Supersymmetry and Supergravity — Problem Sheet 3

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These problems refer to material covered in Lectures 1 through 16. They are due by Saturday before the class on week 7 by 11 am. Links to submit:
TA A. Boido: https://cloud.maths.ox.ac.uk/index.php/s/WP8kazik5pNZjmi
TA J. McGovern: https://cloud.maths.ox.ac.uk/index.php/s/oBKgZcaE9F4bw3z

1 Action of Lorentz generators in superspace

Let us consider flat superspace, defined as (super-Poincaré group)/(Lorentz group), with coset representative

$$G(x,\theta,\overline{\theta}) = \exp(-i\,x^\mu\,P_\mu + i\,\theta^\alpha\,Q_\alpha + i\,\overline{\theta}_{\dot\alpha}\,\overline{Q}^{\dot\alpha}) \ . \tag{1}$$

Recall that the super-Poincaré group acts on superspace from the left. An element g_0 of the super-Poincaré group induces a motion $(x^\mu,\theta^\alpha,\overline{\theta}_{\dot\alpha}) \to (x'^\mu,\theta'^\alpha,\overline{\theta}'_{\dot\alpha})$ in superspace according to

$$g_0^{-1}\,G(x,\theta,\overline{\theta}) = G(x',\theta',\overline{\theta}')\,H(x,\theta,\overline{\theta};g_0) \ , \tag{2}$$

where $H(x,\theta,\overline{\theta};g_0)$ is a compensating Lorentz transformation.

1.a Consider

$$g_0 = \exp(\tfrac{1}{2}\,i\,\lambda^{\mu\nu}\,J_{\mu\nu}) \ , \tag{3}$$

where $\lambda^{\mu\nu} = \lambda^{[\mu\nu]}$ are constant parameters. Working at linear order in $\lambda^{\mu\nu}$, determine the associated motion in superspace: $(x^\mu,\theta^\alpha,\overline{\theta}_{\dot\alpha}) \to (x'^\mu,\theta'^\alpha,\overline{\theta}'_{\dot\alpha}) = (x^\mu+\delta x^\mu,\theta^\alpha+\delta\theta^\alpha,\overline{\theta}_{\dot\alpha}+\delta\overline{\theta}_{\dot\alpha})$.

1.b The differential operator associated to this motion is superspace is

$$\delta x^\mu\,\partial_\mu + \delta\theta^\alpha\,\frac{\partial}{\partial\theta^\alpha} + \delta\overline{\theta}_{\dot\alpha}\,\frac{\partial}{\partial\overline{\theta}_{\dot\alpha}} \equiv \tfrac{1}{2}\,i\,\lambda^{\mu\nu}\,\mathbf{J}_{\mu\nu} \ , \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu} \ . \tag{4}$$

Verify that $\mathbf{J}_{\mu\nu}$ is given by the expression

$$\mathbf{J}_{\mu\nu} = -i\,(x_\mu\,\partial_\nu - x_\nu\,\partial_\mu) + i\,(\sigma_{\mu\nu})_\beta{}^\alpha\,\theta^\beta\,\frac{\partial}{\partial\theta^\alpha} + i\,(\overline{\sigma}_{\mu\nu})^{\dot\beta}{}_{\dot\alpha}\,\overline{\theta}_{\dot\beta}\,\frac{\partial}{\partial\overline{\theta}_{\dot\alpha}} \ . \tag{5}$$

Hint: The following commutators can be useful:

$$[J_{\mu\nu},P_\rho] = i\,\eta_{\mu\rho}\,P_\nu - i\,\eta_{\nu\rho}\,P_\mu \ , \quad [J_{\mu\nu},Q_\alpha] = -i\,(\sigma_{\mu\nu})_\alpha{}^\beta\,Q_\beta \ , \quad [J_{\mu\nu},\overline{Q}^{\dot\alpha}] = -i\,(\overline{\sigma}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,\overline{Q}^{\dot\beta} \ . \tag{6}$$

2 Chiral superspace and chiral superfields

The "standard coordinates" $(x^\mu,\theta^\alpha,\overline{\theta}_{\dot\alpha})$ in superspace are defined by the coset representative $G(x,\theta,\overline{\theta})$ in (1). We can also introduce "chiral coordinates" $(y^\mu,\vartheta^\alpha,\overline{\vartheta}_{\dot\alpha})$ and "antichiral coordinates" $(\widehat{y}^\mu,\widehat{\vartheta}^\alpha,\widehat{\overline{\vartheta}}_{\dot\alpha})$. They are defined by the relations

$$\begin{aligned}\exp(-i\,x^\mu\,P_\mu + i\,\theta^\alpha\,Q_\alpha + i\,\overline{\theta}_{\dot\alpha}\,\overline{Q}^{\dot\alpha}) &= \exp(i\,\vartheta^\alpha\,Q_\alpha)\,\exp(-i\,y^\mu\,P_\mu)\,\exp(i\,\overline{\vartheta}_{\dot\alpha}\,\overline{Q}^{\dot\alpha})\\ &= \exp(i\,\widehat{\overline{\vartheta}}_{\dot\alpha}\,\overline{Q}^{\dot\alpha})\,\exp(-i\,\widehat{y}^\mu\,P_\mu)\,\exp(i\,\widehat{\vartheta}^\alpha\,Q_\alpha)\end{aligned} \tag{7}$$

2.a Compute $(y^\mu, \vartheta^\alpha, \overline{\vartheta}_{\dot\alpha})$ and $(\widehat{y}^\mu, \widehat{\vartheta}^\alpha, \overline{\widehat{\vartheta}}_{\dot\alpha})$ in terms of $(x^\mu, \theta^\alpha, \overline{\theta}_{\dot\alpha})$. You should find

$$\vartheta^\alpha = \theta^\alpha \ , \qquad \overline{\vartheta}_{\dot\alpha} = \overline{\theta}_{\dot\alpha} \ , \qquad y^\mu = x^\mu + i\,\theta\,\sigma^\mu\,\overline{\theta} \ , \tag{8}$$

$$\widehat{\vartheta}^\alpha = \theta^\alpha \ , \qquad \overline{\widehat{\vartheta}}_{\dot\alpha} = \overline{\theta}_{\dot\alpha} \ , \qquad \widehat{y}^\mu = x^\mu - i\,\theta\,\sigma^\mu\,\overline{\theta} \ . \tag{9}$$

Verify that y^μ and $\widehat{y}^\mu$ are complex conjugates of each other.

2.b Act on

$$\exp(i\,\vartheta^\alpha\,Q_\alpha)\,\exp(-i\,y^\mu\,P_\mu)\,\exp(i\,\overline{\vartheta}_{\dot\alpha}\,\overline{Q}^{\dot\alpha}) \tag{10}$$

from the <u>left</u> with $\exp(-i\,\xi^\alpha\,Q_\alpha - i\,\overline{\xi}_{\dot\alpha}\,\overline{Q}_{\dot\alpha})$. Find the infinitesimal motion $(\delta y^\mu, \delta\vartheta^\alpha, \delta\overline{\vartheta}^{\dot\alpha})$ induced by this action. The corresponding differential operator is

$$\delta y^\mu\,\frac{\partial}{\partial y^\mu} + \delta\vartheta^\alpha\,\frac{\partial}{\partial\vartheta^\alpha} + \delta\overline{\vartheta}_{\dot\alpha}\,\frac{\partial}{\partial\overline{\vartheta}_{\dot\alpha}} \equiv i\,\xi^\alpha\,\mathbf{Q}_\alpha + i\,\overline{\xi}_{\dot\alpha}\,\overline{\mathbf{Q}}^{\dot\alpha} \ . \tag{11}$$

Use this relation to find the expressions of the differential operators $\mathbf{Q}_\alpha$, $\overline{\mathbf{Q}}_{\dot\alpha}$ in the coordinate system $(y^\mu, \vartheta^\alpha, \overline{\vartheta}^{\dot\alpha})$.

In a similar way, consider now the <u>right</u> action of $\exp(-i\,\xi^\alpha\,Q_\alpha - i\,\overline{\xi}_{\dot\alpha}\,\overline{Q}_{\dot\alpha})$ on (10). Find the corresponding motion $(\delta_{\rm R} y^\mu, \delta_{\rm R}\vartheta^\alpha, \delta_{\rm R}\overline{\vartheta}^{\dot\alpha})$, where R is stand for "right action". The associated differential operator is

$$\delta_{\rm R} y^\mu\,\frac{\partial}{\partial y^\mu} + \delta_{\rm R}\vartheta^\alpha\,\frac{\partial}{\partial\vartheta^\alpha} + \delta_{\rm R}\overline{\vartheta}_{\dot\alpha}\,\frac{\partial}{\partial\overline{\vartheta}_{\dot\alpha}} \equiv -\overline{\xi}_{\dot\alpha}\,\overline{D}^{\dot\alpha} - \xi^\alpha\,D_\alpha \ . \tag{12}$$

Use this relation to find the expression of the SUSY covariant derivatives D_α, $\overline{D}_{\dot\alpha}$ in the coordinate system $(y^\mu, \vartheta^\alpha, \overline{\vartheta}^{\dot\alpha})$.

Check that the expressions for $\mathbf{Q}_\alpha$, $\overline{\mathbf{Q}}_{\dot\alpha}$, D_α, $\overline{D}_{\dot\alpha}$ in the coordinate system $(y^\mu, \vartheta^\alpha, \overline{\vartheta}^{\dot\alpha})$ can also be obtained from the expressions in the coordinate system $(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha})$, which read

$$\begin{aligned} \mathbf{Q}_\alpha &= i\left[\frac{\partial}{\partial\theta^\alpha} - i\,(\sigma^\mu\,\overline{\theta})_\alpha\,\frac{\partial}{\partial x^\mu}\right] , & \overline{\mathbf{Q}}_{\dot\alpha} &= i\left[-\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + i\,(\theta\,\sigma^\mu)_{\dot\alpha}\,\frac{\partial}{\partial x^\mu}\right] , \\ D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\,(\sigma^\mu\,\overline{\theta})_\alpha\,\frac{\partial}{\partial x^\mu} \ , & \overline{D}_{\dot\alpha} &= -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} - i\,(\theta\,\sigma^\mu)_{\dot\alpha}\,\frac{\partial}{\partial x^\mu} \ . \end{aligned} \tag{13}$$

Change coordinates using the transformation between $(y^\mu, \vartheta^\alpha, \overline{\vartheta}^{\dot\alpha})$ and $(x^\mu, \theta^\alpha, \overline{\theta}^{\dot\alpha})$ and the chain rule.

2.c A chiral superfield is expanded in chiral coordinates as

$$\Phi(y, \vartheta\,\overline{\vartheta}) = X(y) + \sqrt{2}\,\vartheta\,\psi(y) + \vartheta\,\vartheta\,F(y) \ . \tag{14}$$

Derive the expansion of Φ in the standard coordinates $(x, \theta, \overline{\theta})$:

$$\begin{aligned} \Phi &= X(x) + i\,\theta\,\sigma^\mu\,\overline{\theta}\,\partial_\mu(x) + \tfrac{1}{4}\,(\theta\,\theta)\,(\overline{\theta}\,\overline{\theta})\,\partial^\mu\partial_\mu X(x) \\ &\quad + \sqrt{2}\,\theta^\alpha\psi_\alpha(x) - \tfrac{\sqrt{2}}{2}\,i\,(\theta\theta)\,(\partial_\mu\psi\,\sigma^\mu\,\overline{\theta}) + \theta\,\theta\,F(x) \ . \end{aligned} \tag{15}$$

Hint: the following identities can be useful,

$$\theta^\alpha\,\theta^\beta = -\frac{1}{2}\,\epsilon^{\alpha\beta}\,\theta\,\theta \ , \qquad (\theta\,\sigma^\mu\,\overline{\theta})(\theta\,\sigma^\nu\,\overline{\theta}) = -\frac{1}{2}\,(\theta\,\theta)\,(\overline{\theta}\,\overline{\theta})\,\eta^{\mu\nu} \ . \tag{16}$$

3 Supersymmetric version of the Higgs mechanism in SQED

Let us consider SQED with one flavor. The model is a gauge theory with gauge group $U(1)$, one chiral multiplet (X^+, ψ^+, F^+) of charge $+1$ and one chiral multiplet (X^-, ψ^-, F^-) of charge -1. We set the superpotential to zero and we do not include any FI term. After eliminating the auxiliary fields, the Lagrangian for the component fields reads

$$\begin{aligned}\mathcal{L} = & -D^\mu \overline{X^+} D_\mu X^+ - D^\mu \overline{X^-} D_\mu X^- - i\,\overline{\psi^+}\,\overline{\sigma}^\mu D_\mu \psi^+ - i\,\overline{\psi^-}\,\overline{\sigma}^\mu D_\mu \psi^- \\ & - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} - i\,\overline{\lambda}\,\overline{\sigma}^\mu \partial_\mu \lambda + \Big[i\sqrt{2}\, g\, (\overline{X^+}\, \psi^+ \lambda - \overline{X^-}\, \psi^- \lambda) + \text{h.c.} \Big] \\ & - \tfrac{1}{2} g^2 (\overline{X^+} X^+ - \overline{X^-} X^-)^2 \ . \end{aligned} \tag{17}$$

The gauge covariant derivatives are

$$D_\mu X^\pm = \partial_\mu X^\pm \pm i\, g\, A_\mu X^\pm \ , \qquad D_\mu \psi^\pm = \partial_\mu \psi^\pm \pm i\, g\, A_\mu \psi^\pm \ . \tag{18}$$

3.a Let v be a positive constant. Verify that the VEVs

$$\langle X^+ \rangle = v \ , \qquad \langle X^- \rangle = v \tag{19}$$

give a supersymmetric vacuum of the model and that the $U(1)$ gauge symmetry is spontaneously broken in this vacuum.

3.b We want to study the spectrum of scalar modes in the spontaneously broken phase. To this end, it is convenient to perform the following field redefinitions,

$$X^+ = \frac{e^{ia}}{2} (S + \varphi_1 + i\,\varphi_2) \ , \qquad X^- = \frac{e^{-ia}}{2} (S - \varphi_1 + i\,\varphi_2) \ , \tag{20}$$

where a, S, φ_1, φ_2 are real scalar fields. Rewrite the kinetic terms for $X^\pm$ and the scalar potential, which are given by

$$\mathcal{L}_{\text{kin},X^\pm} = -D^\mu \overline{X^+} D_\mu X^+ - D^\mu \overline{X^-} D_\mu X^- \ , \qquad V = \tfrac{1}{2} g^2 (\overline{X^+} X^+ - \overline{X^-} X^-)^2 \ , \tag{21}$$

in terms of the real scalars a, S, φ_1, φ_2 and the gauge field A_μ. Verify that the real scalars S, φ_1, φ_2 have canonical kinetic terms and that A_μ and a enter $\mathcal{L}_{\text{kin},X^\pm}$ in the combination $V_\mu = A_\mu + g^{-1} \partial_\mu a$.

3.c In terms of the scalar fields a, S, φ_1, φ_2, the field that gets a non-zero VEV is S. Find $\langle S \rangle$ by comparing (19) and (20). We expand the scalar field S around its VEV as $S = \langle S \rangle + \delta S$. Use the expressions for $\mathcal{L}_{\text{kin},X^\pm}$ and V derived in the previous point to verify that: the vector V^μ has $m^2 = 4\, g^2 v^2$; the scalar φ_1 has $m^2 = 4\, g^2 v^2$; the scalars δS, φ_2, a are massless.

3.d We now consider the spectrum of fermions in the model. To this end, it is convenient to perform the field redefinition

$$\begin{pmatrix} \psi^+ \\ \psi^- \\ \lambda \end{pmatrix} = \mathcal{U} \begin{pmatrix} \Psi_0 \\ \Psi_1 \\ \Psi_2 \end{pmatrix} \ , \qquad \mathcal{U} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2} e^{i\pi/4} & -\frac{1}{2} e^{-i\pi/4} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} e^{i\pi/4} & \frac{1}{2} e^{-i\pi/4} \\ 0 & \frac{1}{\sqrt{2}} e^{i\pi/4} & \frac{1}{\sqrt{2}} e^{-i\pi/4} \end{pmatrix} \ . \tag{22}$$

Verify that $\mathcal{U}$ is unitary, which implies that Ψ_0, Ψ_1, Ψ_2 have canonical kinetic terms. Plugging (19) into the Yukawa couplings in (17), verify that the mass terms are diagonal when written in terms of the new fermionic fields Ψ_0, Ψ_1, Ψ_2. Verify that Ψ_0 is massless, while Ψ_1 and Ψ_2 both have $|m| = 2\,g\,v$.

Interpretation: The massless real scalar a is the pseudo-Goldstone boson. It is "eaten" by the massless vector A_μ to yield a massive vector. The latter is described by the gauge-invariant combination $V_\mu = A_\mu + g^{-1}\,\partial_\mu a$. The massive vector V_μ, together with the real scalar φ_1 and the fermions Ψ_1, Ψ_2 forms a massive vector multiplet of 4d $\mathcal{N} = 1$ supersymmetry. The massless scalars δS and φ_2, together with the fermion Ψ_0, form a massless chiral multiplet. This example illustrates some general feature of the SUSY version of the Higgs mechanism. If gauge invariance is broken, but SUSY is unbroken, for each broken generator of the gauge group a massless vector multiplet eats a massless chiral multiplet to yield a massive vector multiplet.