM} = \delta_{ab}\left[-\frac{1}{4\,g^2}\,F^{a\mu\nu}\,F^b_{\mu\nu} + \frac{1}{2\,g^2}\,D^a\,D^b - \frac{1}{g^2}\,i\,\lambda^a\,\sigma^\mu\,D_\mu\overline{\lambda}^b + \frac{1}{64\pi^2}\,\theta\,\epsilon_{\mu\nu\rho\sigma}\,F^{a\mu\nu}\,F^{b\rho\sigma}\right]$$

$$\mathscr{L}_K = -\,D^\mu\overline{X}_i\,D_\mu X^i + i\,D_\mu\overline{\psi}_i\,\overline{\sigma}^\mu\,\psi^i + \overline{F}_i\,F^i + i\sqrt{2}\,\left[\overline{X}_i\,(t_a)^i{}_j\,\psi^j\,\lambda^a - \overline{\lambda}^a\,\overline{\psi}_i\,(t_a)^i{}_j\,X^j\right] + D^a\,\overline{X}_i\,(t_a)^i{}_j\,X^j$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{bc}{}^a\,A^b_\mu\,A^c_\nu\;, \qquad D_\mu X^i = \partial_\mu X^i + i\,A^a_\mu\,(t_a)^i)_j\,X^j$$

Reminder on SUSY gauge theory actions

- The θ term is a total derivative. It is invisible in perturbation theory: we ignore it, for the time being
- To set up perturbation theory it is more convenient to work with canonically normalized gauge fields
- After rescaling the gauge fields (and their SUSY partners), the classical action takes the form

$$\mathscr{L}_{\rm SYM} = \delta_{ab}\left[-\tfrac{1}{4}\,F^{a\mu\nu}\,F^{b}_{\mu\nu}+\tfrac{1}{2}\,D^a\,D^b - i\,\lambda^a\,\sigma^\mu\,D_\mu\bar\lambda^b\right]$$

$$\mathscr{L}_K = -\,D^\mu\overline{X}_i\,D_\mu X^i + i\,D_\mu\overline{\psi}_i\,\overline{\sigma}^\mu\,\psi^i + \overline{F}_i\,F^i + i\sqrt{2}\,g\,\left[\overline{X}_i\,(t_a)^i{}_j\,\psi^j\,\lambda^a - \bar\lambda^a\,\overline{\psi}_i\,(t_a)^i{}_j\,X^j\right] + g\,D^a\,\overline{X}_i\,(t_a)^i{}_j\,X^j$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\,f_{bc}{}^a\,A^b_\mu\,A^c_\nu\ , \qquad D_\mu X^i = \partial_\mu X^i + i\,g\,A^a_\mu\,(t_a)^i{}_j\,X^j$$

- Let us assume for simplicity that there is no FI term. The D-term in the scalar potential is

$$V_D = g^2\,\delta^{ab}\left[\overline{X}_i\,(t_a)^i{}_j\,X^j\right]\left[\overline{X}_k\,(t_b)^k{}_\ell\,X^\ell\right]$$

- Remarks: in the classical action, SUSY implies that

1. the Yukawa coupling $\overline{X}\,\psi\,\lambda$ is equal to $\sqrt{2}\,g$
2. the coefficient of the quartic $[\overline{X}\,t\,X]\,[\overline{X}\,t\,X]$ term in the scalar potential is g^2
3. the gauge-invariant term $\left[\overline{X}_i\,\delta^i{}_j\,X^j\right]\left[\overline{X}_k\,\delta^k{}_\ell\,X^\ell\right]$ is absent from the scalar potential

Quantum corrections and SUSY

In the classical action, SUSY implies that

1. the Yukawa coupling $\overline{X}\psi\lambda$ is equal to $\sqrt{2}\,g$
2. the coefficient of the quartic $[\overline{X}\,t\,X]\,[\overline{X}\,t\,X]$ term in the scalar potential is g^2
3. the gauge-invariant term $\left[\overline{X}_i\,\delta^i{}_j\,X^j\right]\left[\overline{X}_k\,\delta^k{}_\ell\,X^\ell\right]$ is absent from the scalar potential

- Are these properties preserved by perturbative quantum corrections?
- This is non-obvious, because renormalization introduces all possible couplings that are compatible with renormalizability and symmetries, a priori with arbitrary coefficients
- One way to diagnose whether the above properties hold in the quantum theory is to study the beta functions of the gauge coupling, the Yukawa coupling, and quartic couplings in the scalar potential

Reminder on beta functions

- The gauge coupling, the Yukawa couplings, and the (scalar)^4 couplings are dimensionless in the classical Lagrangian. Their classical mass dimensions are 0
- Quantum effects break the classical scale invariance of these couplings
- According to the renormalization program, physical quantities are not expressed in terms of the bare gauge coupling g_0 in the bare Lagrangian, but rather in terms of a renormalized coupling $g(\mu)$. Here μ is the renormalization scale. Similar remarks apply to Yukawa couplings and (scalar)^4 couplings
- Let us denote schematically as λ_i the set of dimensionless couplings in the gauge theory with matter
- The dependence of $\lambda_i(\mu)$ on μ is governed by the beta function β_{λ_i}

$$\mu \frac{d\lambda_i(\mu)}{d\mu} = \beta_{\lambda_i}(\{\lambda_j(\mu)\})$$

- $\beta_{\lambda_i} = \beta_{\lambda_i}(\{\lambda_j\})$ is a function of the couplings $\lambda_j(\mu)$. It does not depend on the UV cutoff that regularizes the theory, and it does not depend explicitly on the scale μ
- In general, beta functions are scheme-dependent. Their leading 1-loop terms, however, are universal

Beta function of the gauge coupling

- The 1-loop beta function for the gauge coupling in a generic (non-necessarily SUSY) renormalizable gauge theory can be written as

$$\text{1-loop:}\quad \beta_g(g) = -\frac{g^3}{16\,\pi^2}\,b$$

- The coefficient b receives various contributions:

$$b = \frac{11}{3}\,T(\text{adj}) - \frac{2}{3}\sum_{\text{ferm}} T(\mathbf{r}) - \frac{1}{3}\sum_{\text{compl.scal.}} T(\mathbf{r})$$

1. The first term comes from gauge fields and ghosts
2. The second term is a sum over the representations $\mathbf{r}$ of (positive chirality) Weyl fermions
3. The third term is a sum over the representations $\mathbf{r}$ of complex scalars

- Group theory notation:

$$\mathrm{tr}_{\mathbf{r}}(t_a\,t_b) = T(\mathbf{r})\,\delta_{ab}$$

Beta function of the gauge coupling

$$b = \frac{11}{3} T(\text{adj}) - \frac{2}{3} \sum_{\text{ferm}} T(\mathbf{r}) - \frac{1}{3} \sum_{\text{compl.scal.}} T(\mathbf{r})$$

- In a SUSY theory we can rearrange the sum into multiplets
- The vector multiplet is always in the adjoint representation. The gauge fields and ghosts contribute $\frac{11}{3} T(\text{adj})$. The gaugini are also in the adjoint representation. They contribute $-\frac{2}{3} T(\text{adj})$. In total

vector multiplet: $\qquad b = 3\, T(\text{adj})$

- A chiral multiplet in the representation $\mathbf{r}$ contains one complex scalar and one Weyl fermion (we should not count the auxiliary fields, because it does not have independent dofs). The scalar contributes $-\frac{1}{3} T(\mathbf{r})$; the fermion gives $-\frac{2}{3} T(\mathbf{r})$. Then

chiral multiplet in rep $\mathbf{r}$: $\qquad b = -\, T(\mathbf{r})$

Example: SQCD

- Reminder: FI terms are not allowed. We take a zero superpotential for simplicity. The model has gauge group $SU(N_c)$ and matter chiral superfields $Q^I{}_{\hat{I}}$ and $\widetilde{Q}^{\hat{I}'}{}_I$. We have an $SU(N_f) \times SU(N_f)'$ global symmetry

	$SU(N_c)$	$SU(N_f)$	$SU(N_f)'$
$Q^I{}_{\widehat{I}}$	$\square$	$\overline{\square}$	$\bullet$
$\widetilde{Q}^{\widehat{I}'}{}_I$	$\overline{\square}$	$\bullet$	$\square$

- In our previous equations, X^i denotes collectively all matter fields. For SQCD: $X^i \to (Q, \widetilde{Q})$

Example: SQCD

- In SQCD we only find fields in the adjoint or fund/antifun representations of $SU(N_c)$. We can use

$$SU(N_c): \qquad T(\text{adj}) = N_c \quad , \qquad T(\square) = T(\overline{\square}) = 1/2$$

- We then have

vector multiplet of $SU(N_c)$: $\qquad b = 3\,N_c$

chiral multiplet in the fund or antifund of $SU(N_c)$: $\qquad b = -\,1/2$

- Counting both the Q's and the $\widetilde{Q}$'s, we have a total of $2\,N_f$ chiral multiplets in the fund or antifund rep of $SU(N_c)$. The total 1-loop beta function coefficient is then

$$\text{SCQD:} \qquad b = 3\,N_c - N_f$$

Beta function for Yukawa couplings

- The 1-loop beta function for Yukawa couplings has the schematic form $\beta_y \sim y^3 + g^2\, y$
- To be more precise, we use the notation of ME Machacek, MT Vaughn “Two-loop renormalization group equations in a general quantum field theory (II). Yukawa couplings” Nuclear Physics B, 1984
- Consider a general renormalizable model (not necessarily SUSY) in which we have a set of <u>real</u> scalar fields φ_A and 2-component fermions χ^x. We encode the Yukawa couplings in a complex symmetric matrix Y^A_{xy}

$$\mathscr{L} \supset -\, Y^A_{xy}\, \chi^x\, \chi^y\, \varphi_A + \mathrm{h\,.\,c\,.} \qquad D_\mu \chi^x = \partial_\mu \chi^x + i\, g\, (t_a)^x{}_y\, \chi^y\, A^a_\mu$$

- We have introduced the notation χ^x which stands collectively for all the fermions. In a SUSY model χ^x contains both the fermions from chiral multiplets and the gaugini
- The representation $(t_a)^x{}_y$ is in general reducible

Beta function for Yukawa couplings

- The 1-loop beta function for Y^A_{xy} is given by

$$(4\pi)^2 \beta_{Y^A_{xy}} = \frac{1}{2}(Y^A)_{xz}(Y^{B\dagger})^{zw}(Y^B)_{wy} + \frac{1}{2}(Y^B)_{xz}(Y^{B\dagger})^{zw}(Y^A)_{wy} + 2(Y^B)_{xz}(Y^{A\dagger})^{zw}(Y^B)_{wy}$$
$$+ (Y^B)_{xy}(Y^{B\dagger})^{zw}(Y^A)_{wz} - 3g^2(Y^A)_{xz}(t_a t_a)^z{}_y - 3g^2(Y^A)_{zy}(t_a t_a)^z{}_x$$

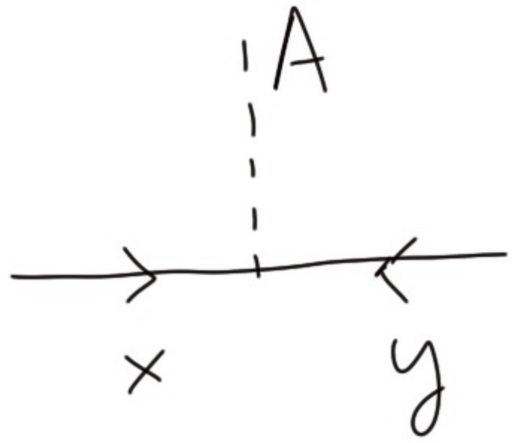
- We write $(Y^A)_{xy}$ to emphasize that we think of Y^A as a matrix with entries Y^A_{xy}
- Repeated A, B or a, b indices are contracted with δ
- The quantity $(t_a t_a)^x{}_y$ is the quadratic Casimir of the (generically reducible) representation of the fermions. In each "irreducible subblock" $(t_a t_a)^x{}_y$ reduces to a multiple of the identity matrix
- It is convenient to use a diagrammatic notation to describe the various terms in the 1-loop beta function

Beta function for Yukawa couplings

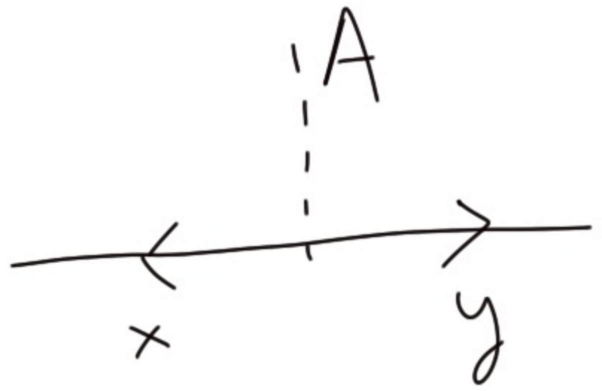
$$(4\pi)^2\,\beta_{Y^A_{xy}} = \frac{1}{2}\,(Y^A)_{xz}\,(Y^{B\dagger})^{zw}\,(Y^B)_{wy} + \frac{1}{2}\,(Y^B)_{xz}\,(Y^{B\dagger})^{zw}\,(Y^A)_{wy} + 2\,(Y^B)_{xz}\,(Y^{A\dagger})^{zw}\,(Y^B)_{wy}$$
$$+\,(Y^B)_{xy}\,(Y^{B\dagger})^{zw}\,(Y^A)_{wz} - 3\,g^2\,(Y^A)_{xz}\,(t_a\,t_a)^z{}_y - 3\,g^2\,(Y^A)_{zy}\,(t_a\,t_a)^z{}_x$$

The vertices that originate from the Yukawa couplings $\mathscr{L} \supset -\,Y^A_{xy}\,\chi^x\,\chi^y\,\varphi_A + \mathrm{h.c.}$ and the gauge covariant derivative are of the form

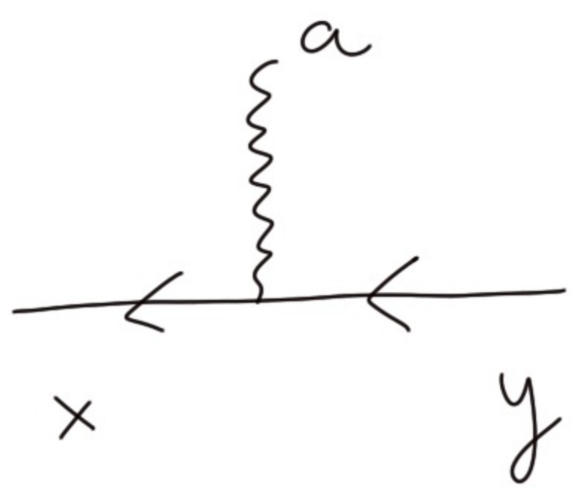
Y^A_{xy}



$(Y^A)^{\dagger\,xy}$



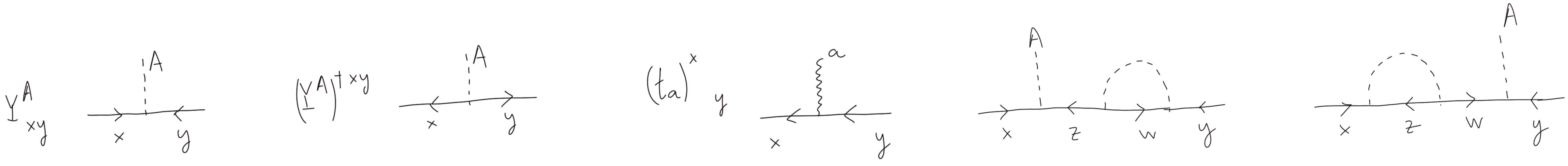
$(t_a)^x{}_y$



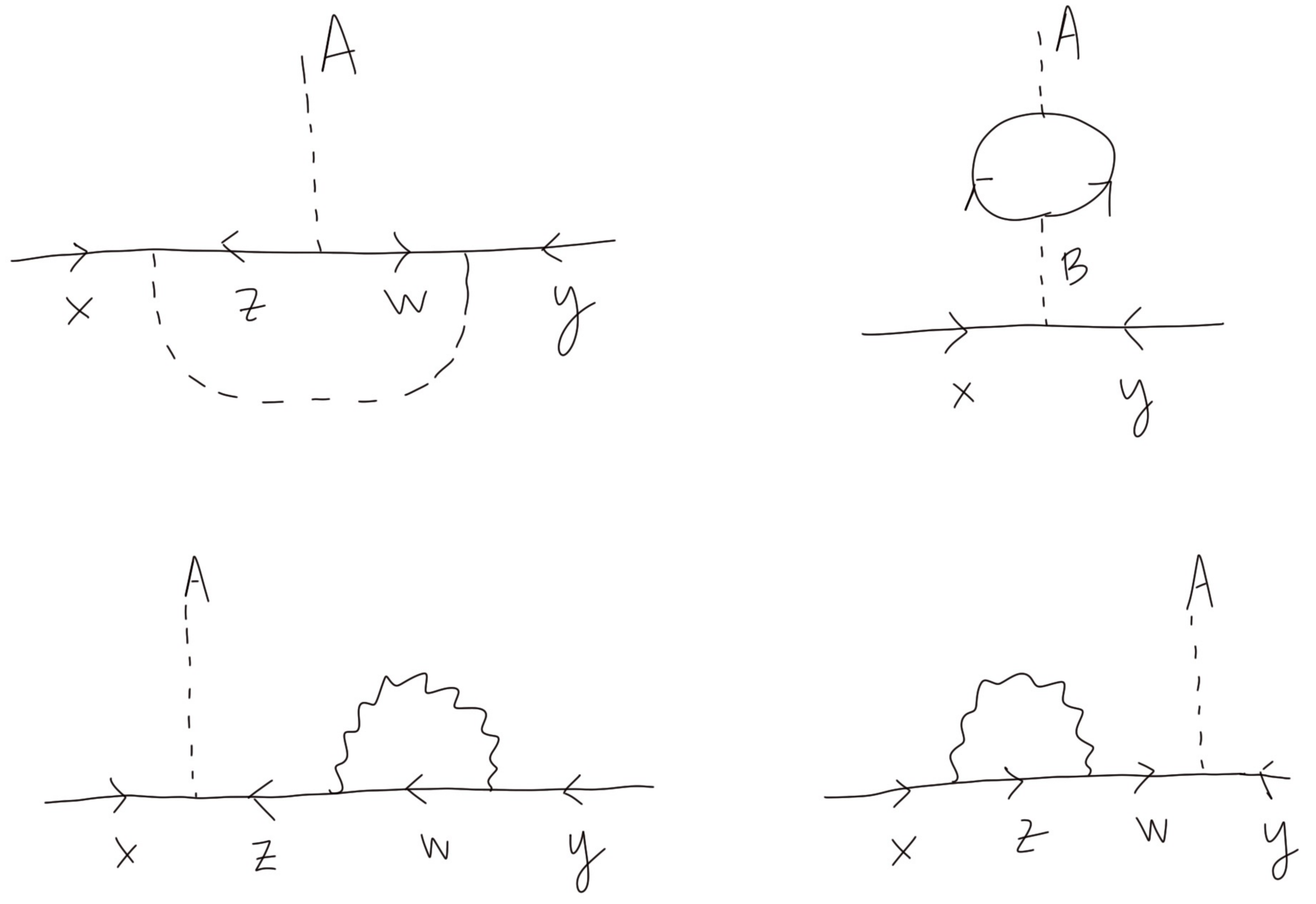
- Ingoing arrows stand for spinors with undotted Weyl indices; outgoing arrows stand for spinors with dotted Weyl indices
- We don't put an arrow on the scalar leg, because the scalars are real. Idem for gauge bosons
- Our discussion is a a bit schematic. A thorough discussion fo Feynman rules in 2-component notation can be found e.g. in arXiv 0812.1594

Beta function for Yukawa couplings

$$(4\pi)^2\,\beta_{Y^A_{xy}} = \frac{1}{2}\,(Y^A)_{xz}\,(Y^{B\dagger})^{zw}\,(Y^B)_{wy} + \frac{1}{2}\,(Y^B)_{xz}\,(Y^{B\dagger})^{zw}\,(Y^A)_{wy} + 2\,(Y^B)_{xz}\,(Y^{A\dagger})^{zw}\,(Y^B)_{wy}$$
$$+\,(Y^B)_{xy}\,(Y^{B\dagger})^{zw}\,(Y^A)_{wz} - 3\,g^2\,(Y^A)_{xz}\,(t_a\,t_a)^z{}_y - 3\,g^2\,(Y^A)_{zy}\,(t_a\,t_a)^z{}_x$$



The various terms correspond to the following diagrams.
These diagrams reflect the actual 1-loop Feynman diagrams used in the computation of the beta function
(Some diagrams that one might draw are absent. This is due to some gauge choices in the propagators.
The final 1-loop beta function is gauge invariant.)



Beta function for Yukawa couplings

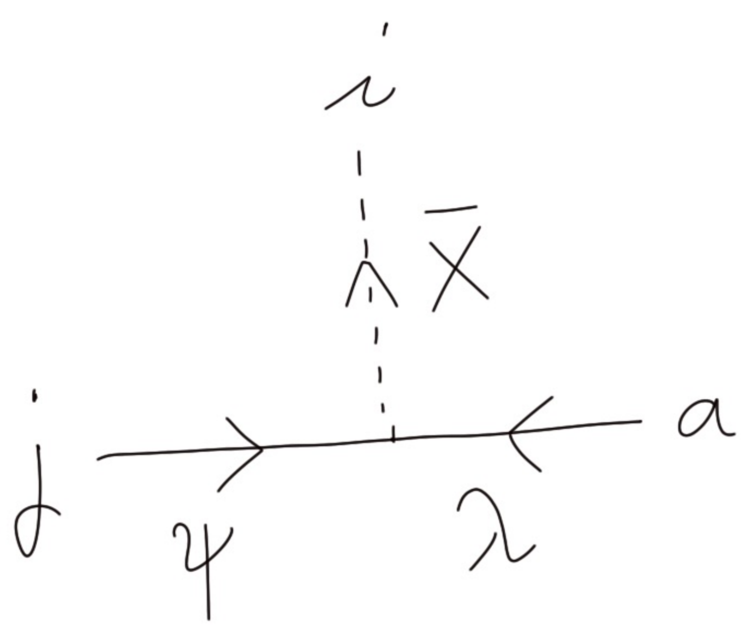
- In a SUSY gauge theory the Yukawa couplings have a special form:

$$\mathscr{L}_K \supset i\sqrt{2}\, g\, \overline{X}_i\, (t_a)^i{}_j \psi^j\, \lambda^a + \mathrm{h.c.}$$

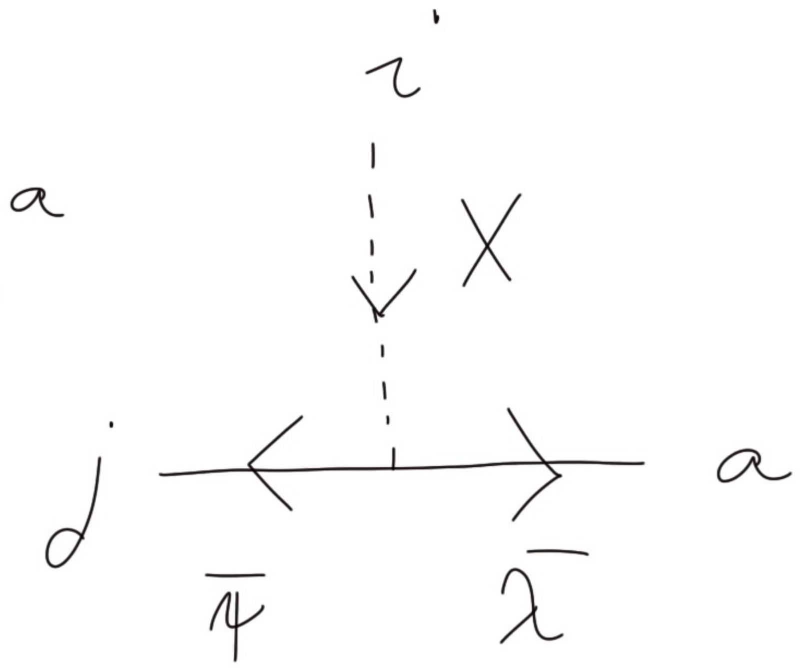
1. They only involve a gaugino and a fermion from a chiral multiplet (and never two gaugini or two ψ's
2. Their index structure is determined by the gauge generators
3. Their coefficient is equal to the gauge coupling constant (up to a numerical constant)

- Diagrammatically:

$$\overline{X}_i (t_a)^i{}_j \psi^j \lambda^a$$



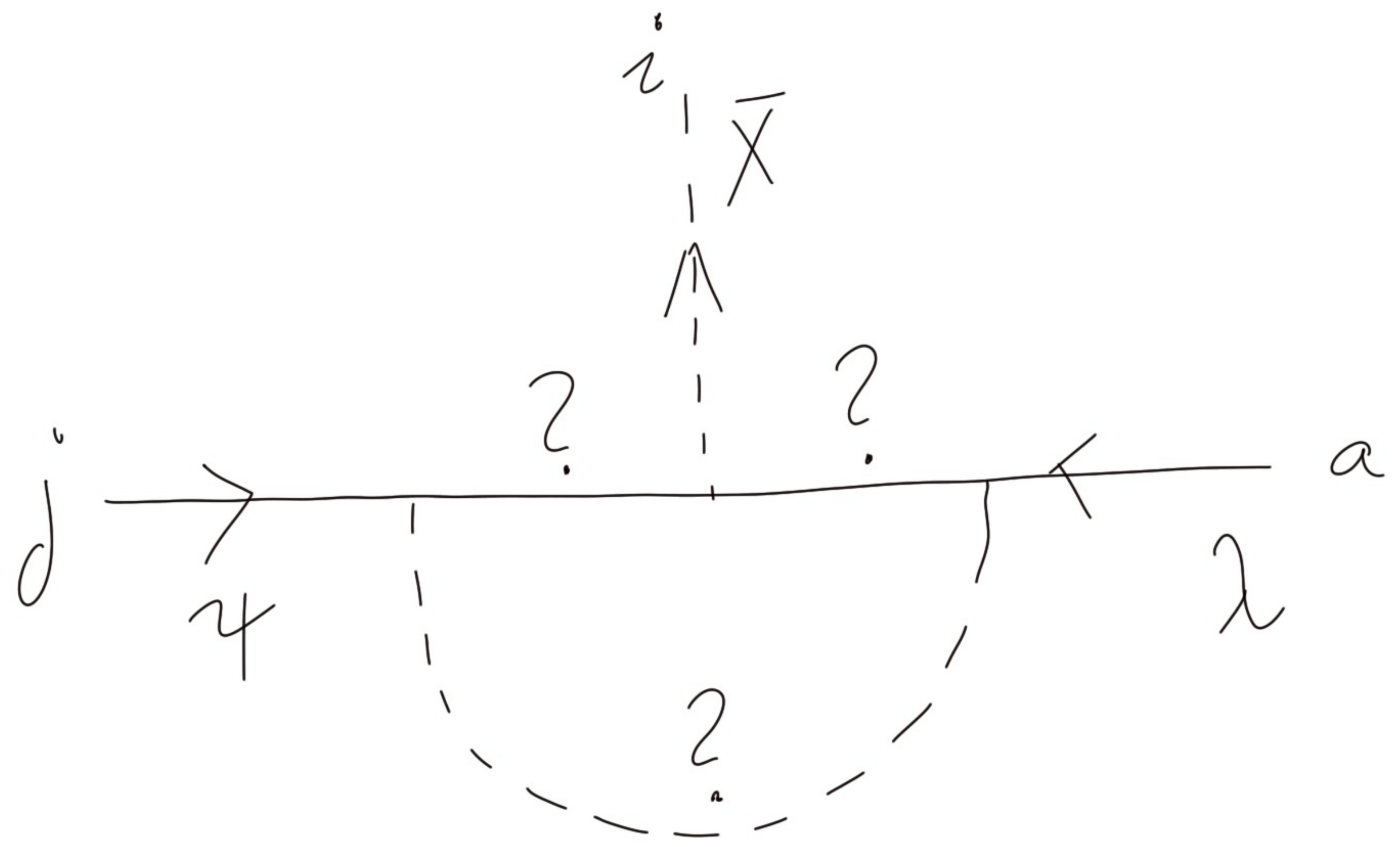
$$X^i (t_a)^j{}_i \overline{\psi}_j \overline{\lambda}^a$$

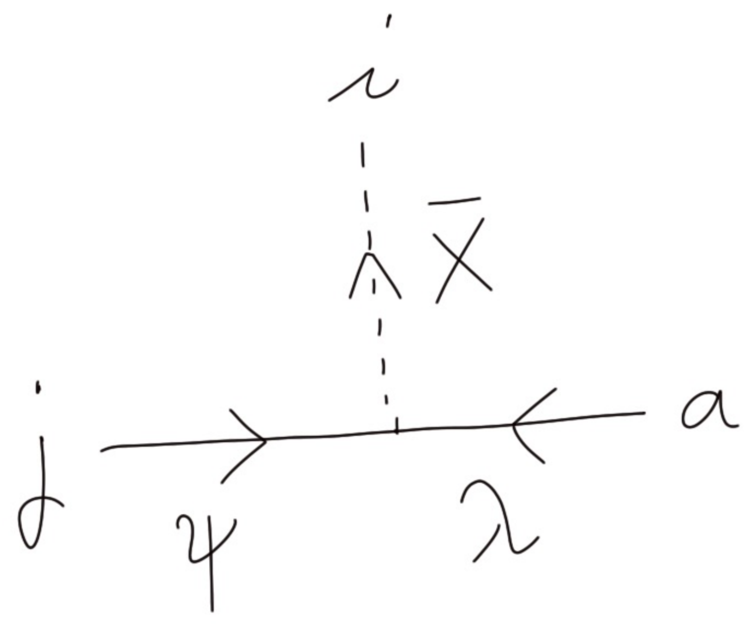


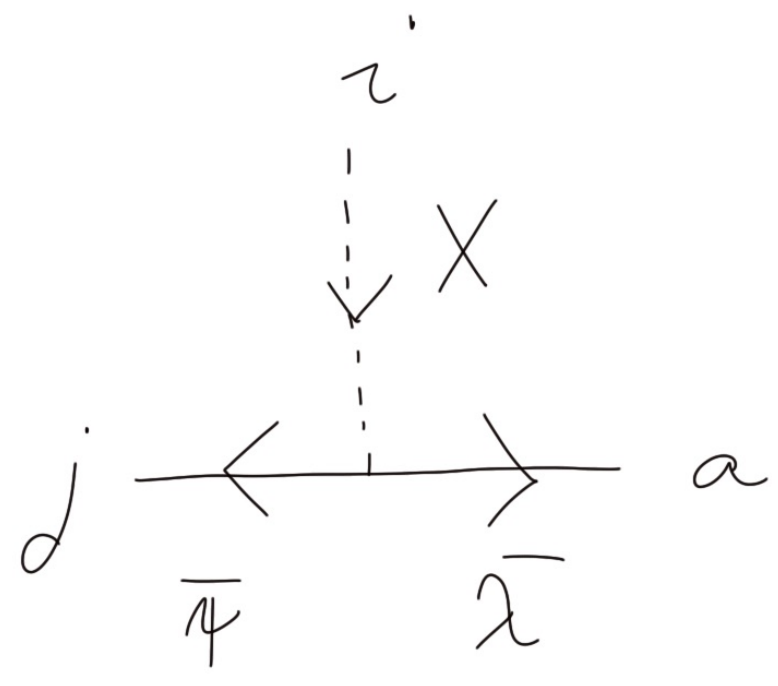
- We put an arrow on the complex scalar. It fits the arrow on ψ to describe the "flow" of "gauge charge" (notice the pattern of ingoing/outgoing arrows VS lower/upper indices)

Beta function for Yukawa couplings

- We can specialize the general expression for the beta function to the case of a SUSY gauge theory
- All terms are automatically proportional to g^3
- One of the diagrams does not contribute, because there is no way to match the arrows of all fermions and scalars





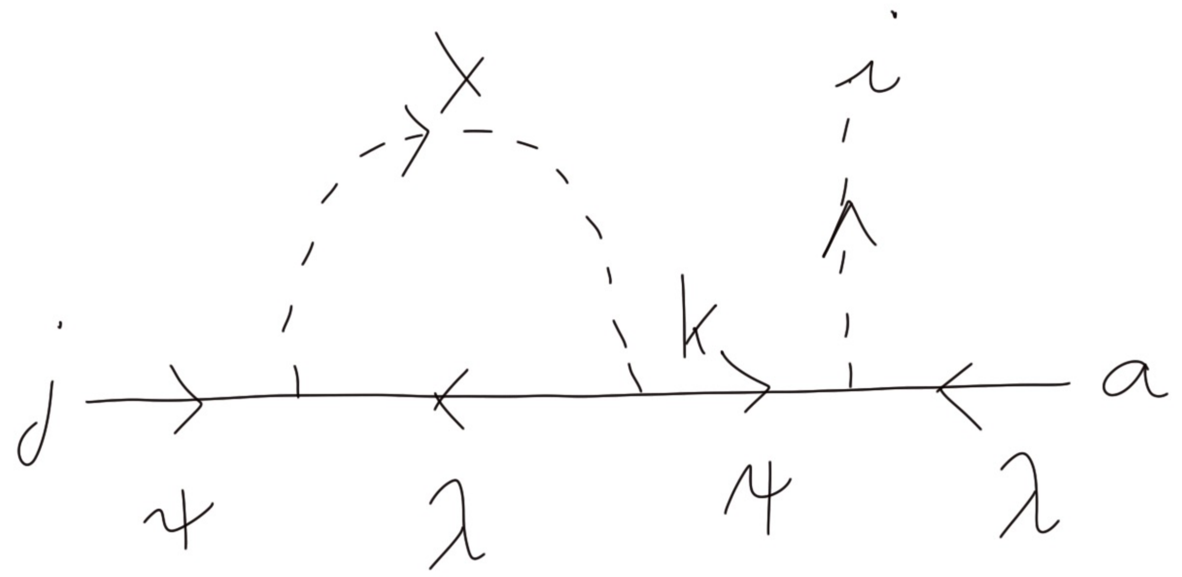


Beta function for Yukawa couplings

- The other diagrams generate various group-theoretical factors through contractions of the gauge generators
- For an irreducible representation $\mathbf{r}$ of the gauge group, we use the notation

$$\mathrm{tr}_{\mathbf{r}}(t_a\, t_b) = T(\mathbf{r})\,\delta_{ab} \qquad\qquad (t_a\, t_a)^i{}_j = C_2(\mathbf{r})\,\delta^i{}_j$$

- Compared to the 1-loop beta function of the gauge coupling (that only contains T's) some of the diagrams generate $C_2(\mathbf{r})$ factors in the 1-loop beta function for Yukawa couplings. For example:



$$(t_b t_b)^k{}_j\,(t_a)^i{}_k \to C_2(r)(t_a)^i{}_j$$

- At the end, however, the numerical factors among the various diagrams conspire in such a way that all $C_2(\mathbf{r})$ factors can be reabsorbed and recast in terms of $T(\mathbf{r})$ factors

Beta function for Yukawa couplings

- In conclusion, one finds that:
  1. The index structure of the Yukawa couplings is preserved, in the sense that $\beta_{Y^A_{xy}}$ has the same index structure as Y^A_{xy}, roughly
  $$\beta_{Y^A_{xy}} = g^3\,(\text{coeff})\,Y^A_{xy}$$
  2. The numerical coefficient in front of the tensor structure matches exactly the 1-loop coefficient of the beta function for the gauge coupling
- This means that 1-loop effects do not spoil the special relations among couplings that we have at tree level

Beta function for (scalar)^4 couplings

- Similar results are found in the analysis of the 1-loop beta function for (scalar)^4 couplings
- In a SUSY gauge theory with gauge group $SU(N_c)$ and fund/antifund matter we can write down two distinct index structures for terms in the scalar potential

$$\delta^{ab}\,[\overline{X}_i\,(t_a)^i{}_j\,X^j][\overline{X}_k\,(t_b)^k{}_\ell\,X^\ell] \qquad \text{or} \qquad [\overline{X}_i\,\delta^i{}_j\,X^j][\overline{X}_k\,\delta^k{}_\ell\,X^\ell]$$

(For other gauge groups and representations we might have more options)

- One has to verify that
  1. The coefficient for the structure $[\overline{X}_i\,\delta^i{}_j\,X^j][\overline{X}_k\,\delta^k{}_\ell\,X^\ell]$ in the (scalar)^4 1-loop beta function is zero (in this way this SUSY breaking coupling is not generated at 1-loop)
  2. The coefficient for the structure $\delta^{ab}\,[\overline{X}_i\,(t_a)^i{}_j\,X^j][\overline{X}_k\,(t_b)^k{}_\ell\,X^\ell]$ in the (scalar)^4 1-loop beta function matches exactly with the 1-loop coefficient in the gauge coupling beta function
- Both these properties are satisfied, but only thanks to cancellations among different diagrams

Supersymmetry and supergravity

Lecture 24

Holomorphic gauge coupling

- We have seen that the 1-loop beta function for the gauge coupling in a SUSY gauge theory has the form

$$\beta(g) = -\frac{g^3}{16\,\pi^2}\,b$$

where the 1-loop coefficient b receives contributions both from the vector multiplet and the chiral multiplets

vector multiplet: $b = 3\,T(\mathrm{adj})$

chiral multiplet in rep $\mathbf{r}$: $b = -\,T(\mathbf{r})$

- Can we use the power of SUSY/holomorphy to to beyond 1-loop?
- Yes, but we have to be careful about notion of gauge coupling we analyze

Holomorphic gauge coupling

- In order to use the power of holomorphy, we need to use the holomorphic gauge coupling
- This is defined by writing the SYM term with an overall $1/g^2$, combining the kinetic term with the theta term in the holomorphic combination τ
- In superspace:

$$S_{\text{SYM}} = \int d^4x \, d^2\theta \, \frac{-i\,\tau}{16\pi T(\mathbf{R})} \, \mathrm{Tr}_{\mathbf{R}} \, (\mathscr{W}^{\alpha} \mathscr{W}_{\alpha}) + \mathrm{h.c.}$$

$$\mathrm{Tr}_{\mathbf{R}} \, (t_a \, t_b) = T(\mathbf{R}) \, \delta_{ab} \ , \qquad \tau = \frac{\theta}{2\pi} + i \, \frac{4\pi}{g^2}$$

- In components:

$$\mathscr{L}_{\text{SYM}} = \delta_{ab} \left[-\frac{1}{4\,g^2} F^{a\mu\nu} F^b_{\mu\nu} + \frac{1}{2\,g^2} D^a D^b - \frac{1}{g^2} i \, \lambda^a \sigma^\mu D_\mu \bar{\lambda}^b + \frac{1}{64\pi^2} \theta \, \epsilon_{\mu\nu\rho\sigma} F^{a\mu\nu} F^{b\rho\sigma} \right]$$

- NB: the gauge fields are <u>not</u> canonically normalized

Holomorphic gauge coupling

- The coupling g runs with the renormalization scale μ according to the beta function

$$\mu \frac{dg}{d\mu} = \beta = -\frac{b}{16\pi^2} g^3 \qquad \text{at 1-loop}$$

- This is an ODE for $g(\mu)$ which we can solve as

$$\frac{1}{g(\mu)^2} = -\frac{b}{8\pi^2} \log \frac{\Lambda_{\mathbb{R}}}{\mu} \qquad \text{(1-loop running)}$$

- The integration constant $\Lambda_{\mathbb{R}}$ is a real, positive dimensionful parameter: the intrinsic scale of the non-Abelian gauge theory
- Because of the non-trivial beta function, a classical dimensionless parameter g is traded for a dimensionful scale $\Lambda_{\mathbb{R}}$: this is the phenomenon of **dimensional transmutation**
- NB: this is a feature of QFT in general, not specific to SUSY. For example, in real-world QCD the intrinsic scale is approx $\Lambda_{\mathbb{R}} \approx 200$ MeV (see for instance Peskin Schroeder Section 17.2)

Holomorphic gauge coupling

- The theta angle is the coefficient of a topological term in the action. It does not run when we change the renormalization scale μ
- We can combine the running $g(\mu)$ with the theta angle to define a running holomorphic coupling

$$\tau_{\text{1-loop}} = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g(\mu)^2} \quad , \qquad \frac{1}{g(\mu)^2} = -\,\frac{b}{8\pi^2}\,\log\frac{\Lambda_{\mathbb{R}}}{\mu}$$

- We can rewrite this quantity as

$$\tau_{\text{1-loop}}(\mu) = \frac{b}{2\pi i}\,\log\frac{\Lambda}{\mu}$$

where we have introduced the holomorphic version of the real intrinsic scale $\Lambda_{\mathbb{R}}$,

$$\Lambda := \Lambda_{\mathbb{R}}\, e^{i\theta/b}$$

Wilsonian effective action and τ

- We are mainly interested in theories that are asymptotically free:
  - the gauge coupling goes to zero at high energies
  - as we lower the energy scale, the coupling grows stronger and stronger
- We can safely do perturbation theory in the UV. In a renormalizable model all UV divergences can be reabsorbed. The gauge theory can be defined without reference to a UV completion
- At lower energies, perturbation theory breaks down, and non-perturbative effects can become important
- Problem: Integrate out the high-energy modes above some scale μ, $E > \mu$; what can we say about the gauge coupling in the Wilsonian effective action for energies $E < \mu$?

Wilsonian effective action and τ

- The Wilsonian effective action is well-defined and does not suffer from IR ambiguities/pathologies
- Since SUSY is unbroken, the Wilsonian effective action must fit into the general structure of a SUSY gauge theory

$$S_{\text{eff}} = \int d^4x \, d^2\theta \, \frac{-i\,\tau_{\text{eff}}}{16\pi} \delta_{ab} \mathscr{W}^{a\alpha} \mathscr{W}^b_\alpha + \text{h.c.}$$

- In the UV gauge theory τ is just a constant. In the IR, the effective τ can depend on Λ (or rather Λ/μ for dimensional reasons). However, SUSY of the low-energy Wilsonian action only allows dependence on Λ/μ and not on its complex conjugate (holomorphy)
- We have already found the 1-loop expression for τ. Let us therefore parametrize the full τ_{eff} as a sum of two terms:

$$\tau_{\text{eff}} = \frac{b}{2\pi i} \log \frac{\Lambda}{\mu} + f\left(\frac{\Lambda}{\mu}\right)$$

where f is an unspecified holomorphic function

Aside: shifts of the theta angle and instantons

- Let us recall how the theta angle enters the action:

$$S_\theta = \theta \int d^4x\, \delta_{ab}\, \frac{1}{64\pi^2}\, \epsilon_{\mu\nu\rho\sigma}\, F^{a\mu\nu}\, F^{b\rho\sigma} \quad , \qquad S_\theta = \theta\, n \quad , \qquad n = \int d^4x\, \delta_{ab}\, \frac{1}{64\pi^2}\, \epsilon_{\mu\nu\rho\sigma}\, F^{a\mu\nu}\, F^{b\rho\sigma}$$

- The integrand in the expression of n can be written as a total derivative:

$$\frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma}\, \delta_{ab}\, T(\mathbf{r})\, F^{a\mu\nu}\, F^{b\rho\sigma} = \partial_\mu \mathrm{Tr}_{\mathbf{r}}\Big(4\, \epsilon^{\mu\nu\rho\sigma}\, A_\nu\, \partial_\rho A_\sigma + \frac{2}{3}\, A_\nu\, A_\rho\, A_\sigma\Big)$$

- This is why the theta term does not change the EOMs and does not affect the perturbative expansion
- Caveat: The object inside ∂_μ is only locally defined. We cannot apply Stokes' theorem and conclude that n is zero!
- Indeed, n turns out to be an integer, the so-called **instanton number**
- In performing the path integral, we have to sum over various sectors, with all possible instanton numbers. Usual perturbation theory is done around $A^a_\mu = 0$ and is implicitly done in the $n = 0$ sector

Aside: shifts of the theta angle and instantons

$$S_\theta = \theta\, n \quad , \qquad n = \int d^4x\, \delta_{ab}\, \frac{1}{64\pi^2}\, \epsilon_{\mu\nu\rho\sigma}\, F^{a\mu\nu}\, F^{b\rho\sigma}$$

- The action enters the path integral via e^{iS}. The theta term thus contributes a phase factor $e^{in\theta}$. We see that $\theta \to \theta + 2\pi$ is a symmetry
- Another point of view:
    - in perturbation theory: the theta term has no effect; we can freely shift θ by any amount, and the perturbation expansion does not change
    - including non-perturbative instanton effects: we can no longer shift θ by an arbitrary amount, but we can still perform discrete shifts by integer multiples of 2π

Wilsonian effective action and τ

$$\tau_{\text{eff}} = \frac{b}{2\pi i}\,\log\frac{\Lambda}{\mu} + f\Big(\frac{\Lambda}{\mu}\Big)$$

- Recall the definition $\Lambda := \Lambda_{\mathbb{R}}\, e^{i\theta/b}$. A shift $\theta \to \theta + 2\pi$ is equivalent to a phase rotation $\Lambda \to \Lambda\, e^{2\pi i/b}$
- In the 1-loop term, $\Lambda \to \Lambda\, e^{2\pi i/b}$ implies $\frac{b}{2\pi i}\,\log\frac{\Lambda}{\mu} \to \frac{b}{2\pi i}\,\log\frac{\Lambda}{\mu} + 2\pi$. This is exactly the behavior that we want for the holomorphic gauge coupling
- This means that under $\Lambda \to \Lambda\, e^{2\pi i/b}$ the quantity $f(\Lambda/\mu)$ must be invariant. It can depend on Λ only via Λ^b
- The weak coupling limit is $\Lambda \to 0$ (notice that as $g \to 0^+$, in our conventions $\tau \to +i\infty$). The correction term $f(\Lambda/\mu)$ must have a regular expansion around $\Lambda = 0$
- Final expression:

$$\tau_{\text{eff}} = \frac{b}{2\pi i}\,\log\frac{\Lambda}{\mu} + \sum_{n=1}^{\infty} a_n \Big(\frac{\Lambda^b}{\mu^b}\Big)^n$$

Wilsonian effective action and τ

$$\tau_{\text{eff}} = \frac{b}{2\pi i}\,\log\frac{\Lambda}{\mu} + \sum_{n=1}^{\infty} a_n \left(\frac{\Lambda^b}{\mu^b}\right)^n$$

- The definition of Λ and the expression of the 1-loop $g(\mu)$ imply

$$\frac{1}{g(\mu)^2} = -\frac{b}{8\pi^2}\,\log\frac{\Lambda_{\mathbb{R}}}{\mu}\ , \qquad \Lambda := \Lambda_{\mathbb{R}}\,e^{i\theta/b}\ , \qquad \frac{\Lambda^b}{\mu^b} = e^{-8\pi^2/g(\mu)^2}\,e^{i\theta}$$

- Notice the exponential suppression e^{-1/g^2}: these effects are invisible in perturbation theory!
- Interpretation:
    - The holomorphic gauge coupling function is renormalized at 1-loop
    - It receives no other corrections in perturbation theory
    - It can receive non-perturbative corrections

The coefficients a_n are in principle well-defined, but extremely hard to compute. They have been computed by Seiberg and Witten in some 4d $\mathcal{N} = 2$ SUSY gauge theories

The NSVZ beta function

- There is another important formula for the beta function of a 4d $\mathcal{N} = 1$ SUSY gauge theory
- It is not the beta function for the holomorphic gauge coupling, but rather for the canonical gauge coupling. We have to write the action as

$$\mathcal{L}_{\text{SYM}} = \delta_{ab}\left[-\frac{1}{4}F^{a\mu\nu}F^{b}_{\mu\nu} + \frac{1}{2}D^a D^b - i\,\lambda^a\,\sigma^\mu\,D_\mu\bar{\lambda}^b + \frac{1}{64\pi^2}\,\theta\,\epsilon_{\mu\nu\rho\sigma}F^{a\mu\nu}F^{b\rho\sigma}\right]$$

with factors of g inside the field strength and the covariant derivatives

- Using techniques based on instanton methods, Novikov, Shifman, Vainshtein, and Zakharov proved the NSVZ formula for the "exact" beta function:

$$\beta_{\text{NSVZ}}(g) = -\frac{g^3}{16\pi^2}\,\frac{3\,T(\text{adj}) - \sum_i T(\mathbf{r}_i)\,(1-\gamma_i)}{1 - T(\text{adj})\,g^2/(8\pi^2)}$$

- The derivation goes beyond the scope of these lectures

The NSVZ beta function

$$\beta_{\rm NSVZ}(g) = -\frac{g^3}{16\pi^2}\,\frac{3\,T(\mathrm{adj}) - \sum_i T(\mathbf{r}_i)\,(1-\gamma_i)}{1 - T(\mathrm{adj})\,g^2/(8\pi^2)}$$

- We have already encountered the group-theoretical constants $\mathrm{Tr}_{\rm r}(t_a\,t_b) = T(\mathbf{r})\,\delta_{ab}$
- The sum is over all matter chiral superfields
- The quantity γ_i is the “anomalous dimension” of the i-th matter chiral superfield Φ^i. It is defined in terms of the wavefunction renormalization factor

$$\Phi^i_{\rm bare} = (Z^i)^{1/2}\,\Phi^i_{\rm r}\ \ \text{(no sum on } i\text{)} \quad , \qquad \gamma^i = \frac{1}{2}\,\frac{\partial \log Z^i}{\partial \log \mu}$$

- Recall that SUSY allows wavefunction renormalization for chiral superfields
- NB: the anomalous dimensions γ_i are themselves non-trivial functions of g, so the NSVZ formula is not fully explicit!

The NSVZ beta function

$$\beta_{\mathrm{NSVZ}}(g) = -\frac{g^3}{16\pi^2}\frac{3\,T(\mathrm{adj}) - \sum_i T(\mathbf{r}_i)\,(1-\gamma_i)}{1 - T(\mathrm{adj})\,g^2/(8\pi^2)}$$

- If one expands the beta function in powers of g, the coefficients of the 1-loop and 2-loop terms are universal: they do not depend on the regularization and renormalization scheme
- The NSVZ beta function has been tested at 2-loops in several examples
- Beyond the 2-loop coefficient, the beta function coefficients start to be scheme dependent. So $\beta_{\mathrm{NSVZ}}(g)$ is indeed exact to all orders in perturbation theory, but in a particular scheme which is not known independently in closed form

The NSVZ beta function

$$\beta_{\mathrm{NSVZ}}(g) = -\frac{g^3}{16\pi^2}\,\frac{3\,T(\mathrm{adj}) - \sum_i T(\mathbf{r}_i)\,(1-\gamma_i)}{1 - T(\mathrm{adj})\,g^2/(8\pi^2)}$$

- The NSVZ beta function is particularly useful in arguing for IR fixed points of the RG flow
- In some favorable cases, one can deduce what the anomalous dimensions at putative fixed point must be (this is usually done using properties of the superconformal algebra)
- Armed with the knowledge of γ^i, one can verify that the numerator of the NSVZ formula is zero, and thus be sure that the beta function is zero to all orders in perturbation theory
- We will not see explicit examples of these techniques because they would require us to develop several new tools…

Holomorphy vs NSVZ

To recap:

- The holomorphic gauge coupling function is corrected at 1-loop, plus possibly by non-perturbative effects. In particular, its 2-loop coefficient is zero
- The canonical gauge coupling function has a beta function given by NSVZ. It predicts a non-zero 2-loop coefficient
- Recall that the 1-loop and 2-loop coefficients are scheme-independent

Question: What is the relation between the holomorphic and the canonical gauge couplings? It must explain why the former does not receive corrections at 2 loops, while the latter does

Holomorphy vs NSVZ

- In the classical theory, all we need to move from the holomorphic to the canonical framework is a simple rescaling of the gauge field:

$$\frac{1}{4g^2}\,\delta_{ab}\,F^a_{\mu\nu}\,F^{b\mu\nu} = \frac{1}{4}\,\delta_{ab}\,F^a_{\mu\nu\mathrm{c}}\,F^{b\mu\nu}{}_{\mathrm{c}} \quad , \quad A^a_\mu = g\,A^a_{\mu\,\mathrm{c}} \quad , \quad F^a_{\mu\nu} = g_{\mathrm{c}}\,F^a_{\mu\nu\mathrm{c}}\,,$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f_{bc}{}^a\,A^b_\mu\,A^c_\nu\,, \qquad F^a_{\mu\nu\mathrm{c}} = \partial_\mu A^a_{\nu\,\mathrm{c}} - \partial_\nu A^a_{\mu\,\mathrm{c}} - g_{\mathrm{c}} f_{bc}{}^a\,A^b_{\mu\,\mathrm{c}}\,A^c_{\nu\,\mathrm{c}}$$

- It turns out that this operation is not so innocent in the quantum theory
- The change of variables $A^a_\mu = g\,A^a_{\mu\,\mathrm{can}}$ in the path integral has a non-trivial Jacobian
- If matter chiral superfields are present, they also contribute to the Jacobian in the path integral measure
- This effect is related to quantum anomalies (which we will discuss briefly later)

Holomorphy vs NSVZ

- Recall that in our conventions the holomorphic coupling is $\tau = \frac{\theta}{2\pi} + i\,\frac{4\pi}{g^2}$ and therefore $\mathrm{Re}\frac{\tau}{4\pi i} = \frac{1}{g^2}$
- A careful analysis of the Jacobian reveals the relation between the holomorphic and canonical couplings

$$\frac{1}{g_{\mathrm{c}}^2} = \mathrm{Re}\,\frac{\tau}{4\pi i} - \frac{1}{8\pi^2}\left[2\,T(\mathrm{adj})\,\log g_{\mathrm{c}} - \sum_i T(\mathbf{r}_i)\,\log Z^i\right] \qquad (*)$$

- We know that the holomorphic coupling satisfies $\tau_{\text{1-loop}}(\mu) = \frac{b}{2\pi i}\,\log\frac{\Lambda}{\mu}$ and therefore (omitting the 1-loop label)

$$\frac{d\tau}{d\log\mu} = -\frac{b}{2\pi i}\ , \qquad \frac{d}{d\log\mu}\mathrm{Re}\,\frac{\tau}{4\pi i} = \frac{b}{8\pi^2} = \frac{3\,T(\mathrm{adj}) - \sum_i T(\mathbf{r}_i)}{8\pi^2}$$

- We also know that $\frac{d\log Z^i}{d\log\mu} = 2\,\gamma^i$. Taking the derivative of (*) wrt $\log\mu$, we get an equation for $\frac{dg_{\mathrm{c}}}{d\log\mu}$ in terms of g_{c}, γ^i, and group theoretical constant. Solving for $\frac{dg_{\mathrm{c}}}{d\log\mu}$ one recovers the NSVZ formula

Supersymmetry and supergravity

Lecture 25

SUSY gauge theories and anomalies

- Let us consider a SUSY gauge theory without superpotential. It is specified by:
    1. a choice of gauge group G
    2. a collection of chiral superfields $\Phi^i = (X^i, \psi^i_\alpha, F^i)$ in some representation of the gauge group (possibly reducible)
- At the level of the classical action, any choice of gauge group and representation is allowed and gives a SUSY Lagrangian
- At the quantum level, some choices can be inconsistent due to anomalies

Chiral gauge theories

- A gauge theory is called non-chiral if, for every positive-chirality Weyl fermion in a representation $\mathbf{r}$ of the gauge group G, there is another positive-chirality Weyl fermion in the representation $\overline{\mathbf{r}}$
- In a non-chiral gauge theory the total representation of the Weyl fermions is necessarily real, because it is of the form $\mathbf{R} = \oplus_i \left(\mathbf{r}_i \oplus \overline{\mathbf{r}}_i\right)$
- A non-chiral gauge theory can be formulated in terms of 4-component Dirac spinors with “vector” couplings to the gauge fields (i.e. without any factor of the chirality matrix γ_5). Examples include QED and QCD

Chiral gauge theories

- A gauge theory is called chiral if the positive-chirality Weyl fermion do not appear up in pairs $(\mathbf{r}, \bar{\mathbf{r}})$. The representation $\mathbf{R}$ can be complex
- Chiral gauge theories violate parity (but preserve CP)
- The SM is an example of chiral gauge theory

Gauge anomalies

- For some choice of gauge group G and representation $\mathbf{R}$ the gauge theory is inconsistent because of a (perturbative) gauge anomaly
- The anomaly arises because the action is invariant under a gauge transformation, but the fermion measure $\mathscr{D}\psi\mathscr{D}\overline{\psi}$ in the path integral is not invariant
- In perturbation theory, gauge anomalies arise from triangle diagrams with three external gauge fields
- The condition for the cancellation of gauge anomalies is

$$\mathrm{Tr}_{\mathbf{R}}\left(t_{(a}\,t_b\,t_{c)}\right) = 0$$

- $\mathbf{R}$ is the total representation of the Weyl fermions (usually reducible). The adjoint indices a, b, c run over all generators of the gauge group G (which can contain both Abelian factors and simple non-Abelian factors)

Gravitational anomalies

- If we couple a Weyl spinor ψ_α both to a gauge field and to a non-trivial spacetime metric, the measure $\mathscr{D}\psi\,\mathscr{D}\overline{\psi}$ has a non-zero anomalous variation. We cannot make it invariant under both gauge transformations and diffeomorphisms. We have a <u>mixed gauge-gravitational</u> anomaly
- If we are only interested in studying QFT in a rigid, flat spacetime, we can allow a gravitational anomaly. The gauge theory is still consistent as a QFT
- If we want to couple the QFT to dynamical gravity, the gravitational anomaly must be canceled. The condition for this to happen is

$$\mathrm{Tr}_{\mathbf{R}}\, t_a = 0$$

Witten's $SU(2)$ anomaly

- There is another, more subtle, anomaly that can destroy the consistency of a gauge theory in which the gauge group contains an $SU(2)$ factor
- Witten proved that:

a gauge theory with gauge group $SU(2)$ and an <u>odd number</u> of Weyl fermions in the fundamental rep (the doublet) is inconsistent

- This anomaly cannot be seen in perturbation theory/triangle diagrams. It originates from an ambiguity in the sign of the measure $\mathscr{D}\psi\,\mathscr{D}\overline{\psi}$ under "large $SU(2)$ gauge transformations".
- There is a generalization of Witten's anomaly to other representations of $SU(2)$, as well as to the gauge groups $USp(2N)$ for some choices of representations. NB: $USp(2) \cong SU(2)$
- Witten's anomaly is rooted in the topological fact that $\pi_4(USp(2N)) \cong \mathbb{Z}_2$

Example: the SM is anomaly-free

- All perturbative gauge anomalies in the SM cancel, and the cancellation is non-trivial because the electroweak sector is chiral
- The SM is free from Witten's $SU(2)$ anomaly, because it contains an even number of $SU(2)$ gauge doublets: three generations of left-handed quarks Q^i and three generations of left-handed leptons L^i ($i = 1,2,3$)
- All mixed gauge-gravitational anomalies cancel. This is good news, because we need to couple the SM fields to gravity

A different kind of anomaly: ABJ anomaly

- Let us consider a gauge theory with gauge group G (generally consisting of both Abelian and non-Abelian factors) and a collection of <u>massless</u> Weyl fermions in a representation $\mathbf{R}$ of G (generally reducible)
- Suppose that G and the representation have been chosen in such a way that the model is free of gauge anomalies and Witten's $SU(2)$ anomaly
- We now consider a <u>global</u> symmetry of this system (as opposed to a gauge symmetry) and show that, while this global symmetry holds at the classical level, it is destroyed by a quantum anomaly

A different kind of anomaly: ABJ anomaly

- The classical Lagrangian

$$\mathscr{L}_{\psi} = -\, i\, \overline{\psi}_i\, \overline{\sigma}^{\mu}\, D_{\mu}\psi^i \quad , \qquad\qquad D_{\mu}\psi^i = \partial_{\mu}\psi^i + i\, A^a_{\mu}\, (t_a)^i{}_j\, \psi^j$$

is invariant under a $U(1)$ <u>global</u> symmetry that rotates all ψ's with the same phase: infinitesimally

$$\delta\psi^i_{\alpha} = i\, \omega_{U(1)}\, \psi^i_{\alpha}$$

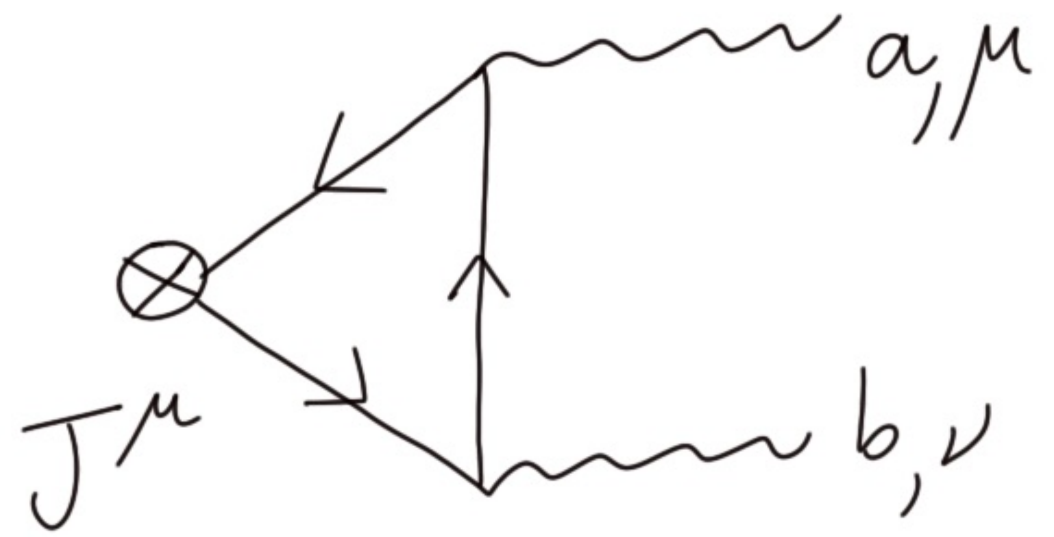
- The Noether current associated to this $U(1)$ global symmetry is

$$J^{\mu} = \overline{\psi}_i\, \overline{\sigma}^{\mu}\, \psi^i$$

A different kind of anomaly: ABJ anomaly

- In the classical theory $\partial_\mu J^\mu = 0$. In the quantum theory the divergence is non-zero due to the **Adler–Bell–Jackiw** (ABJ) anomaly

$$\partial_\mu J^\mu = \frac{1}{32\pi^2}\, \epsilon^{\mu\nu\rho\sigma}\, \mathrm{Tr}\,(F_{\mu\nu}\, F_{\rho\sigma})$$



- This effect arises from 1-loop triangle diagrams with one insertion of the operator J^μ and two dynamical gauge bosons
- The quantity $\mathrm{Tr}\,(F_{\mu\nu}\, F_{\rho\sigma})$ is an operator in the gauge theory because we are performing the path integral over A_μ^a. The non-conservation of the current holds as an operator equation inside correlation functions. We cannot simply turn off A_μ^a as if it were a background field
- Conclusion: the $U(1)$ global symmetry $\delta\psi_\alpha^i = i\,\omega_{U(1)}\,\psi_\alpha^i$ is a symmetry of the classical theory, but it is explicitly broken by quantum effects in the quantum theory

A different kind of anomaly: ABJ anomaly

Caveats:

- The fact that a global $U(1)$ symmetry suffers from an ABJ anomaly does not render the gauge theory inconsistent
- The ABJ anomaly is 1-loop exact: it is not corrected by higher loops or by non-perturbative effects
- The ABJ anomaly is "additive". Our formula gives the contribution of one Weyl fermion of charge $+1$ under the global $U(1)$. For the contribution of one Weyl fermion of charge q, simply insert a prefactor q. The contributions of various fermions are added up. We can write

$$\partial_\mu J^\mu = \frac{1}{32\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\sum_f q_f\,T(\mathbf{r}_f)\,\delta_{ab}\,F^a_{\mu\nu}\,F^b_{\rho\sigma}$$

(sum over pos.-chirality Weyl fermions)

- Recall the def. $\mathrm{Tr}_{\mathbf{r}_f}(t_a\,t_b) = T(\mathbf{r}_f)\,\delta_{ab}$

A different kind of anomaly: ABJ anomaly

$$\partial_\mu J^\mu = \frac{1}{32\pi^2}\,\epsilon^{\mu\nu\rho\sigma} \sum_f q_f\, T(\mathbf{r}_f)\,\delta_{ab}\,F^a_{\mu\nu}\,F^b_{\rho\sigma}$$

A technical point:

- In most non-SUSY applications, the formula for the ABJ anomaly is stated in terms of <u>Dirac</u> fermions running in the loop (as opposed to Weyl). For one Dirac fermion of charge $+1$ under the global $U(1)$, the prefactor $\frac{1}{32\pi^2}$ turns into $\frac{1}{16\pi^2}$
- For applications to SUSY we need the formula in its "Weyl version" with $\frac{1}{32\pi^2}$, because the gaugino λ^a does not have any "partner" $\widetilde{\lambda}^a$ to be paired up in a Dirac 4-component fermion

ABJ anomaly and theta angle

- The ABJ anomaly can also be regarded as an effect that originates from the non-invariance of the path integral measure for chiral fermions under a global $U(1)$ transformation
- By a careful treatment of the path integral measure (which needs to be suitably regularized to be computed), one can prove that

$$\mathscr{D}\psi' \, \mathscr{D}\overline{\psi}' = \mathscr{D}\psi \, \mathscr{D}\overline{\psi} \, \exp\left[-i\,\omega_{U(1)} \int d^4x \, \frac{1}{32\pi^2} \, \epsilon^{\mu\nu\rho\sigma} \sum_f q_f \, T(\mathbf{r}_f) \, \delta_{ab} \, F^a_{\mu\nu} \, F^b_{\rho\sigma} \right]$$

- The notation is a bit schematic: $\mathscr{D}\psi \, \mathscr{D}\overline{\psi}$ stands for the path integral measure on all the Weyl fermions labeled by f on the RHS (and their complex conjugates)

ABJ anomaly and theta angle

$$\mathscr{D}\psi' \, \mathscr{D}\overline{\psi}' = \mathscr{D}\psi \, \mathscr{D}\overline{\psi} \; \exp\left[-\, i\, \omega_{U(1)} \int d^4x \, \frac{1}{32\pi^2} \, \epsilon^{\mu\nu\rho\sigma} \sum_f q_f \, T(\mathbf{r}_f) \, \delta_{ab} \, F^a_{\mu\nu} \, F^b_{\rho\sigma} \right] \quad \text{(*)}$$

- Notice that the anomaly in the path integral measure is equivalent to a change in the action. Indeed, the path integrand is $\mathscr{D}\psi \, \mathscr{D}\overline{\psi} \, e^{iS}$, and we see that the net effect of (*) is the same as a shift in the action:

$$S \to S' = S - \omega_{U(1)} \int d^4x \, \frac{1}{32\pi^2} \, \epsilon^{\mu\nu\rho\sigma} \sum_f q_f \, T(\mathbf{r}_f) \, \delta_{ab} \, F^a_{\mu\nu} \, F^b_{\rho\sigma}$$

ABJ anomaly and theta angle

$$S \to S' = S - \omega_{U(1)} \int d^4x \, \frac{1}{32\pi^2} \, \epsilon^{\mu\nu\rho\sigma} \sum_f q_f \, T(\mathbf{r}_f) \, \delta_{ab} \, F^a_{\mu\nu} \, F^b_{\rho\sigma} \quad (*)$$

- There is another term in the action with a similar structure: the theta angle term

$$S_\theta = \int d^4x \, \frac{\theta}{64\pi^2} \, \epsilon^{\mu\nu\rho\sigma} \, \delta_{ab} \, F^a_{\mu\nu} \, F^b_{\rho\sigma}$$

- Formally, we can compensate the shift (*) if we promote the constant θ to a **spurion**, i.e. a background field that also transforms. More precisely, in order to cancel (*) we need the shift

$$\theta \to \theta' = \theta + 2\,\omega_{U(1)} \sum_f q_f \, T(\mathbf{r}_f)$$

ABJ anomaly and theta angle

To summarize:

- If a global $U(1)$ symmetry suffers from an ABJ anomaly, it is explicitly broken by quantum effects
- Nonetheless, we can formally restore this global $U(1)$ symmetry if we use a spurion
- The spurion is the theta angle:

$$\theta \to \theta' = \theta + 2\,\omega_{U(1)} \sum_f q_f \, T(\mathbf{r}_f)$$

R-symmetry in SUSY gauge theories

- A $U(1)_R$ R-symmetry acts on the fields in a chiral multiplet and on the SUSY parameter as follows:

field/param	X^i	ψ^i_α	F^i	ξ
charge	$R[X^i]$	$R[X^i]-1$	$R[X^i]-2$	1

$$\delta X^i = \sqrt{2}\,\xi\,\psi^i$$

$$\delta\psi^i_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar\xi)_\alpha\,\partial_\mu X^i + \sqrt{2}\,F^i\,\xi_\alpha$$

$$\delta F^i = i\sqrt{2}\,\bar\xi\,\bar\sigma^\mu\,\partial_\mu\psi^i$$

- For a vector multiplet, one has

field/param	A_μ	λ_α	D	ξ
charge	0	1	0	1

$$\delta A_\mu = i\,\bar\xi\,\bar\sigma_\mu\,\lambda - i\,\bar\lambda\,\bar\sigma_\mu\,\xi$$

$$\delta\lambda_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F_{\mu\nu} + i\,D\,\xi_\alpha$$

$$\delta D = \bar\xi\,\bar\sigma^\mu\,\partial_\mu\lambda + \partial_\mu\bar\lambda\,\bar\sigma^\mu\,\xi$$

- Since A_μ and D are real fields, they must have charge 0. There is no freedom in the R-symmetry charge assigments in a vector multiplet

Some classical global symmetries of SQCD

- We already know that massless SQCD has gauge group $SU(N_c)$ and a flavor symmetry group $SU(N_f) \times SU(N_f)'$
- Let us now consider candidate global $U(1)$ symmetries. At the **classical** level, we have the following table

	$SU(N_c)$	$SU(N_f)$	$SU(N_f)'$	$U(1)_B$	$U(1)_A^{\text{cl}}$	$U(1)_R^{\text{cl}}$
$(\,Q^I{}_{\widehat{I}}\,,\,\psi^I{}_{\widehat{I}}\,)$	$(\square,\square)$	$(\overline{\square},\overline{\square})$	$(\bullet,\bullet)$	$(+1,+1)$	$(+1,+1)$	$(0,-1)$
$(\,\widetilde{Q}^{\widehat{I}'}{}_I\,,\,\widetilde{\psi}^{\widehat{I}'}{}_I\,)$	$(\overline{\square},\overline{\square})$	$(\bullet,\bullet)$	$(\square,\square)$	$(-1,-1)$	$(+1,+1)$	$(0,-1)$
$(\,A^a_\mu\,,\,\lambda^a\,)$	(adj,adj)	$(\bullet,\bullet)$	$(\bullet,\bullet)$	$(0,0)$	$(0,0)$	$(0,+1)$

- NB: we have chosen a classical reference R-symmetry; any linear combination of generators of the form $t_{U(1)_R^{\text{cl}}} + \alpha\, t_{U(1)_B} + \beta\, t_{U(1)_A^{\text{cl}}}$ is an equally good classical R-symmetry

Symmetries that survive the ABJ anomaly

- The symmetries $SU(N_f) \times SU(N_f)'$ and $U(1)_B$ do not suffer from ABJ anomalies
- The classical symmetries $U(1)_A^{\text{cl}}$ and $U(1)_R^{\text{cl}}$ are separately destroyed by ABJ anomalies

$$\text{Tr}\,(t_{U(1)_A^{\text{cl}}}\, t_a^{SU(N_c)}\, t_b^{SU(N_c)}) = (\psi) + (\widetilde{\psi}) + (\lambda) = (+1)\, N_f\, T(\square)\, \delta_{ab} + (+1)\, N_f\, T(\overline{\square})\, \delta_{ab} + (0)\, T(\text{adj}) = N_f$$

$$\text{Tr}\,(t_{U(1)_R^{\text{cl}}}\, t_a^{SU(N_c)}\, t_b^{SU(N_c)}) = (\psi) + (\widetilde{\psi}) + (\lambda) = (-1)\, N_f\, T(\square)\, \delta_{ab} + (-1)\, N_f\, T(\overline{\square})\, \delta_{ab} + (+1)\, T(\text{adj}) = -N_f + N_c$$

- A linear combination of the form $t_{U(1)_R^{\text{cl}}} + \beta\, t_{U(1)_A^{\text{cl}}}$ is free of ABJ anomalies: we have to take

$$\beta = (N_f - N_c)/N_f$$

Symmetries that survive the ABJ anomaly

- The table below summarizes the global continuous symmetries of the quantum theory
- It is also useful, however, to "recycle" the classical symmetry $U(1)_A^{\rm cl}$. We now know that it is explicitly broken by the ABJ anomaly, but we also know that we can formally restore it if we treat the theta angle as a spurion $\theta \to \theta' = \theta + 2\,\omega_{U(1)} \sum_f q_f\, T(\mathbf{r}_f)$

	$SU(N_c)$	$SU(N_f)$	$SU(N_f)'$	$U(1)_B$	$U(1)_R$	$U(1)_A^{\rm cl}$
$(\,Q^I{}_{\widehat{I}}\,,\,\psi^I{}_{\widehat{I}}\,)$	$(\square,\square)$	$(\overline{\square},\overline{\square})$	$(\bullet,\bullet)$	$(+1,+1)$	$(1-\frac{N_c}{N_f},-\frac{N_c}{N_f})$	$(+1,+1)$
$(\,\widetilde{Q}^{\widehat{I}'}{}_I\,,\,\widetilde{\psi}^{\widehat{I}'}{}_I\,)$	$(\overline{\square},\overline{\square})$	$(\bullet,\bullet)$	$(\square,\square)$	$(-1,-1)$	$(1-\frac{N_c}{N_f},-\frac{N_c}{N_f})$	$(+1,+1)$
$(\,A^a_\mu\,,\,\lambda^a\,)$	$(\mathrm{adj},\mathrm{adj})$	$(\bullet,\bullet)$	$(\bullet,\bullet)$	$(0,0)$	$(0,+1)$	$(0,0)$
shift in θ	—	—	—	—	—	$\theta \to \theta + 2\,N_f\,\omega_{U(1)}$

Supersymmetry and supergravity

Lecture 26

SUSY breaking overview

- Spontaneous SUSY breaking

  The theory is SUSY invariant, but the vacuum is not

  - Tree-level SUSY breaking

    The classical Lagrangian is SUSY invariant, but has no SUSY vacua

  - Dynamical SUSY breaking

    The classical Lagrangian is SUSY invariant and has a SUSY vacuum; non-perturbative effects lift the vacuum and break SUSY spontaneously

- Explicit SUSY breaking

  We introduce terms in the classical Lagrangian that are not SUSY invariant

  - Soft SUSY breaking

    The SUSY-breaking terms are chosen so that the cancellation of quadratic divergences persists

Spontaneous SUSY breaking

Some general remarks that hold both for tree-level breaking and dynamical breaking:

- The vacuum energy is the order parameter for spontaneous SUSY breaking. From the SUSY algebra ($H = P^{\mu=0}$)

$$4\,H = Q_1\,(Q_1)^\dagger + (Q_1)^\dagger\,Q_1 + Q_2\,(Q_2)^\dagger + (Q_2)^\dagger\,Q_2$$

$$\text{SUSY is unbroken} \quad \Leftrightarrow \quad \begin{cases} Q_1\,|0\rangle = 0 \\ Q_2\,|0\rangle = 0 \end{cases} \quad \Leftrightarrow \quad \langle 0\,|\,H\,|\,0\rangle$$

- We are only interested in vacua the do no break Poincaré symmetry: only scalar fields can get non-zero VEVs, and the VEVs are constant in spacetime
- The vacuum energy is the same as the value of the scalar potential at the vacuum

Spontaneous SUSY breaking

- The scalar potential of a SUSY model is of the form

$$V = V_F + V_D$$

  where both V_F and V_D are separately non-negative. In order to have zero vacuum energy, they have to be zero separately

- In a renormalizable model, the kinetic terms are canonical and the gauge coupling function is a constant

$$\mathscr{L} \supset -\delta_{i\bar{j}}\, D_\mu X^i\, D^\mu \overline{X}^{\bar{j}} - \frac{1}{4}\,\delta_{ab}\, F^a_{\mu\nu}\, F^{b\mu\nu} + \dots$$

- In this case

$$V_F = \delta_{i\bar{j}}\, F^i\, \overline{F}^{\bar{j}} \quad , \quad V_D = \frac{1}{2}\,\delta^{ab}\, D_a\, D_b \qquad \text{where} \qquad F^i = -\,\delta^{i\bar{j}}\,\frac{\partial \overline{W}}{\partial \overline{X}^{\bar{j}}} \quad , \quad D_a = -\,g\,\overline{X}_i\,(t_a)^i{}_j\,X^j - p_a$$

- Here the constants p_a are FI terms. They are only allowed for $U(1)$ factors in the gauge group

Spontaneous SUSY breaking

- In a non-renormalizable model (which can emerge for example as a low-energy Wilsonian effective action) we still have $V = V_F + V_D$ with both terms separately non-negative
- The quantities V_F, V_D are now constructed in terms of the Kähler metric, the real part of the gauge coupling function, and the moment maps $\mathscr{P}_a$ (see Lecture 20)

$$V_F = G^{i\bar{j}} \frac{\partial W}{\partial \Phi^i} \frac{\partial \overline{W}}{\partial \overline{\Phi}^{\bar{j}}} \quad , \qquad V_D = \frac{1}{2} (\mathrm{Re} f)^{-1ab} \mathscr{P}_a \mathscr{P}_b$$

- It is still true that we have zero vacuum energy if and only if we satisfy all F-term conditions $\partial_{\Phi^i} W = 0$ and all D-term conditions $\mathscr{P}_a = 0$
- To have spontaneous SUSY breaking we must violate at least one of these conditions

Tree-level SUSY breaking

- Let us examine two examples of tree-level SUSY breaking:
  - O'Raifeartaigh models: F-term SUSY breaking
  - Gauge theories with FI parameters: D-term SUSY breaking

Example of O'Raifeartaigh model

- Let us study a renormalizable model with no vector superfields and three chiral superfields
- The Kähler potential is canonical; the superpotential is

$$W = -\kappa^2\,\Phi_1 + m\,\Phi_2\,\Phi_3 + \frac{1}{2}\,y\,\Phi_1\,\Phi_3^2$$

- For simplicity, we take the parameters κ, m, y to be real
- The scalar potential has only the V_F term; it reads

$$V_F = |\kappa^2 - \frac{1}{2}\,y\,\overline{X}_3^2|^2 + |m\,\overline{X}_3|^2 + |m\,\overline{X}_2 + y\,\overline{X}_1\,\overline{X}_3|^2$$

- The three quantities inside the absolute values are the three F-term conditions. For generic κ, m, y we cannot set all three F-terms to zero simultaneously

Example of O'Raifeartaigh model

$$V_F = |\kappa^2 - \frac{1}{2} y \overline{X}_3^2|^2 + |m \overline{X}_3|^2 + |m \overline{X}_2 + y \overline{X}_1 \overline{X}_3|^2$$

- We cannot find SUSY vacua, but the model still has non-SUSY vacua, which are determined by minimizing V_F
- If the mass parameter is large enough, the solution is

$$\text{minimize } V_F \ : \quad X_2 = 0 \ , \quad X_3 = 0 \ , \quad X_1 \text{ undetermined}$$

- The value of V_F at the minumum is $V_F = \kappa^4$
- At the <u>classical level</u>, the scalar potential has a flat direction (X_1). Giving different VEVs to X_1 we get inequivalent vacua (for example, the masses of the various particles in the model depend on the VEV $\langle X_1 \rangle$)

Example of O'Raifeartaigh model

- For example, if we choose $\langle X_1 \rangle = 0$ and we study small fluctuations, the mass spectrum of the theory is
  - (real) scalars: $0\,,\,0\,,\,m^2\,,\,m^2\,,\,m^2 - y\,\kappa^2\,,\,m^2 + y\,\kappa^2$
  - fermions: $0\,,\,m\,,\,m$
- These masses satisfy a so-called "sum rule"

$$\mathrm{Tr}(M^2_{\mathrm{scalars}}) = 2\,\mathrm{Tr}(M^2_{\mathrm{fermions}})$$

- Sum rules of this form are a generic feature of tree-level SUSY breaking

Example of O'Raifeartaigh model

- Caveat: we have seen that X_1 is a flat direction classically
- In the classical theory the flat direction X_1 is associated to a massless complex scalar. This was the origin of the two 0's in the scalar spectrum
- Quantum effects lift this flat direction: Coleman-Weinberg potential
- One can compute 1-loop corrections to the 2-point function of the quantum field δX_1 that describes fluctuations around the VEV $\langle X_1 \rangle = 0$
- One finds a non-zero correction that gives a positive mass-squared

$$\delta m^2 = \frac{y^4 \kappa^4}{48\pi^2 m^2}$$

- This means that $\langle X_1 \rangle = 0$ is a stable non-SUSY vacuum
- NB: the fermion mass spectrum had a massless fermion at tree-level; this fermion remains massless even after δX_1 acquires a mass. This is due to the Goldstino theorem (see below)

Example of model with FI term

- As an example of tree-level SUSY breaking with FI terms, let us consider SQED
- It is a SUSY gauge theory with gauge group $U(1)$ and two chiral superfields $\Phi^{\pm}$ of charges ± 1. We include a non-zero FI term p
- We consider massive SQED by turning on the (gauge-invariant) superpotential

$$W = m\,\Phi^{+}\,\Phi^{-}$$

- The total scalar potential is a sum of two F-terms and one D-term:

$$V = |\,m\,\overline{X^{+}}\,|^2 + |\,m\,\overline{X^{-}}\,|^2 + \left[g(\overline{X^{+}}\,X^{+} - \overline{X^{-}}\,X^{-}) - p\right]^2$$

- For generic non-zero m and p we cannot set the two F-terms and the D-term to zero simultaneously

Example of model with FI term

A more explicit expression of the scalar potential is

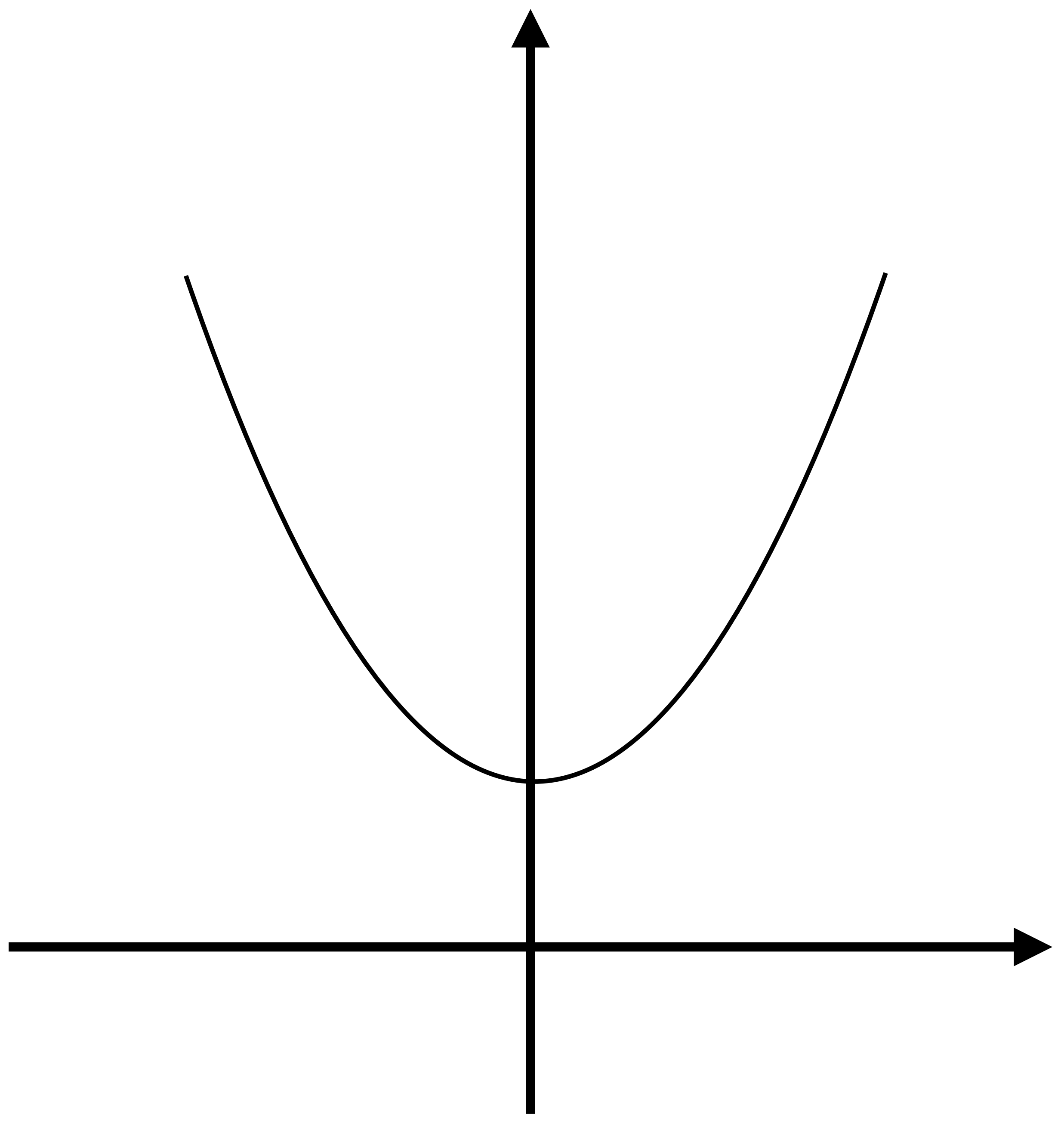
$$V = p^2 + (m^2 - 2\,g\,p)\,|\,X^+|^2 + (m^2 + 2\,g\,p)\,|\,X^-|^2 + g^2\left(\,|\,X^+|^2 - |\,X^-|^2\,\right)^2$$

We have two qualitatively different cases:

- Case A: $2\,g\,|\,p\,| < m^2$

Both complex scalars have a positive mass-squared, so they are stable at zero. The value of the potential at zero is $V = p^2$. SUSY is broken, but the $U(1)$ gauge symmetry remains unbroken. The photon and the gaugino remain massless, while the partners $\psi^\pm$ of $X^\pm$ have mass m. We have a sum rule

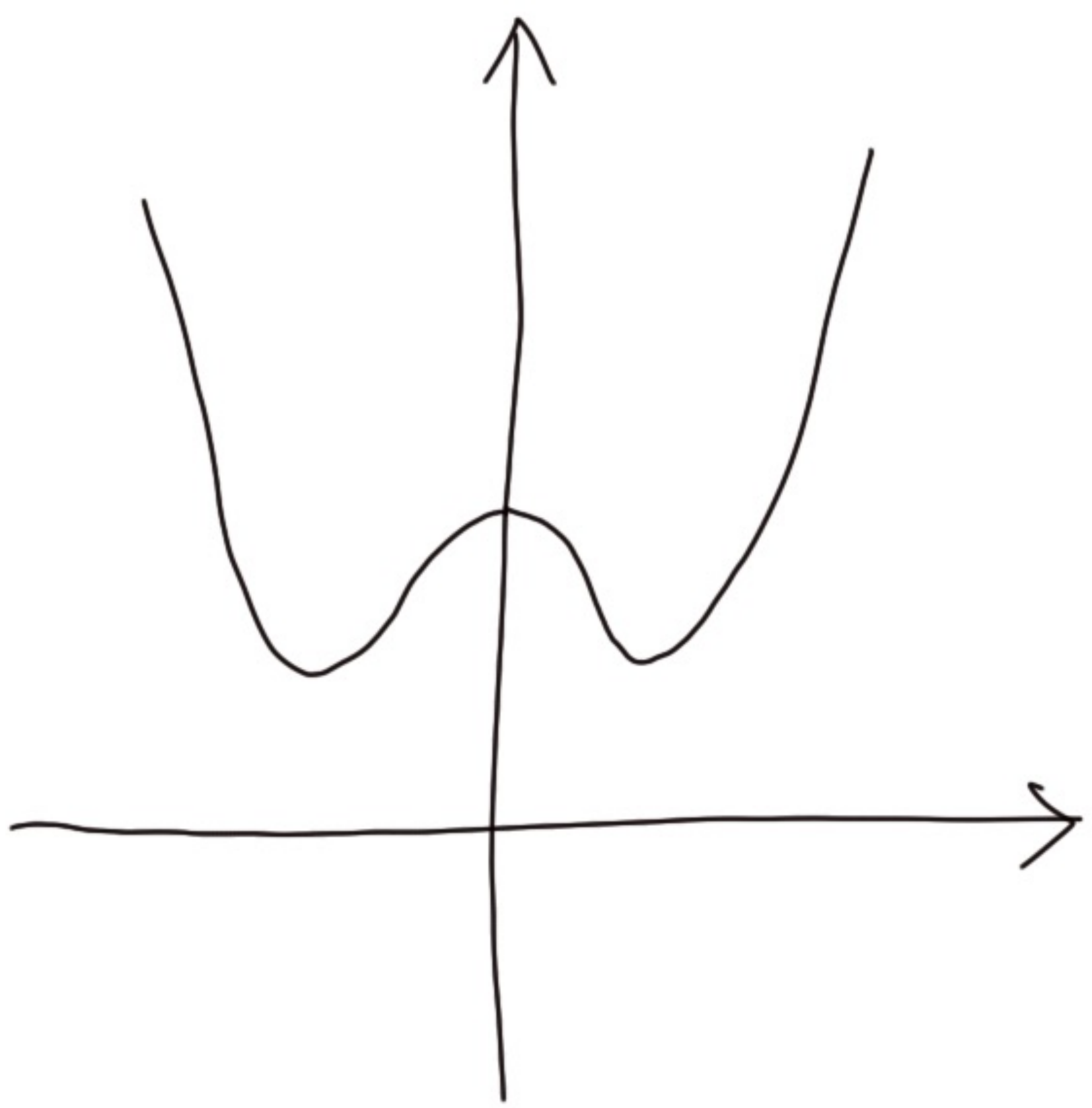
$$(m^2 - 2\,g\,p) + (m^2 + 2\,g\,p) = 2\,m^2$$



Example of model with FI term

- Case B: $2\,g\,|p| > m^2$

One of the two complex scalars acquire a term with a negative mass-squared, which means that zero is an unstable point. The scalar is driven towards a non-zero VEV. The value of V at the minimum is non-zero. Both SUSY and $U(1)$ gauge symmetry are broken. The mass spectrum is more complicated, but let us hightlight two features:



- There is still a sum rule for the masses of scalars vs fermions
- There is still a massless fermion (it is no longer the gaugino, but rather a suitable linear combination of λ and $\psi^{\pm}$

Drawbacks of tree-level SUSY breaking

- The scale of SUSY breaking is governed by a parameter in the classical Lagrangian
- If we want a large hierarchy of scales (for example SUSY breaking at a scale much lower than GUT scale or Planck scale), we have to tune the parameters by hand. This is not a “natural” solution to SUSY breaking
- Tree-level SUSY breaking generically leads to sum rules for the traces of the masses of scalars and fermions. These sum rules are usually an obstacle in constructing realistic models (they would imply light superpartners of known particles, which haven’t been observed in experiment)

The appeal of dynamical SUSY breaking

- In dynamical SUSY breaking, SUSY is preserved at classical level, but is spontaneously broken once quantum effects are taken into account
- Because of the non-renormalization theorem for the superpotential, if the classical potential is zero for some choice of VEVs, it remains zero at all orders in perturbation theory
- Our only hope are non-perturbative corrections
- They are much harder to study, but they lead naturally to large hierarchies
- This is the case because they are suppressed by exponential factors of the schematic form e^{-1/g^2}
- A study of dynamical SUSY breaking goes beyond the scope of these lectures

The (classical) Goldstino theorem

- Let us consider a SUSY gauge theory with canonical Kähler potential and constant gauge coupling function
- The mass matrix for all the fermions in the model is extracted by the following terms in the Lagrangian:

The Yukawa couplings that originate from $\int d^4x\, d^2\theta\, d^2\overline{\theta}\, \Phi^\dagger\, e^{2V}\, \Phi$:

$$\mathscr{L} \supset i\sqrt{2}\, g\, \left[\overline{X}_i\, (t_a)^i{}_j\, \psi^j\, \lambda^a - \overline{\lambda}^a\, \overline{\psi}_i\, (t_a)^i{}_j\, X^j\right]$$

The mass terms/Yukawa couplings that originate from $\int d^4x\, d^2\theta W + \mathrm{h.c.}$

$$\mathscr{L} \supset -\frac{1}{2}\, W_{ij}\, \psi^i\, \psi^j - \frac{1}{2}\, \overline{W}^{ij}\, \overline{\psi}_i\, \overline{\psi}_j$$

The (classical) Goldstino theorem

- In total, we can write the mass terms as

$$\frac{1}{2}\,(\lambda^a\ \psi^i)\begin{pmatrix} M_{ab} & M_{aj} \\ M_{ib} & M_{ij}\end{pmatrix}\begin{pmatrix}\lambda^b \\ \psi^j\end{pmatrix} = \frac{1}{2}\,(\lambda^a\ \psi^i)\begin{pmatrix} 0 & i\,g\sqrt{2}\,\overline{X}_k\,(t_a)^k{}_j \\ i\,g\sqrt{2}\,\overline{X}_k\,(t_b)^k{}_i & -W_{ij}\end{pmatrix}\begin{pmatrix}\lambda^b \\ \psi^j\end{pmatrix}$$

- NB: here and in the following, when we write a scalar we mean its VEV. Thus, W_{ij} stands for the second derivative of the superpotential, evaluated at the VEV
- Claim: the following vector is an eigenvalue of the mass matrix with eigenvalue 0:

$$\begin{pmatrix} \frac{i}{\sqrt{2}}\,D^a \\ F^i \end{pmatrix} \quad \text{where} \quad F^i = -\,\overline{W}^i(\overline{X}) \quad , \quad D_a = -\,g\,\overline{X}_i\,(t_a)^i{}_j\,X^j - p_a$$

- NB: D^b and F^i take their on-shell values and are evaluated at the VEV

The (classical) Goldstino theorem

- The relation we have to prove is

$$0 = \begin{pmatrix} 0 & i\,g\sqrt{2}\,\overline{X}_k\,(t_a)^k{}_j \\ i\,g\sqrt{2}\,\overline{X}_k\,(t_b)^k{}_i & -W_{ij} \end{pmatrix} \begin{pmatrix} \frac{i}{\sqrt{2}}\,D^b \\ F^j \end{pmatrix} = \begin{pmatrix} i\,g\sqrt{2}\,\overline{X}_k\,(t_a)^k{}_j\,F^j \\ -g\,\overline{X}_k\,(t_a)^k{}_i\,D^b - W_{ij}\,F^j \end{pmatrix}$$

- Recall: the superpotential is gauge invariant

$$\delta X^i = i\,\epsilon^a\,(t_a)^i{}_j\,X^j \qquad 0 = \delta W = \frac{\partial W}{\partial X^i}\,\delta X^i = i\,\epsilon^a\,W_i\,(t_a)^i{}_j\,X^j$$

- Take the complex conjugate: $0 = -\,i\,\epsilon^a\,\overline{X}_i\,(t_a)^i{}_j\,\overline{W}^j$
- Recall that $F^i = -\,\overline{W}^i(\overline{X})$. We obtain $\overline{X}_k\,(t_a)^k{}_j\,F^j = 0$

The (classical) Goldstino theorem

- The relation we have to prove is

$$0 = \begin{pmatrix} 0 & i\,g\sqrt{2}\,\overline{X}_k\,(t_a)^k{}_j \\ i\,g\sqrt{2}\,\overline{X}_k\,(t_b)^k{}_i & -W_{ij} \end{pmatrix} \begin{pmatrix} \frac{i}{\sqrt{2}}\,D^b \\ F^j \end{pmatrix} = \begin{pmatrix} i\,g\sqrt{2}\,\overline{X}_k\,(t_a)^k{}_j\,F^j \\ -g\,\overline{X}_k\,(t_a)^k{}_i\,D^b - W_{ij}\,F^j \end{pmatrix}$$

- Next we use the fact that the vacuum must be a configuration in which the gradient of the scalar potential is zero. The scalar potential is

$$V = \overline{F}_i\,F^i + \frac{1}{2}\,\delta_{ab}\,D^a\,D^b = W_i\,\overline{W}^i + \frac{1}{2}\,\delta_{ab}\,[g\,\overline{X}_i\,(t^a)^i{}_j\,X^j + p^a]\,[g\,\overline{X}_k\,(t^b)^k{}_\ell\,X^\ell + p^b]$$

- If we take a derivative wrt X^i and evaluate at the VEV, we get

$$0 = \frac{\partial V}{\partial X^i} = \frac{\partial W_h}{\partial X^i}\,\overline{W}^j + \delta_{ab}\,[g\,\overline{X}_k\,(t^a)^k{}_\ell\,X^\ell + p^a]\,g\,\overline{X}_j\,(t^b)^j{}_i = W_{ih}\,(-F^j) + \delta_{ab}\,(-D^a)\,g\,\overline{X}_j\,(t^b)^j{}_i$$

- We have verified the claim
- NB: in order for $(\frac{i}{\sqrt{2}}\,D^a\;\;F^i)^T$ to be a non-trivial eigenvector, it has to be not identically zero, which means that at least one of the D-terms or F-terms must be non-zero in the vacuum. This is exactly the condition for SUSY breaking

The (classical) Goldstino theorem

- Let us perform a <u>unitary</u> rotation from the original basis of fermions (λ^a, ψ^i) to a new basis (Ψ^0, Ψ^X) where Ψ^0 corresponds to the special eigenvector $(\frac{i}{\sqrt{2}} D^a \; F^i)^T$ and Ψ^X denotes collectively all other fermions

$$\begin{pmatrix} \lambda^a \\ \psi^i \end{pmatrix} = \begin{pmatrix} \frac{i}{\sqrt{2}} D^a & U^a{}_X \\ F^i & V^i{}_X \end{pmatrix} \begin{pmatrix} \Psi^0 \\ \Psi^X \end{pmatrix} \qquad \begin{pmatrix} \Psi^0 \\ \Psi^X \end{pmatrix} = \begin{pmatrix} \frac{-i}{\sqrt{2}} D^b \, \delta_{ba} & \overline{F}_i \\ (U^\dagger)^a{}_X & (V^\dagger)^i{}_X \end{pmatrix} \begin{pmatrix} \lambda^a \\ \psi^i \end{pmatrix}$$

- We know that the fermion

$$\Psi^0 = -\frac{i}{\sqrt{2}} \langle D^a \rangle \, \delta_{ab} \, \lambda^b + \langle \overline{F}_i \rangle \, \psi^i$$

is a massless fermion. (We have reintroduced the angular brackets in our notation to emphasize that we are taking the VEV; it was implicit in the previous slides). It is known as the **Goldstino**, by analogy with the massless Goldstone boson that appears in the spontaneous breaking of a continuous global symmetry

The (classical) Goldstino theorem

- We should split the scalars into VEV and fluctuations, schematically $X^i = \langle X^i \rangle + \Delta X^i$
- The SUSY variation of the Goldstino is computed from

$$\delta\psi^i = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\, D_\mu X^i + \sqrt{2}\,F^i\,\xi_\alpha \qquad\qquad \delta\lambda^a_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\, F^a{}_{\mu\nu} + i\,D^a\,\xi_\alpha$$

- Isolating the terms without fluctuations, and the terms with fluctuations, we can write

$$\delta\Psi^0 = -\frac{i}{\sqrt{2}}\,\langle D^a\rangle\,\delta_{ab}\,[i\,\langle D^b\rangle\xi + \text{fluct.}] + \langle \bar{F}_i\rangle\,[\sqrt{2}\,\langle F^i\rangle\,\xi + \text{fluct.}]$$

$$= \sqrt{2}\,\Big[\langle \bar{F}_i\rangle\langle F^i\rangle + \frac{1}{2}\,\delta_{ab}\,\langle D^a\rangle\,\langle D^b\rangle\Big]\,\xi + \text{fluct.} = \sqrt{2}\,V_0\,\xi + \text{fluct.}$$

where V_0 is the vacuum energy

- The Goldstino SUSY variation is **inhomogeneous** (it contains a constant, field-independent shift)
- This is the analog of the fact that the Goldstone bosons transform inhomogeneously under the spontaneously broken global symmetry

The (quantum) Goldstino theorem

- Our discussion so far was based on a Lagrangian description
- In dynamical SUSY breaking scenarios, SUSY is spontaneously broken by non-perturbative effects. How can we be sure that there is going to be a Goldstino?
- We need a non-perturbative version of the Goldstino theorem

The (quantum) Goldstino theorem

Sketch of the proof:

- Any SUSY theory has a supersymmetry current J^μ_α. It lies in a "supermultiplet of currents" that also contains the stress-tensor $T_{\mu\nu}$ (this is a general fact that we have not derived in these lectures)
- If we act with a supercharge on the supersymmetry current J^μ_α, we get $T_{\mu\nu}$

$$\{Q_\alpha, \bar{J}^\mu_{\dot\beta}(x)\} = \sqrt{2}\, \sigma^\nu_{\alpha\dot\beta}\, T^\mu{}_\nu(x)$$

- Now we take the VEV. In order to preserve Poincaré symmetry, the VEV of $T_{\mu\nu}$ must be a constant times the metric, $\langle T_{\mu\nu}\rangle = E\,\eta_{\mu\nu}$. (Cfr. with the cosmological constant term in GR). We get

$$\langle 0 |\, \{Q_\alpha, \bar{J}^\mu_{\dot\beta}(x)\}\, | 0 \rangle = \sqrt{2}\, \sigma^\mu_{\alpha\dot\beta}\, E$$

The (quantum) Goldstino theorem

Sketch of the proof:

$$\langle 0 | \{Q_\alpha, \bar{J}^\mu_{\dot\beta}(x)\} | 0 \rangle = \sqrt{2}\, \sigma^\mu_{\alpha\dot\beta}\, E$$

- Since x is arbitrary, we can set $x = 0$
- Next, write the supercharge Q_α as the integral of the timelike component of supersymmetry current on the spatial slice at $x^0 = 0$. We get a relation of the form

$$\sqrt{2}\, E\, \sigma^\mu_{\alpha\dot\beta} = \int d^3x\, \langle 0 | J^0_\alpha(0,\mathbf{x})\, \bar{J}^\mu_{\dot\beta}(0) + \bar{J}^\mu_{\dot\beta}(0) J^0_\alpha(0,\mathbf{x}) | 0 \rangle$$

- Equivalently, we can do the integral over d^4x, inserting a delta function $\delta(x^0)$

$$\sqrt{2}\, E\, \sigma^\mu_{\alpha\dot\beta} = \int d^4x\, \delta(x^0)\, \langle 0 | J^0_\alpha(x)\, \bar{J}^\mu_{\dot\beta}(0) + \bar{J}^\mu_{\dot\beta}(0) J^0_\alpha(x) | 0 \rangle$$

The (quantum) Goldstino theorem

Sketch of the proof:

$$\sqrt{2}\, E\, \sigma^{\mu}_{\alpha\dot\beta} = \int d^4x\, \delta(x^0)\, \langle 0 \,|\, J^0_\alpha(x)\, \bar{J}^{\mu}_{\dot\beta}(0) + \bar{J}^{\mu}_{\dot\beta}(0) J^0_\alpha(x) \,|\, 0 \rangle$$

- Finally, this can be recast as a time-ordered 2-point function:

$$\sqrt{2}\, E\, \sigma^{\mu}_{\alpha\dot\beta} = \int d^4x\, \partial_\nu\, \langle 0 \,|\, T\, J^\nu_\alpha(x)\, \bar{J}^{\mu}_{\dot\beta}(0) \,|\, 0 \rangle$$

- Notice that the time ordering symbol has a Heaviside theta function $\Theta(x^0)$ whose derivative ∂_ν is non-zero for $\nu = 0$ only and gives back $\delta(x^0)$. We do not get contributions from $\partial_\nu J^\nu_\alpha(x)$ because the supersymmetry current is conserved

The (quantum) Goldstino theorem

Sketch of the proof:

$$\sqrt{2}\, E\, \sigma^{\mu}_{\alpha\dot{\beta}} = \int d^4x\, \partial_\nu \, \langle 0 \,|\, T\, J^{\nu}_{\alpha}(x)\, \bar{J}^{\mu}_{\dot{\beta}}(0) \,|\, 0 \rangle$$

- SUSY is broken iff $E \neq 0$, iff the integral is non-zero. But it is an integral of a total derivative. If it is non-zero, it must be because the quantity $\langle 0 \,|\, T\, J^{\nu}_{\alpha}(x)\, \bar{J}^{\mu}_{\dot{\beta}}(0) \,|\, 0 \rangle$ does not fall off to infinity sufficiently rapidly
- In general $\langle 0 \,|\, T\, J^{\nu}_{\alpha}(x)\, \bar{J}^{\mu}_{\dot{\beta}}(0) \,|\, 0 \rangle$ receives contribution from all the possible intermediate states (single-particle, multiparticle, etc). One can prove that, if we want $\langle 0 \,|\, T\, J^{\nu}_{\alpha}(x)\, \bar{J}^{\mu}_{\dot{\beta}}(0) \,|\, 0 \rangle$ to fall off at infinity slowly enough to give a non-zero integral, then we must have a massless 1-particle state of spin 1/2
- Moreover, this massless fermion Ψ^0 is characterized by the fact it can be generated from the vacuum by acting with the SUSY current

$$\langle 0 \,|\, \bar{J}^{\mu}_{\dot{\alpha}} \,|\, \Psi^0_{\beta} \rangle = f\, \sigma^{\mu}_{\beta\dot{\alpha}}$$ with f a non-zero constant; actually one proves $f^2 = E$

The (quantum) Goldstino theorem

Sketch of the proof:

- This is the non-perturbative definition of the Goldstino: a massless fermion whose 1-particle states $|\Psi_\alpha^0\rangle$ is such that the SUSY current has a non-zero matrix element between $|\Psi_\alpha^0\rangle$ and the vacuum
- This is analogous to the non-perturbative definition of the Goldstone boson: a massless scalar whose 1-particle states $|\varphi\rangle$ are such that the SUSY current has a non-zero matrix element between $|\varphi\rangle$ and the vacuum. Schematically

$$\langle 0|J^\mu|\varphi\rangle \sim f p^\mu \qquad (p^\mu \text{ is the momentum of the state } |\varphi\rangle)$$

Supersymmetry and supergravity

Lecture 27

Soft SUSY breaking

- Soft SUSY breaking is a special kind of explicit SUSY breaking
- Let us consider the Wess-Zumino model. We know that, because of the non-renormalization theorem, the mass parameter and couplings do not receive independent renormalizations, other than wavefunction renormalization
- In terms of Feynman diagrams at 1-loop, one find “miraculous” cancellations between bosons and fermions. The cancellations remove pieces that are quadratically divergent in the momentum cutoff, all well as pieces that diverge logarithmically, and finite pieces
- Is there a way to break SUSY explicitly, but keep the cancellation of the quadratically divergent pieces?

Soft SUSY breaking

- This question is relevant for phenomenology
- We have seen that tree-level SUSY breaking has drawbacks (problem of scales, sum rules for masses…) that make it not suitable to construct models for SUSY breaking in pheno
- It is expected that SUSY is dynamically <u>spontaneously</u> broken by non-perturbative effects in a “hidden sector”. SUSY breaking effects are trasmitted to the “visible sector” (the Standard Model) via non-renormalizable interactions or loop effects
- We can parametrize our ignorance about these mechanisms by including an <u>explicit</u> SUSY breaking in the “visible sector” model, but doing it softly

Soft SUSY breaking

- It turns out that one can organize the possible soft SUSY breaking terms using background superfields in superspace
- Let us consider a background chiral superfield Φ_{bkg}. So far, we have only considered situations in which the only non-zero component of Φ_{bkg} is its X component, given by a constant,

$$\Phi_{\text{bkg}} = X_{\text{bkg}} \ , \qquad \partial_\mu X_{\text{bkg}} = 0$$

- This was dictated by the fact that we wanted to preserve SUSY. We can see this from the variation of the ψ component

$$\delta\psi_{\text{bkg}} = i\sqrt{2}\,\sigma^\mu\,\bar{\xi}\,\partial_\mu X_{\text{bkg}} + \sqrt{2}\,\xi\,F_{\text{bkg}}$$

- In order to describe explicit SUSY breaking, we now allow for a non-zero F_{bkg}

$$\Phi_{\text{bkg}} = X_{\text{bkg}} + \theta^2\,F_{\text{bkg}} \ , \qquad \partial_\mu X_{\text{bkg}} = 0 \ , \qquad \partial_\mu F_{\text{bkg}} = 0$$

- NB: We are breaking SUSY, but we are preserving Poincaré

An analogy

- We can describe explicity breaking of Lorentz symmetry in a Lorentz-covariant way
- For example, we can define the constant vector field $v^\mu = (1,0,0,0)$ that singles out the time direction, and write terms in the action such as

$$(v^\mu \, \partial_\mu \phi)^2$$

- Here we are describing explicit SUSY breaking is superspace using background superfields with an explicit θ, $\bar{\theta}$ dependence

How to generate soft SUSY breaking terms

- We describe explicit supersymmetry breaking is superspace. To this end, we need background superfields that have an explicit θ dependence (but no x dependence)
- We consider a suitable term in superspace and we promote the constant parameter in the term to a background superfield with explicit θ dependence
- Integration in θ, $\bar{\theta}$ yields soft SUSY breaking terms

Overview of soft terms

bkgr superfield	superfield term	in components	mass dim.'s	
$U_i{}^j = (m^2)_i{}^j\,\theta^2\,\overline{\theta}^2$	$\int d^2\theta\, d^2\overline{\theta}\, U_i{}^j\, (\Phi^\dagger\, e^{2V})_i\, \Phi^j$	$(m^2)_i{}^j\, \overline{X}_i\, X^j$	$[U_i{}^j] = 0$	$[(m^2)_i{}^j] = 2$
$U_{ij} = b_{ij}\,\theta^2$	$\int d^2\theta\, U_{ij}\, \Phi^i\, \Phi^j + \text{h.c.}$	$b_{ij}\, X^i\, X^j + \text{h.c.}$	$[U_{ij}] = 1$	$[b_{ij}] = 2$
$U_{ijk} = a_{ijk}\,\theta^2$	$\int d^2\theta\, U_{ijk}\, \Phi^i\, \Phi^j\, \Phi^k + \text{h.c.}$	$a_{ijk}\, X^i\, X^j\, X^k + \text{h.c.}$	$[U_{ijk}] = 0$	$[a_{ijk}] = 1$
$U_i = e_i\,\theta^2$	$\int d^2\theta\, U_i\, \Phi^i + \text{h.c.}$	$e_i\, X^i + \text{h.c.}$	$[U_i] = 2$	$[e_i] = 3$
$U_\lambda = M_\lambda\,\theta^2$	$\int d^2\theta\, U_\lambda\, \delta_{ab}\, \mathcal{W}^a\, \mathcal{W}^b + \text{h.c.}$	$M_\lambda\, \delta_{ab}\, \lambda^a\, \lambda^b + \text{h.c.}$	$[U_\lambda] = 0$	$[M_\lambda] = 1$

Notation: Φ^i = chiral superfields; e^{2V} = factors for gauge invariance in superspace; $\mathscr{W}^a_\alpha$ = chiral superfields with the field strength of the gauge fields

Overview of soft terms

bkgr superfield	superfield term	in components	mass dim.'s	
$U_i{}^j = (m^2)_i{}^j\, \theta^2\, \overline{\theta}^2$	$\int d^2\theta\, d^2\overline{\theta}\, U_i{}^j\, (\Phi^\dagger\, e^{2V})_i\, \Phi^j$	$(m^2)_i{}^j\, \overline{X}_i\, X^j$	$[U_i{}^j] = 0$	$[(m^2)_i{}^j] = 2$
$U_{ij} = b_{ij}\, \theta^2$	$\int d^2\theta\, U_{ij}\, \Phi^i\, \Phi^j + \text{h.c.}$	$b_{ij}\, X^i\, X^j + \text{h.c.}$	$[U_{ij}] = 1$	$[b_{ij}] = 2$
$U_{ijk} = a_{ijk}\, \theta^2$	$\int d^2\theta\, U_{ijk}\, \Phi^i\, \Phi^j\, \Phi^k + \text{h.c.}$	$a_{ijk}\, X^i\, X^j\, X^k + \text{h.c.}$	$[U_{ijk}] = 0$	$[a_{ijk}] = 1$
$U_i = e_i\, \theta^2$	$\int d^2\theta\, U_i\, \Phi^i + \text{h.c.}$	$e_i\, X^i + \text{h.c.}$	$[U_i] = 2$	$[e_i] = 3$
$U_\lambda = M_\lambda\, \theta^2$	$\int d^2\theta\, U_\lambda\, \delta_{ab}\, \mathcal{W}^a\, \mathcal{W}^b + \text{h.c.}$	$M_\lambda\, \delta_{ab}\, \lambda^a\, \lambda^b + \text{h.c.}$	$[U_\lambda] = 0$	$[M_\lambda] = 1$

- Why are these terms soft? The proof relies on power-counting for Feynman diagrams in superspace
- Heuristic explanation: these soft terms come from terms in superspace that are renormalizable; the mass dimension of the background superfield U is non-negative

Supersymmetry and supergravity

Lecture 28

SUSY quantum mechanics

- SUSY exists in various spacetime dimensions, including in 0+1 dimensions, i.e. quantum mechanics
- In 0+1 dimensions, the analog of the 4d Poincaré group consists of time translations (generated by $P^0 = H$). There are no “rotations”, no “boosts”, no notion of spin in the usual sense. We still distinguish bosons and fermions by their statistics (commuting vs anticommuting)
- We know that the structure of the SUSY algebra in 4d is very roughly $[P, Q] = 0$ and $\{Q, Q\} \sim P + Z$
- The SUSY algebra in 0+1 dimensions has a similar structure:

$$[Q^I, H] = 0 \quad , \quad \{Q^I, Q^J\} = H\,\delta^{IJ} + Z^{IJ} \quad , \quad I = 1, \ldots, \mathcal{N}$$

- The operators Q^I are Hermitian, $(Q^I)^\dagger = Q^I$ (they are “real” supercharges). The central charges Z^{IJ} commute with everything. Contrary to their 4d cousins, they are symmetric, $Z^{IJ} = Z^{JI}$

SUSY quantum mechanics

General properties of a SUSY QM model (for simplicity, without central charges):

1. The Hilbert space decomposes into the direct sum of "bosonic" and "fermionic" subspaces
$$\mathscr{H} = \mathscr{H}^{\mathrm{B}} \oplus \mathscr{H}^{\mathrm{F}}$$
2. The fermion number operator $(-1)^F$ is defined to be the operator that has eigenvalue $+1$ with eigenspace $\mathscr{H}^{\mathrm{B}}$ and eigenvalue -1 with eigenspace $\mathscr{H}^{\mathrm{F}}$
3. The Hamiltonian commutes with $(-1)^F$: it sends bosonic states to bosonic states, and fermionic states to fermionic states

SUSY quantum mechanics

General properties of a SUSY QM model (for simplicity, without central charges):

4. The model has supercharge operators Q^I defined in the Hilbert space that satisfy the following properties:
   - they are Hermitian
   - they anticommute with $(-1)^F$: they enchange bosons and fermions
   - they commute with the Hamiltonian: $[Q^I, H] = 0$
   - their anticommutator gives the Hamiltonian: $\{Q^I, Q^J\} = H\,\delta^{IJ}$

Spectrum in SUSY QM

- We are interested in the situation in which the Hamiltonian H has a **discrete spectrum**:

$$E_0 < E_1 < \ldots < E_n < \ldots$$

- The groundstate energy E_0 is always non-negative. It is 0 iff SUSY is unbroken. This follows from $\{Q^I, Q^J\} = H\,\delta^{IJ}$:

$$H = (Q^1)^2 = \frac{1}{2}\,Q^1\,(Q^1)^\dagger + \frac{1}{2}\,(Q^1)^\dagger\,Q^1$$

Spectrum in SUSY QM

- Let us fix our attention on the supercharge $\mathcal{Q} = Q^1$
- If out SUSY QM has extended SUSY, and if we choose Q^2, or Q^3, … we get similar conclusions
- The crucial properties of $\mathcal{Q}$ are

$$\mathcal{Q}^\dagger = \mathcal{Q} \quad , \qquad \mathcal{Q}^2 = H$$

- Claim #1: eigenstates with non-zero energy always come in boson-fermion pairs
- Claim #2: all energy eigenstates of zero energy are annihilated by $\mathcal{Q}$

Spectrum in SUSY QM

Claim #1: eigenstates with non-zero energy always come in boson-fermion pairs

- Suppose $|\Psi^{\mathrm{B}}\rangle$ is a bosonic eigenstate of energy $E_n \neq 0$. Let us define $|\Psi^{\mathrm{F}}\rangle := \mathcal{Q}\,|\Psi^{\mathrm{B}}\rangle$. This state:
  - Non-zero: its norm-squared is

$$\langle\Psi^{\mathrm{B}}|\,\mathcal{Q}^{\dagger}\,\mathcal{Q}\,|\Psi^{\mathrm{B}}\rangle = \langle\Psi^{\mathrm{B}}|\,\mathcal{Q}^{2}\,|\Psi^{\mathrm{B}}\rangle = E_n\,\langle\Psi^{\mathrm{B}}\,|\,\Psi^{\mathrm{B}}\rangle$$

  - A fermionic eigenstate of energy E_n
  - Such that, acting with $\mathcal{Q}$ on it, we recover $|\Psi^{\mathrm{B}}\rangle$ (up to a non-zero constant)

$$\mathcal{Q}\,|\Psi^{\mathrm{F}}\rangle = \mathcal{Q}^{2}\,|\Psi^{\mathrm{B}}\rangle = E_n\,|\Psi^{\mathrm{B}}\rangle$$

- The states $|\Psi^{\mathrm{B}}\rangle$, $|\Psi^{\mathrm{F}}\rangle := \mathcal{Q}\,|\Psi^{\mathrm{B}}\rangle$ form a “long” multiplet of the supercharge $\mathcal{Q}$

Spectrum in SUSY QM

Claim #2: all energy eigenstates of zero energy are annihilated by $\mathcal{Q}$

- This follows from

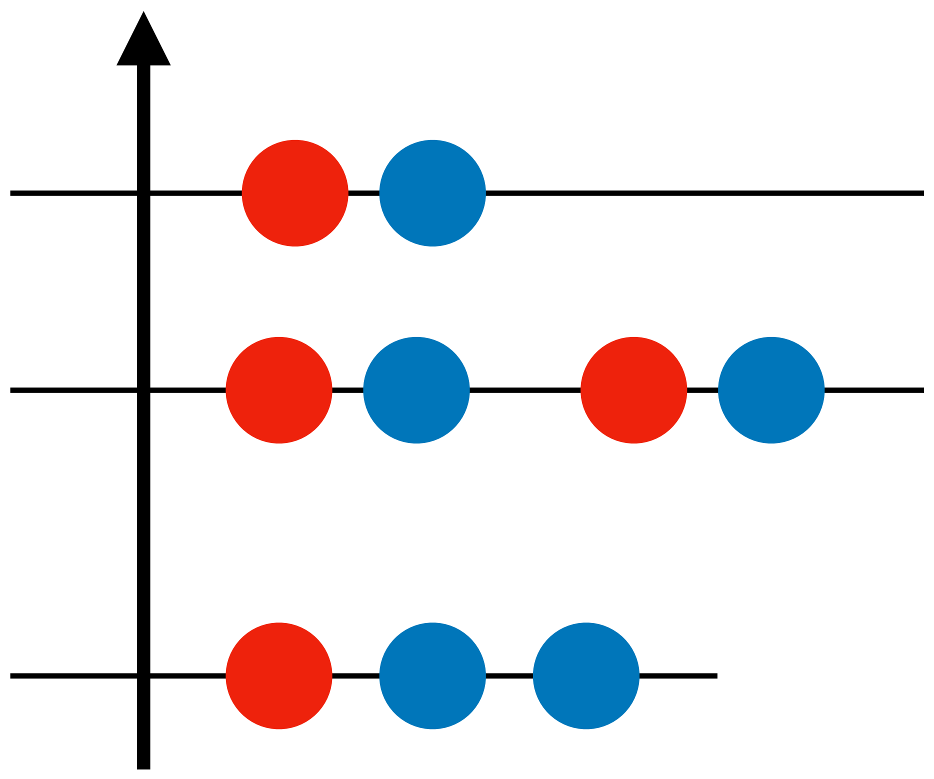
$$\langle \Psi | H | \Psi \rangle = \langle \Psi | \mathcal{Q}^\dagger \mathcal{Q} | \Psi \rangle = \| \mathcal{Q} | \Psi \rangle \|^2$$

- We can say that eigenstate of zero energy form a “short” multiplet of the supercharge $\mathcal{Q}$

The Witten index

- Non-zero energy eigenstates always appear in boson-fermions pairs
- States with zero energy need not be paired up
- In particular, the number of bosonic groundstates can be different from the number of fermionic groundstates, and we can define the Witten index

$$I = n_0^{\mathrm{B}} - n_0^{\mathrm{F}}$$



The Witten index as a trace

- We can write the Witten index as a trace over the Hilbert space

$$I = \mathrm{Tr}_{\mathscr{H}}(-1)^F e^{-\beta H} \qquad \beta > 0$$

- The positive parameter β gives a "convergence factor" that makes the trace well defined
- The RHS is actually independent of β: taking the trace, the contributions of states with $E_n > 0$ cancel in pairs between bosons and fermions. Only groundstates give a non-zero contribution
- We can then take the limit $\beta \to 0^+$. For this reason the Witten index is usually written simply as

$$I = \mathrm{Tr}_{\mathscr{H}}(-1)^F$$

The Witten index in four dimensions

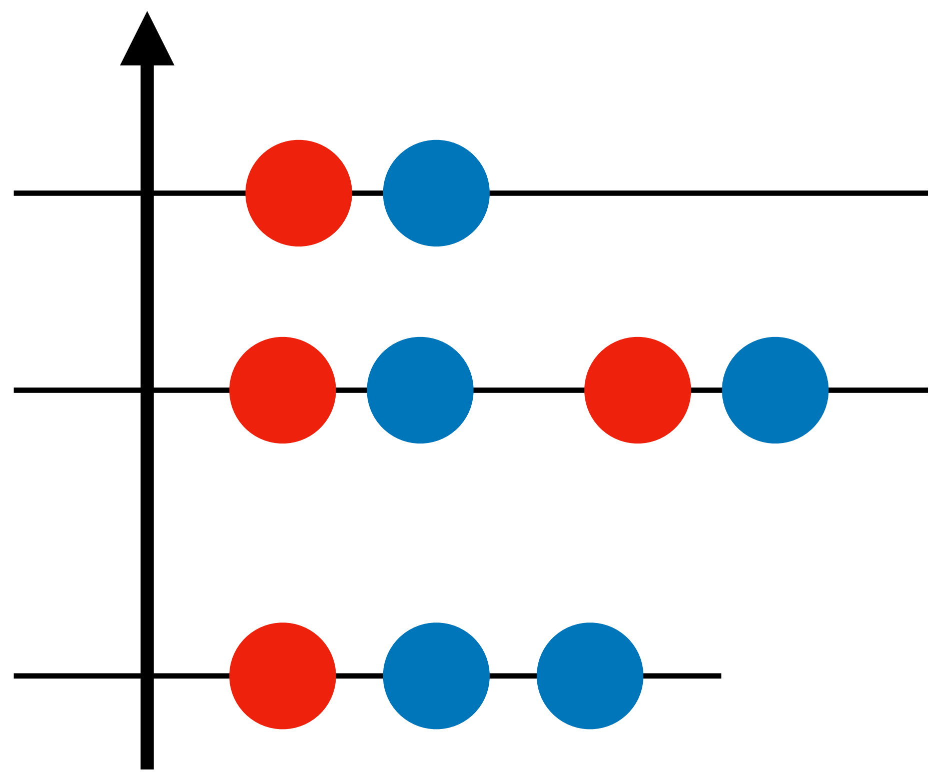
- Can we define the Witten index in a 4d SUSY QFT?
- We need to be able to count state in a meaningful way, which requires a discrete energy spectrum
- To obtain a discrete spectrum, we put the QFT on a spatial "box" of side L. More precisely, we consider the theory on T^3 with periodic boundary conditions for all fields (bosons and fermions)
- SUSY would be violated if we used different boundary conditions for bosons and fermions
- The box breaks Lorentz symmetry, but translational invariance is preserved. This is necessary, because we know that $\{Q, Q\} \sim P$ so breaking translations would break SUSY

The Witten index in four dimensions

- The role of the supercharge $\mathbb{Q}$ can be played by any linear combination of the Q_α's, $\overline{Q}_{\dot{\alpha}}$'s that satisfies the properties

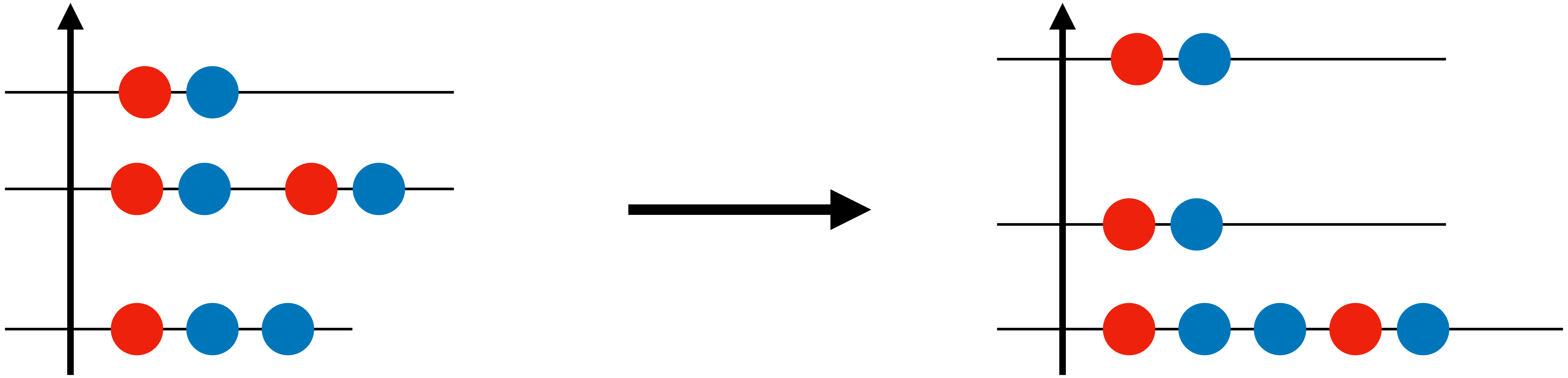
$$\mathbb{Q}^\dagger = \mathbb{Q} \ , \qquad \mathbb{Q}^2 = H \equiv P^0$$

- Just like in the QM toy model, states with non-zero energy are paired up in boson-fermion pairs, while states with zero energy are not necessarily paired up



The Witten index is robust

- Let us imagine a small deformation of some parameter in the SUSY QFT (say, a superpotential coupling) or a small deformation in the size L of the "box"
- What can happen to the spectrum of the theory?
- States with positive energy can start shifting. It might happen that some of them go down to zero energy. This must happen to a boson-fermion pair, so the index $I = n_0^{\mathrm{B}} - n_0^{\mathrm{F}}$ is not affected
- States with zero energy might acquire a non-zero energy. If they do, they must do so in pairs, because they cannot leave the energy level $E_0 = 0$ without a superpartner under the action of $\mathcal{Q}$. So again $I = n_0^{\mathrm{B}} - n_0^{\mathrm{F}}$ is not affected
- Lesson: the Witten index is invariant under small deformations of the parameters



The Witten index can rule out SUSY breaking

- Suppose we are able to compute the Witten index of a 4d SUSY QFT in a box of size L, and we get a non-zero result: $I = n_0^{\mathrm{B}} - n_0^{\mathrm{F}} \neq 0$
- We must have $n_0^{\mathrm{B}} \neq 0$ and/or $n_0^{\mathrm{F}} \neq 0$
- As we increase the size of the box, some of the groundstates might pair up and go to a higher energy level, but this cannot happen to all of them, because $I = n_0^{\mathrm{B}} - n_0^{\mathrm{F}} \neq 0$. SUSY is unbroken in a box of size L, for any finite L
- This means that the vacuum energy of the theory on a box of size L is exactly zero, for any finite L
- The vacuum energy must remain zero in the infinite volume limit. As a result, SUSY is unbroken in the infinite volume limit
- This is a non-perturbative argument. It can be used to rule out dynamical SUSY breaking by non-perturbative effects (no matter how small or hard to compute)

Supersymmetry and supergravity

Lecture 29

The MSSM: main idea

- The Minimal Supersymmetric Standard Model is a SUSY gauge theory with the same gauge group as the Standard Model, and minimal field content to account for all known particles
- The SUSY particles of known particles are often called superpartner or “sparticles”
- The SUSY partners of leptons, quarks are usually called sleptons, squarks
- The SUSY partners of gauge bosons have a -ino suffix, e.g. photino, guino, W-ino, etc…

The MSSM: main idea

- The MSSM is usually thought of as a low-energy effective action of a more complete UV theory
- In the most promising scenarios, SUSY is spontaneouly broken by non-perturbative effects (dynamical SUSY breaking) in a “hidden sector”
- SUSY breaking is then transmitted to the “visible sector”, i.e. the MSSM
- In order to describe these phenomena in a model-independent way, we consider the MSSM supplemented by soft SUSY breaking terms

Reminder on the SM

- The SM is a gauge theory with gauge group $SU(3) \times SU(2) \times U(1)_Y$
- The $SU(3)$ factor is the color gauge symmetry of QCD
- The $SU(2) \times U(1)_Y$ factor is the gauge symmetry of electroweak interactions. It is spontaneouly broken to a $U(1)$ gauge group, identified with electromagnetism
- Caveat on notation: so far, following Wess-Bagger, we have used a bar to denote the complex conjugate of a Weyl spinor (Hermitian conjugate if it is an operator). E.g. $\overline{\psi}_{\dot{\alpha}} = (\psi_\alpha)^\dagger$. In the MSSM literature, it is customary to use the bar as part of the name of a field. One then uses $\dagger$ for Hermitian conjugation
- Example: given a positive-chirality Weyl spinor $\overline{e}_\alpha$ (the bar is part of the name), its conjugate is

$$(\overline{e}^\dagger)_{\dot{\alpha}} = (\overline{e}_\alpha)^\dagger$$

Reminder on the SM

$$(\underbrace{\square}_{\text{rep of } SU(3)}, \underbrace{\square}_{\text{rep of } SU(2)})_{\underbrace{1/6}_{U(1)_Y \text{ charge}}}$$

The matter content of the SM is summarized as follows

left-handed quarks	Q^i	$(\square, \square)_{1/6}$
right-handed up-type quarks	$\bar{u}^i$	$(\overline{\square}, \bullet)_{-2/3}$
right-handed down-type quarks	$\bar{d}^i$	$(\overline{\square}, \bullet)_{+1/3}$
left-handed leptons	L^i	$(\bullet, \square)_{-1/2}$
right-handed electron-type leptons	$\bar{e}^i$	$(\bullet, \bullet)_1$
Higgs doublet	H	$(\bullet, \square)_{-1/2}$

Remarks:

- All spinors are positive-chirality Weyl spinors. The bars are part of the name. We suppress spacetime and gauge indices.
- The index $i = 1,2,3$ is a generation label
- The SM is a chiral gauge theory. It is free of gauge anomalies, gravitational anomalies, Witten's $SU(2)$ anomaly

Reminder on the SM

We consider all possible terms in the Lagrangian that are allowed by gauge symmetry and renormalizability

1. YM terms for the $SU(3) \times SU(2) \times U(1)_Y$ gauge fields
2. Theta angle terms (they are only relevant for non-perturbative effects)
3. Kinetic terms for the fermions. We write them in a basis that is diagonal wrt the generation index i

$$\mathscr{L} \supset -i\, Q_i^\dagger\, \bar{\sigma}^\mu\, D_\mu Q^i - i\, \bar{u}_i^\dagger\, \bar{\sigma}^\mu\, D_\mu \bar{u}^i - i\, \bar{d}_i^\dagger\, \bar{\sigma}^\mu\, D_\mu \bar{d}^i + \dots$$

4. Kinetic term and scalar potential for the Higgs doublet:

$$\mathscr{L} \supset -D^\mu H^\dagger D_\mu H - V \quad , \qquad V = -m^2 H^\dagger H + \lambda\, (H^\dagger H)^2$$

Reminder on the SM

5. Yukawa couplings. Since we have already diagonalized the kinetic terms, these couplings in general mix the generation labels i, j

$$\mathscr{L} \supset (y_u)_{ij}\, Q^i\, H^\dagger\, \bar{u}^j + (y_d)_{ij}\, Q^i\, H\, \bar{d}^j + (y_e)_{ij}\, L^i\, H\, \bar{e}^j + \mathrm{h.c.}$$

- If we restore the fundamental indices of $SU(2)$ gauge, the $SU(2)$ index contraction is $(Q^i)^I\,(H^\dagger)_I$ in the first term, and $\epsilon_{IJ}\,(Q^i)^I\,H^J$, $\epsilon_{IJ}\,(L^i)^I\,H^J$ in the second and third terms. Here ϵ_{IJ} is the antisymmetric invariant of the $SU(2)$ gauge group
- In the SM there is no notion of "holomorphy", and we can write Yukawa couplings of positive-chirality Weyl spinors both to H and to $H^\dagger$
- Curiosity: if we turn off <u>all</u> Yukawa couplings, we get a global $SU(3)^5$ symmetry rotating the $i = 1,2,3$ labels of the five fermions Q^i, $\bar{u}^i$, $\bar{d}^i$, L^i, $\bar{e}^i$. This enhanced global symmetry at Yukawa = 0 explains why it is natural for Yukawa couplings to be very small (e.g. $\sim 10^{-5}$ for the electron)

After EW breaking

- The "wrong" sign of the mass of the Higgs induces a VEV which breaks the $SU(2)$ gauge symmetry. The gauge fields of $SU(2) \times U(1)_Y$ are reorganized into the massless neutral photon A_μ, the massive neutral Z_μ, and the massive charged $W_\mu^\pm$
- Let us give a name to the components of our $SU(2)$ doublets:

$$H = \begin{pmatrix} H^0 \\ H^- \end{pmatrix} \;,\quad Q^i = \begin{pmatrix} u^i \\ d^i \end{pmatrix} \quad,\quad L^i = \begin{pmatrix} \nu^i \\ e^i \end{pmatrix}$$

- Up to a gauge transformation we align the VEV of the Higgs as

$$\langle H \rangle \propto \begin{pmatrix} v \\ 0 \end{pmatrix}$$

After EW breaking

- The Yukawa couplings yield mass terms: up to numerical factors
$$\mathscr{L} \supset v\,(y_u)_{ij}\, u^i\, \bar{u}^j + v\,(y_d)_{ij}\, d^i\, \bar{d}^j + v\,(y_e)_{ij}\, e^i\, \bar{e}^j + \mathrm{h.c.}$$
- Observed particles correspond to mass eigenstates. We diagonalize the mass terms using independent <u>unitary</u> rotations on u^i, $\bar{u}^i$, d^i, $\bar{d}^i$, e^i, $\bar{e}^i$
- The kinetic terms remain diagonal in the generation indices i
- The couplings of the fermions to A_μ, Z_μ also remain diagonal. In the SM, there are no flavor changing neutral currents (FCNC) at tree-level
- The coupling of the leptons to $W^\pm_\mu$ are also diagonal. The couplings of quarks to $W^\pm_\mu$, however, are not diagonal in the generation indices:
$$V^i{}_j\, W^+_\mu\, u^\dagger_i\ \bar{\sigma}^\mu\, d^j \qquad V^i{}_j \text{ : unitary CKW matrix}$$

Some flavor structure in the SM

- The SM has important accidental symmetries:
  - three separate “lepton numbers” for each generation $i = 1,2,3$
  - “baryon number”
- Flavor changing neutral currents (FCNC) are naturally suppressed: we have seen that they do not arise at tree-level. They arise at 1-loop level, but they are suppressed by the so-called “GIM mechanism”, which is a consequence of the unitarity of the CKM matrix
- None of these appealing features were put in by hand!
- Caveat: I’m describing the SM with massless neutrinos, which is clearly incomplete because we know that neutrinos are massive. Flavor physics is more complicated with massive neutrinos

The matter content of the MSSM

- We use vector superfields for the gauge group $SU(3) \times SU(2) \times U(1)_Y$
- We recycle the table of matter fields we had before, reinterpreting all the entries as chiral superfields

left-handed (s)quarks	Q^i	$(\square, \square)_{1/6}$
right-handed up-type (s)quarks	$\bar{u}^i$	$(\overline{\square}, \bullet)_{-2/3}$
right-handed down-type (s)quarks	$\bar{d}^i$	$(\overline{\square}, \bullet)_{+1/3}$
left-handed (s)leptons	L^i	$(\bullet, \square)_{-1/2}$
right-handed electron-type (s)leptons	$\bar{e}^i$	$(\bullet, \bullet)_1$
Higgs doublet no.1	H_d	$(\bullet, \square)_{-1/2}$
Higgs doublet no.2	H_u	$(\bullet, \square)_{1/2}$

Remarks:

- We have to include two Higgs doublets to avoid gauge anomalies and a Witten $SU(2)$ anomaly

Renormalizable SUSY interactions

We consider all possible terms in the Lagrangian that are allowed by gauge symmetry, SUSY, and renormalizability

1. SYM terms for the $SU(3) \times SU(2) \times U(1)_Y$ gauge fields
2. Theta angle terms (they are only relevant for non-perturbative effects)
3. Kinetic terms for all the non-Higgs chiral superfields. We write them in a basis that is diagonal wrt the generation index i

$$\mathscr{L} \supset \int d^2\theta\, d^2\overline{\theta} \big(Q_i^\dagger\, e^{2V}\, E^i + \overline{u}_i^\dagger\, e^{2V}\, \overline{u}^i + \overline{d}_i^\dagger\, e^{2V}\, \overline{d}^i + \ldots \big)$$

4. Kinetic terms for the Higgs doublets chiral superfields

$$\mathscr{L} \supset \int d^2\theta\, d^2\overline{\theta} \big(H_u^\dagger\, e^{2V}\, H_u + H_d^\dagger\, e^{2V}\, H_d \big)$$

Renormalizable SUSY interactions

5. Superpotential couplings that mimic the SM Yukawa couplings:

$$\mathscr{L} \supset \int d^2\theta \left[(y_u)_{ij}\, Q^i H_u \bar{u}^j + (y_d)_{ij}\, Q^i H_d \bar{d}^j + (y_e)_{ij}\, L^i H_d \bar{e}^j \right] + \mathrm{h.c.}$$

   Notice that holomorphy forbids a term of the form $Q^i H^\dagger \bar{u}$, which is another reason why we need both H_d and H_u

6. Two superpotential couplings involving the Higgses with dimensions of mass:

$$\mathscr{L}_{\mu,\kappa} = \int d^2\theta \left[\mu H_u H_d + \kappa_i L^i H_u \right] + \mathrm{h.c.}$$

7. More "Yukawa-like" terms in the superpotential:

$$\mathscr{L}_{\text{bad Yukawa}} = \int d^2\theta \left[\alpha_{ijk}\, Q^i L^j \bar{d}^k + \beta_{ijk}\, L^i L^j \bar{e}^k + \delta_{ijk}\, \bar{d}^i \bar{d}^j \bar{u}^k \right] + \mathrm{h.c.}$$

“Bad” interactions

- The superpotential terms

$$\alpha_{ijk}\, Q^i\, L^j\, \bar{d}^k + \beta_{ijk}\, L^i\, L^j\, \bar{e}^k + \delta_{ijk}\, \bar{d}^i\, \bar{d}^j\, \bar{u}^k + \kappa_i\, L^i\, H_u$$

  are all “bad”: they induce baryon and lepton violation at tree-level

- The couplings α and δ induce processes that make the proton decay into a meson and an antilepton (e.g. a π^0 and a positron). The experimental bounds on proton decay imply the estimate

$$|\alpha|\,|\delta| < 10^{-25}$$

- There is no obvious reason why the “good” Yukawas should be not too small (to account for known particle masses) while suppressing these “bad” Yukawas

- The μ term $\mu\, H_u\, H_d$ can be problematic. It induces quadratic terms in the scalar potential for the Higgses. μ is a mass parameter and one has to ensure that it is naturally at the EW scale, as opposed to, say, the Planck scale

R-parity

- In order to forbid "bad" terms at tree-level we can <u>postulate</u> new global symmetries
- One promising candidate is "R-parity". In general, it is not related to the $U(1)_R$ symmetries that we have discussed so far
- In superspace, R-parity is by definition a transformation of the form

$$\mathsf{R}\,\Phi(x,\theta,\bar{\theta})\,\mathsf{R}^{-1} = s_\Phi\,\Phi(x,-\theta,-\bar{\theta})$$

where s_Φ is the intrinsic R-parity of the superfield Φ

- If we choose
  - $s_\Phi = +1$ for all vector superfields and for the Higgs chiral superfields H_u, H_d
  - $s_\Phi = -1$ for all the other chiral superfields $Q^i, \bar{u}^i, \bar{d}^i, L^i, \bar{e}^i$

then the known particles of the SM have parity $+1$ while their unobserved SUSY partners have parity -1

R-parity

- $s_\Phi = +1$ for all vector superfields and for the Higgs chiral superfields H_u, H_d
- $s_\Phi = -1$ for all the other chiral superfields $Q^i, \bar{u}^i, \bar{d}^i, L^i, \bar{e}^i$

• If we postulate this discrete global symmetry, the "bad" couplings

$$\alpha_{ijk}\, Q^i\, L^j\, \bar{d}^k + \beta_{ijk}\, L^i\, L^j\, \bar{e}^k + \delta_{ijk}\, \bar{d}^i\, \bar{d}^j\, \bar{u}^k + \kappa_i\, L^i\, H_u$$

are all forbidden. The μ term is still allowed

• Other consequences of R-parity:

- Scattering of known particles can only produce superpartners in pairs
- The lightest superpartner (LSP) is stable: it cannot decay. If it has the right quantum numbers (e.g. has zero electric charge), the LSP can be a candidate for dark matter

MSSM with soft breaking

- The MSSM is supplemented by explicit SUSY breaking by soft terms
- To write the SUSY breaking terms in components, we use the following convention, popular in the MSSM literature:
    - component fields with no tilde are the known particles (or Higgs like scalars)
    - component fields with a tilde are the unobserved superpartners

MSSM with soft breaking

Schematically, they soft SUSY breaking terms fall into the following classes:

- Gaugino mass terms, e.g.

$$\mathscr{L} \supset M \tilde{G} \tilde{G} + \mathrm{h.c.}$$

- where $\tilde{G}$ is the "gluino", i.e. the gaugino partner of the gluon (gauge field of $SU(3)$)
- "A-terms": trilinear couplings among scalar fields, one of which is a Higgs scalar and the other two are sleptons and/or squarks. E.g.

$$\mathscr{L} \supset A_{ij} \tilde{\bar{u}}^i \tilde{Q}^j H_u$$

- "Real" mass terms for the squarks and sleptons, as well as Higgses, e.g.

$$\mathscr{L} \supset (m^2)^i{}_j \tilde{Q}_i^\dagger \tilde{Q}^j \quad , \qquad \mathscr{L} \supset (m_{H_u}^2)^i{}_j H_u^\dagger H_u \ , \qquad \mathscr{L} \supset (m_{H_d}^2)^i{}_j H_d^\dagger H_d$$

- "Complex" mass term for the Higgses,

$$\mathscr{L} \supset b \ \ H_d H_u + \mathrm{h.c.}$$

- This looks like the μ term, but is not a term in the scalar potential, rather than the SUSY superpotential
- The MSSM with R-parity and soft SUSY breaking has 105 more parameters than the SM!

Gauge coupling unification

- One can use the RG equations and the three beta functions for the gauge couplings of the factors $SU(3)$, $SU(2)$, $U(1)_Y$ to extrapolate the experimentally measured values of the couplings to higher and higher energies
- Gauge coupling unification is the idea that the three individual couplings should become one at sufficiently high energies. The low energy gauge group $SU(3) \times SU(2) \times U(1)_Y$ is interpreted in these scenarios as coming from spontaneous breaking of a bigger gauge group (e.g. $SU(5)$). Such scenarios are referred to as Grand Unification Theories (GUT)
- In the SM, as we approach higher energies, the three couplings come close to approaching the same value, but they don't quite match
- In the MSSM, the situation is improved considerably. If the masses of the SUSY partners are approx $3 \text{ GeV} < \mathrm{M}_{\mathrm{SUSY}} < 100 \text{ TeV}$, the three couplings come extremely close to unifying at a scale $M_{\mathrm{GUT}} \approx 2 \times 10^{16} \text{ GeV}$. Cfr with the Plank mass $M_{\mathrm{Pl}} \approx 10^{19} \text{ GeV}$
- Precise tests of gauge coupling unifications are hard because of "threshold effects", which happen as we approach higher and higher energy when we approach the mass of the superpartners (and possibly other unobserved massive particles)

Challenges of the MSSM

- The μ-problem

  The SUSY preserving mass term μ has to satisfy stringent inequalities in relation to the soft SUSY breaking masses b, $m^2_{H_u}$, $m^2_{H_d}$. To get a scalar potential for the Higgses that gives stable vacua with non-sero VEVs (so that EW gauge symmetry is broken) we need

  $$2\,b < 2\,|\,\mu\,|^2 + m^2_{H_d} + m^2_{H_u} \ , \qquad b^2 > (\,|\,\mu^2\,| + m^2_{H_u})(\,|\,\mu^2\,| + m^2_{H_d})$$

  Problem: find a mechanism to engineer μ, b, $m^2_{H_u}$, $m^2_{H_d}$ that obey these constraints

Challenges of the MSSM

- Individual lepton numbers can be violated too much

  In the SM we have individual lepton number conservation for the three generations. This forbids for example a decay of the form $\mu^- \to e^- \gamma$. For generic values of the parameters, the MSSM has a slepton mass matrix that mixed the three generations by an amount that is too large to be compatible on bounds on decays such as $\mu^- \to e^- \gamma$

Challenges of the MSSM

- CP violation can be too strong

  CP violation comes from complex phases in the Lagrangian that cannot be removed by field redefinitions. In the SM, the CKM matrix gives us one complex phase. In the MSSM, the soft SUSY breaking A-terms can potentially give more complex phases. This can result in a violation of CP that is too strong (for example, an electric dipole moment for the neutron that is too large to fit experiment)

Challenges of the MSSM

- FCNCs and not-strong-enough GIM suppression

  In the SM there are not tree-level FCNCs, and the GIM mechanism suppresses them at 1-loop. In the MSSM, the soft SUSY breaking terms can invalidate the GIM suppression mechanism. This would give too strong FCNC effects that contradict for instance the data on $K^0\overline{K}^0$ mixing

Challenges of the MSSM

- Roughly speaking, the soft masses and A-terms are required to be “aligned” to the “good” Yukawa couplings in a special way to avoid unwanted effects
- We are over-simplifying. The phenomenology of the MSSM is extremely rich but we will not explore it further in these lectures

Supersymmetry and supergravity

Lecture 30

Supergravity: main ideas

- So far we have considered theories that are invariant under global (a.k.a. rigid) SUSY
- The SUSY parameters ξ^α, $\bar{\xi}^{\dot{\alpha}}$ are constants, independent of the spacetime position
- NB: In deriving the SUSY current, we have sometimes treated ξ^α, $\bar{\xi}^{\dot{\alpha}}$ as spacetime dependent. This is just a formal trick
- It is natural to ask: can we “gauge SUSY”? This would mean finding a theory that is invariant under a set of SUSY transformations in which the parameters ξ^α, $\bar{\xi}^{\dot{\alpha}}$ can be arbitrary functions of spacetime
- The answer is positive, but there is a price to pay: we must necessarily include dynamical gravity in the picture. The resulting theory is a supergravity theory

Supergravity: main ideas

- Intuitively, we can see this from the SUSY algebra. The commutator of two SUSY variations is a translation. If the parameters of the SUSY transformations are promoted from constants to arbitrary functions of spacetime, the same must happen to the parameters of translations
- A rigid translation with parameter a^μ is $x'^\mu = x^\mu + a^\mu$
- A translation in which the parameter a^μ is promoted to an arbitrary function of x^μ is nothing but a general coordinate transformation $x'^\mu = x'^\mu(x)$ (diffeomorphism)
- A theory that is invariant under diffeomorphisms necessarily contains the metric $g_{\mu\nu}$ as a dynamical field, and is thus a theory of dynamical gravity
- These arguments are heuristic, but lead to the correct conclusions

The gravitino

- Let us consider a QFT in flat spacetime, and let us imagine to couple it to dynamical gravity. At linear order, we consider small metric fluctuations around the Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The symmetric field $h_{\mu\nu}$ describes the graviton, which is a massless particle of helicity ± 2
- At linear order, the metric fluctuation $h_{\mu\nu}$ couples to the stress tensor of the QFT. Schematically:

$$\mathscr{L}_{\text{int,lin}} = h_{\mu\nu} T^{\mu\nu}$$

- If we want to gauge SUSY, we expect to need a field that can couple in a similar way to the SUSY current J^μ_α. We then see that the "gauge field for SUSY" should be a vector-spinor, so that schematically

$$\mathscr{L}_{\text{int,lin}} = \psi^\alpha_\mu J^\mu_\alpha + \overline{\psi}^\mu_{\dot{\alpha}} \bar{J}^{\dot{\alpha}}_\mu$$

- This expectation is correct: all supergravity theories contain a vector-spinor field known as gravitino. One finds that this field corresponds to massless particles of helicities $\pm 3/2$

The gravitino

- We can see why we should expect particles of helicity $\pm 3/2$ from the structure of SUSY multiplets, too
- In 4d $\mathcal{N} = 1$, massless multiplets consists of pairs of helicities $(\lambda, \lambda - 1/2)$. We need helicities ± 2 to describe the graviton. At the same time, we want to avoid particles with helicities higher than 2 in absolute value, because it is not known how to describe interactions for such fields. We must then choose

$$(2,3/2) \quad \text{and} \quad (-3/2,-2) \quad \text{for CPT}$$

The on-shell gravity multiplet

- The graviton has 2 on-shell d.o.f.'s: helicities ± 2
- The gravitino has 2 on-shell d.o.f.'s: helicities $\pm 3/2$
- The on-shell supergravity multiplet of minimal 4d $\mathcal{N} = 1$ supergravity is very simple:

on-shell gravity multiplet $\qquad (g_{\mu\nu}, \psi_{\mu\alpha})$

- In 4d with $\mathcal{N} \geq 2$ and in more than 4 dimensions the on-shell gravity multiplets contain also massless fields corresponding to particles of helicities ± 1 and/or 0

Caveat on notation

- For the rest of this lecture, I will switch to 4-component notation for spinors, including the gravitino and the SUSY parameters
- The 4-component gravitino is a Majorana vector-spinor. It is given in terms of the 2-component gravitino by the usual relation

$$\psi_{\mu\ 4-\mathrm{comp}} = \begin{pmatrix} \psi_{\mu\alpha} \\ \overline{\psi}_{\mu}{}^{\dot{\alpha}} \end{pmatrix}$$

- The SUSY parameter will be denoted ϵ and is a 4-component Majorana fermion
- Both ψ_μ and ϵ are Grassmann-odd
- A bar over a 4-component Majorana fermions denotes either Dirac conjugate, or Majorana conjugate, which are equal thanks to the Majorana condition. Up to convention-dependent phase factors we can write

$$\overline{\psi} = \psi^\dagger \gamma^0 = \psi^T C$$

Spinors in curved spacetime

- In SUGRA, gravity is dynamical. The geometry of spacetime is determined by Einstein’s equation
- In order to formulate SUGRA, we have to consider spinors in an arbitrary curved spacetime
- Let us discuss the “physicist’s approach” to this problem: we will be writing local expressions for the covariant derivatives of spinor fields
- Spinors in curved space(time) can be rigorously defined using a more refined mathematical language
- The new tool we need is a reformulation of GR via the “vielbein formalism”

Vielbein formalism

- The name comes from the German for “many legs”. In 4d, sometimes it is called a vierbein, from the German for “four legs”.
- In the vielbein formalism, the metric $g_{\mu\nu}(x)$ is written in terms of the flat, constant Minkowski metric η_{ab} and a positition-dependent square matrix $e^a{}_\mu(x)$

$$g_{\mu\nu}(x) = \eta_{ab}\, e^a{}_\mu(x)\, e^b{}_\nu(x)$$

- The inverse of $e^a{}_\mu(x)$ is denoted $e_a{}^\mu(x)$ and satisfies

$$e^a{}_\mu(x)\, e_a{}^\nu(x) = \delta^\nu_\mu \;\; , \qquad e^a{}_\mu(x)\, e_b{}^\mu(x) = \delta^a_b$$

- The μ, ν indices are called “curved indices”. They are always raised/lowered with $g_{\mu\nu}(x)$
- The a, b indices are called “flat indices”. They are always raised/lowered with η_{ab}

Spin connection

- Using the vielbein $e^a{}_\mu(x)$ and its inverse $e_a{}^\mu(x)$ one constructs the so-called spin connection $\omega_{\mu ab}(x)$. It has one curved index and two flat indices. It is antisymmetric in its flat indices:

$$\omega_{\mu ab}(x) = -\,\omega_{\mu ba}(x)$$

- There is an explicit formula for $\omega_{\mu ab}(x)$ in terms of $e^a{}_\mu(x)$, $e_a{}^\mu(x)$

$$\omega_\mu{}^{ab} = \frac{1}{2}\, e^{a\nu}\,(\partial_\mu e^b{}_\nu - \partial_\nu\, e^b{}_\mu) - \frac{1}{2}\, e^{b\nu}\,(\partial_\mu e^a{}_\nu - \partial_\nu\, e^a{}_\mu) - \frac{1}{2}\, e^{a\rho}\, e^{b\sigma}\, e^c{}_\mu\,(\partial_\rho e_{c\sigma} - \partial_\sigma e_{c\rho})$$

- Recall: we raise/lower flat indices with η, so $\omega_\mu{}^{ab} = \eta^{ac}\,\eta^{bd}\,\omega_{\mu cd}$, $e^{a\mu} = \eta^{ab}\, e_b{}^\mu$, and so on

Covariant derivative of a spinor field

- Suppose $\epsilon(x)$ is a spinor field, i.e. a spinor that depends on the position x on spacetime
- In an arbitrary curved spacetime, the partial derivative $\partial_\mu \epsilon$ does not transform covariantly. It must be amended with a term including the spin connection constructed from the vielbein:

$$\nabla_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\,\omega_{\mu ab}\,\gamma^{ab}\,\epsilon$$

- Here $\gamma^{ab} = \gamma^{[a}\gamma^{b]}$ where the γ^a are the gamma matrices of Minkowski spacetime

$$\{\gamma^a, \gamma^b\} = 2\,\eta^{ab}$$

- NB: γ^a is a constant matrix that does not depend on x, just like η_{ab}
- The formula for $\nabla_\mu \epsilon$ holds in any dimension for Dirac or Majorana spinors

The universal part of the SUGRA action

- The Einstein-Hilbert term describes the "kinetic terms" of GR in any spacetime dimension. We write it in 4d for definiteness:

$$S_{\mathrm{EH}} = \frac{1}{2\,\kappa^2} \int d^4x \sqrt{-g}\, R$$

- Here R is the Ricci scalar constructed from the metric (which is in turn constructed from the vielbein $e^a{}_\mu$). κ^2 is a parametrization of Newton's constant.
- The analog of the Einstein-Hilbert action for the gravitino is the Rarita-Schwinger action. It contains the "kinetic terms" of the gravitino ψ_μ in all SUGRA models
- The Rarita-Schwinger action reads (we write it in 4d for definiteness)

$$S_{\mathrm{RS}} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, \overline{\psi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho$$

The Rarita-Schwinger term

$$S_{\mathrm{RS}} = -\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,\overline{\psi}_\mu\,\gamma^{\mu\nu\rho}\,\nabla_\nu\psi_\rho$$

- $d^4x\sqrt{-g}$ is the volume form constructed from the metric (constructed from the vielbein), familiar from the Einstein-Hilbert term
- The covariant derivative $\nabla_\nu\psi_\rho$ is defined by

$$\nabla_\nu\psi_\rho = \partial_\nu\psi_\rho + \frac{1}{4}\,\omega_{\nu ab}\,\gamma^{ab}\,\psi_\rho$$

- The object $\gamma^{\mu\nu\rho}$ is by definition

$$\gamma^{\mu\nu\rho} = e_a{}^\mu\, e_b{}^\nu\, e_c{}^\rho\,\gamma^{abc} \quad , \qquad \gamma^{abc} = \gamma^{[a}\,\gamma^b\,\gamma^{c]}$$

- NB: γ^{abc} is a constant, but $\gamma^{\mu\nu\rho}$ depends on x because of the three $e_a{}^\mu(x)$ factors

The action of minimal SUGRA in 4d

The action of 4d minimal SUGRA takes the form

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - \overline{\psi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho \right] + \text{(terms with four } \psi_\mu\text{'s)}$$

It is invariant under local SUSY transformations:

$$\delta_{\mathrm{SUSY}}\, e^a{}_\mu = \frac{1}{2}\, \bar{\epsilon}\, \gamma^a \psi_\mu$$

$$\delta_{\mathrm{SUSY}}\, \psi_\mu = \nabla_\mu \epsilon + \text{(terms with one } \epsilon \text{ and two } \psi_\mu\text{'s)}$$

The action of minimal SUGRA in 4d

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - \overline{\psi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \psi_\rho \right] + \text{(terms with four } \psi_\mu \text{'s)}$$

$$\delta_{\text{SUSY}}\, e^a{}_\mu = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu \quad ,$$

$$\delta_{\text{SUSY}}\, \psi_\mu = \nabla_\mu \epsilon + \text{(terms with one } \epsilon \text{ and two } \psi_\mu \text{'s)}$$

- We do not write down the 4-fermi terms in the action and the 3-fermi term in the SUSY variation because they are quite involved
- The point is: they are completely determined by invariance under local SUSY
- NB: we are describing on-shell SUSY. The algebra of SUSY variations closes only using the equations of motion
- There exist various off-shell formalisms for minimal 4d SUGRA

General structure of SUGRA actions

- The action for any SUGRA model takes the following schematic form:

$$S = S_{\mathrm{B}} + S_{\mathrm{F}}$$

- S_{B} is the part of the action that contains bosonic fields only. It always contains the EH term, plus extra terms in the presence of bosonic matter fields (scalars, vectors)
- S_{F} is the part of the action that contains fermionic fields. It always contains the RS term for the gravitino.
- Simple observation: S_{F} always contains an <u>even</u> number of fermionic fields
- The local SUSY variations in any SUGRA model take the schematic form

$$\delta_{\mathrm{SUSY}}(\text{boson}) = \text{terms with one } \epsilon \text{ and an odd \# of fermions}$$

$$\delta_{\mathrm{SUSY}}(\text{fermion}) = \text{terms with one } \epsilon \text{ and an even \# of fermions}$$

SUSY solutions of SUGRA theories

- We are usually interested in studying solutions of SUGRA theories in which only the bosonic fields are activated (e.g. a non-trivial metric $g_{\mu\nu}(x)$ or a non-trivial profile for scalar fields, if present)
- We set all fermionic fields to zero. This is always consistent with the EOMs because all terms in the action have either 0 fermions, or an even number of fermions
- By definition, we say that a solution to a SUGRA theory is supersymmetric if a spinor $\epsilon(x)$ can be found such that

$$\delta_{\text{SUSY}}(\text{all fields})\Big|_{\text{evaluated at the solution}} = 0$$

SUSY solutions of SUGRA theories

$$\delta_{\mathrm{SUSY}}(\text{all fields})\Big|_{\text{evaluated at the solution}} = 0$$

- In practice, the SUSY variations of bosons evaluated at the solution are automatically zero, because $\delta_{\mathrm{SUSY}}(\mathrm{boson})$ contains an odd number of fermions
- The only variations to check are those of the fermions:

$$\delta_{\mathrm{SUSY}}(\text{all fermionic fields})\Big|_{\text{evaluated at the solution}} = 0$$

- These are the <u>BPS equations</u> of the SUGRA theory

Example in minimal SUGRA

- We have seen that the bosonic action S_{B} is simply the EH term. If we set the gravitino to zero, the EOM for the metric is simply the vacuum Einstein equation without cosmological constant

$$R_{\mu\nu} = 0$$

- What are the BPS equations of this SUGRA model? We take the SUSY variation of the gravitino

$$\delta_{\mathrm{SUSY}}\,\psi_\mu = \nabla_\mu \epsilon + \text{(terms with one } \epsilon \text{ and two } \psi_\mu\text{'s)}$$

- We neglect terms with two gravitini and we get the simple condition

$$\nabla_\mu \epsilon = 0$$

- A spinor field $\epsilon(x)$ that satisfied the above PDE is called a Killing spinor

Trivial example: Minkowski spacetime

- A trivial example of supersymmetric solution is Minkowski spacetime:

$$g_{\mu\nu}(x) = \eta_{\mu\nu}$$

- The vielbein in this case can taken to be a constant: $e^a{}_\mu(x) = \delta^a{}_\mu$
- The spin connection is zero because all derivatives of $e^a{}_\mu(x)$ vanish
- The Killing spinor equation reduces to $\partial_\mu \epsilon(x) = 0$ which is solved by

$$\epsilon(x) = \text{constant}$$

Supergravity and renormalizabity

- Just like GR, SUGRA models are non-renormalizable
- At the quantum level they make sense as low-energy effective actions valid below some cutoff (typically the Planck mass). For energies higher than the cutoff we need a UV completion
- In many interesting cases the UV completion is provided by constructions in string theory or M-theory

Supergravity and string theory

- SUGRA theories are used to describe the low-energy dynamics of string theory and M-theory
- Supersymmetric solutions of SUGRA theory are a powerful window into the dynamics of string/M-theory setups
- Even though SUGRA is only a low-energy approximation of the full string/M-theory setup, supersymmetric solutions of SUGRA are usually protected against quantum corrections
- We can trust the conclusions of an analysis in the SUGRA approximation to learn more about string/M-theory setups

Matter-coupled supergravity

- It is possible to construct matter-coupled 4d $\mathscr{N} = 1$ models that contain the gravity multiplet, together with arbitrary chiral multiplets and vector multiplets
- These models can be quite complicated and we will not be able to study them in detail
- Let us highlight some important qualitative differences between rigid SUSY models and SUGRA models
- For simplicity, we ignore vector multiplets and we consider chiral multiplets only

Matter-coupled supergravity

- In rigid SUSY, the data of a model with chiral multiplets are the Kähler potential K and the superpotential W. The scalar potential is

$$\text{rigid SUSY:} \qquad V = g^{i\bar{j}} \frac{\partial W}{\partial X^i} \frac{\partial \overline{W}}{\partial \overline{X}^{\bar{j}}}$$

- Here $g^{i\bar{j}}$ is the inverse of the Kähler metric $g_{i\bar{j}} = \partial^2 K / (\partial X^i \partial \overline{X}^{\bar{j}})$

- In SUGRA with chiral multiplets, the defining data is still K and W, but the scalar potential has a different form:

$$V = e^K \left(g^{i\bar{j}} D_i W D_{\bar{j}} \overline{W} - 3\,|W|^2 \right) \quad , \qquad D_i W = \frac{\partial W}{\partial X^i} + \frac{\partial K}{\partial X^i} W$$

Matter-coupled supergravity

$$V = e^K \left(g^{i\bar{j}} D_i W D_{\bar{j}} \overline{W} - 3\,|W|^2 \right) \quad , \qquad D_i W = \frac{\partial W}{\partial X^i} + \frac{\partial K}{\partial X^i} W$$

- The potential is not positive-definite!
- The value of the potential at the minumum is the cosmological constant. If we want Minkowski spacetime, we have to choose the scalar VEVs in such a way that $V = 0$
- We can have $V = 0$ and also break SUSY, contrary to rigid models
- Is it possible to have a “super-Higgs” effect: SUSY is spontaneously broken; the would-be Goldstino is eaten by the gravitino, to get a massive spin-3/2 field