Supersymmetry and supergravity

Lecture 1

Introduction

- Symmetries are a powerful tool in the study of quantum systems
- Our focus: relativistic QFTs and their continuous symmetries
- Examples of symmetries familiar from particle physics
  - Poincaré symmetry
  - Internal/flavor symmetries (e.g. baryon, lepton numbers in SM)
- These symmetries map bosons to bosons and fermions to fermions
- **Supersymmetry** (SUSY) is a generalization in which bosons and fermions are transformed into each other
- Bosonic fields/particles are combined with fermionic fields/particles into representations of SUSY (**supermultiplets**)
- When SUSY is combined with GR we get **supergravity** (SUGRA)

Motivation

From a particle physics/phenomenology point of view:

- If SUSY is a symmetry in Nature, it must be broken at the energies we can probe in experiment (observed particles do not fit into supermultiplets)
- Nonetheless, SUSY can help solve some puzzles in the SM. For example:
    - Hierarchy problem

      In the SM, the mass of the Higgs field receives 1-loop corrections that are *quadratic* in the cutoff scale Λ: $\delta m_H^2 \sim \Lambda^2$. Puzzle:

      Why is $m_H \approx 125$ GeV much smaller than $\Lambda_{\text{GUT}} \sim 10^{16}$ GeV or $\Lambda_{\text{Planck}} \sim 10^{19}$ GeV ?

      The new particles and interactions required by (broken) SUSY improve the 1-loop correction to the Higgs mass: $\delta m_H^2 \sim \log \Lambda$
    - GUT scenarios: SUSY can improve gauge coupling unification
    - SUSY can provide candidates for dark matter

Motivation

From a formal QFT point of view:

- SUSY improves the UV behavior of QFTs
  - E.g.: the so-called 4d $\mathcal{N} = 4$ super-Yang-Mills theory is an interacting QFT that is free of UV divergences!
- SUSY allows analytic control beyond perturbative regime
  - understanding of confinement in 4d $\mathcal{N} = 2$ gauge theories (Seiberg-Witten)
  - computing the path integral exactly and analytically using *supersymmetric localization*
  - exploring strong coupling/weak coupling dualities
  - many more examples…

Motivation

From the point of view of string theory:

- String theory “predicts” SUGRA in ten spacetime dimensions (from self-consistency of the theory)
- SUSY gauge theories live on D-branes
- SUSY is a powerful tool for studying string constructions
    - e.g.: the landmark Strominger-Vafa computation of BH microstates in string theory relies on SUSY

SUSY has interesting connections with mathematics

- SUSY quantum mechanics and the proof of the Atiyah-Patodi-Singer index theorem
- Witten’s topological twist of 4d SUSY theories and Jones polynomials in knot theory
- many more…

Historical timeline

- 1967: Coleman-Mandula theorem (see below)
- 1971: SUSY on the 2d worksheet of strings
- 1971: SUSY in 4d QFT in the work of Gol'fand and Likhtman
- 1973: spontaneous SUSY breaking in the work of Volkov and Akulov
- 1974: SUSY in 4d QFT independently rediscovered by Wess and Zumino
- 1975: Haag-Lopuszanski-Sohnius theorem (see below)

Symmetries of the S-matrix

Relativistic QFT in 4d: Poincaré algebra

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\,\eta_{\mu\rho}\,J_{\nu\sigma} - i\,\eta_{\nu\rho}\,J_{\mu\sigma} - i\,\eta_{\mu\sigma}\,J_{\nu\rho} + i\,\eta_{\nu\sigma}\,J_{\mu\rho}$$

$$[J_{\mu\nu}, P_\rho] = i\,\eta_{\mu\rho}\,P_\nu - i\,\eta_{\nu\rho}\,P_\mu \qquad [P_\mu, P_\nu] = 0$$

$$\mu, \nu, \ldots = 0,1,2,3 \qquad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$$

The generators $J_{\mu\nu} = -J_{\nu\mu}$ and P_μ are represented by Hermitian operators acting on the Hilbert space of the theory, $(J_{\mu\nu})^\dagger = J_{\mu\nu}$, $(P_\mu)^\dagger = P_\mu$

We can also have “internal symmetries”: their generators are translational invariant Lorenz scalars

$$[J_{\mu\nu}, T^A] = 0 \qquad [P_\mu, T^A] = 0 \qquad [T^A, T^B] = i f^{AB}{}_C\, T^C$$

Natural questions:

- What is the most general continuous symmetry of a relativistic QFT?
- Can we have extra generators that are not $J_{\mu\nu}$ and P_μ but carry spacetime indices, eg X_μ?

Symmetries of the S-matrix

Coleman-Mandula theorem

Consider a 4d relativistic QFT such that

(i) there are only a finite number of particles associated to 1-particle states of a given mass

(ii) there is an energy gap between the vacuum and 1-particle states

(iii) the S-matrix is non-trivial and analytic

Then the most general symmetry generators of the S-matrix are the Poincaré generators, plus possibly internal symmetries (and the latter generate a finite-dim. compact Lie algebra)

for a proof see e.g. Weinberg vol 3

Intuitive argument: in a 2-to-2 scattering process, Poincaré invariance implies that the amplitude is a function of one variable only (the scattering angle). If we had a conserved charge with spacetime indices, the amplitude woould be further constrained, and it could not be an analytic function.

NB: if we only have massless particles, the most general symmetry is conformal symmetry + internal symmetry

In this case we do have an extra generator with spacetime indices: conformal boost K_μ

Evading the no-go theorem

- We evade the no-go theorem by allowing both for commutation and anticommutation relations between symmetry generators
- Commutator and anticommutator of operators

$$[A, B] = AB - BA \ , \quad \{A, B\} = AB + BA$$

- Anticommutation relations are familiar from canonical quantization of free Dirac field

$$\psi(x) \sim \sum_{s=\pm} \int \frac{d^3\mathbf{p}}{2p^0} \left[b(\mathbf{p}, s) u(\mathbf{p}, s) e^{ipx} + d^\dagger(\mathbf{p}, s) v(\mathbf{p}, s) e^{-ipx} \right]$$

$$\{b(\mathbf{p}, s), b^\dagger(\mathbf{p}', s')\} \sim \delta^3(\mathbf{p} - \mathbf{p}')\, \delta_{s,s'}$$
$$\{d(\mathbf{p}, s), d^\dagger(\mathbf{p}', s')\} \sim \delta^3(\mathbf{p} - \mathbf{p}')\, \delta_{s,s'}$$

- The algebra has both bosonic and fermionic generators, with

[boson, boson] = boson [boson, fermion] = fermion

{ fermion , fermion } = boson

- Mathematical formalism: Lie superalgebras

Evading the no-go theorem

Haag-Lopuszanski-Sohnius theorem

Same assumptions as Coleman-Mandula, but allowing for a Lie superalgebra of symmetry generators.

Conclusion: the most general Lie superalgebra of symmetries of the S-matrix is a “Poincaré supersymmetry algebra”, possibly “extended” and with the inclusion of “central charges” (we will unpack the terminology)

Bosonic generators:

$J_{\mu\nu}$,$P_{\mu\nu}$, and internal symm. gen’s T^A (same properties as in Coleman-Mandula)

Fermionic generators (aka supercharges):

Q_α, $\overline{Q}_{\dot{\alpha}}$ that transform as spinors under Lorenz transformations

NB: if we only have massless particles, we can have *superconformal* symmetry

A first look at the SUSY algebra

Minimal SUSY in 4d: the supercharges form one Weyl spinor (4 real dof's)

$$Q_\alpha\,, \quad \overline{Q}_{\dot\alpha} = (Q_\alpha)^\dagger \;\;, \;\; \alpha,\beta = 1{,}2, \;\; \dot\alpha,\dot\beta = 1{,}2$$

(We will review 2-component notation for 4d spinors later). The supercharges are translationally invariant:

$$[P_\mu, Q_\alpha] = 0 \;\;, \qquad [P_\mu, \overline{Q}_{\dot\alpha}] = 0$$

The crucial part of the SUSY algebra are the QQ anticommutators:

$$\{Q_\alpha, Q_\beta\} = 0 \qquad\qquad \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0 \qquad\qquad \{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\,\sigma^\mu{}_{\alpha\dot\beta}\,P_\mu$$

Remarks:

- The object $\sigma^\mu{}_{\alpha\dot\beta}$ is an invariant tensor of the Lorentz group (a chiral block of 4d gamma matrices). We follow the conventions of Wess-Bagger:

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The anticommutator $\{Q, Q\} \sim P$ is the distinctive feature of supersymmetry algebras (as opposed to more general Lie superalgebras). Motto: "**supercharges are the square root of translations**"
- Single-particle states fall into irreps of the SUSY algebra, called **supermultiplets**

Some physical implications

1. <u>All states in a supermultiplet have the same mass</u>

   Follows from the fact that $P_\mu P^\mu$ commutes with Q_α , $\overline{Q}_{\dot\alpha}$

2. <u>Any supermultiplet contains an equal number of bosonic and fermionic dof's</u>

   Define the fermion number operator $(-)^{N_F}$: it has eigenvalue +1 (resp. −1) on bosonic (resp. fermionic) states. Since acting with one Q_α or $\overline{Q}_{\dot\alpha}$ takes a boson to a fermion and vice versa, we have

$$(-)^{N_F} Q_\alpha + Q_\alpha (-)^{N_F} = 0 \quad , \quad (-)^{N_F} \overline{Q}_{\dot\alpha} + \overline{Q}_{\dot\alpha} (-)^{N_F} = 0$$

   Consider a supermultiplet and restrict to 1-particle states with definite non-zero eigenvalue p_μ of P_μ (we can do it because the supercharges commute with P). We have a finite number of states with definite p_μ so the trace is well-defined

$$\begin{aligned} \mathrm{Tr}\Big((-)^{N_F}\,\{Q_\alpha,\overline{Q}_{\dot\beta}\}\Big) &= \mathrm{Tr}\Big((-)^{N_F}\,Q_\alpha\,\overline{Q}_{\dot\beta} + (-)^{N_F}\,\overline{Q}_{\dot\beta}\,Q_\alpha\Big) && \text{cyclic property of trace} \\ &= \mathrm{Tr}\Big(Q_\alpha\,\overline{Q}_{\dot\beta}\,(-)^{N_F} + Q_\alpha\,(-)^{N_F}\,\overline{Q}_{\dot\beta}\Big) && \overline{Q}_{\dot\beta} \text{ anticommutes with } (-)^{N_F} \\ &= 0 \end{aligned}$$

   Use the SUSY algebra:

$$\sigma^\mu{}_{\alpha\dot\beta}\,\mathrm{Tr}\Big((-)^{N_F}\,P_\mu\Big) = \sigma^\mu{}_{\alpha\dot\beta}\,p_\mu\,\mathrm{Tr}\Big((-)^{N_F}\Big) = 0$$

   Lesson: for a given non-zero eigenvalue p_μ of P_μ we must have an equal number of bosonic and fermionic dof's.

Some physical implications

1. <u>The Hamiltonian in a SUSY theory is a sum of squares</u>

   We just need to spell out the $(\alpha,\dot{\beta}) = (1{,}1)$ and $(\alpha,\dot{\beta}) = (2{,}2)$ components of $\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2\,\sigma^\mu{}_{\alpha\dot{\beta}}\,P_\mu$

$$Q_1\,(Q_1)^\dagger + (Q_1)^\dagger\,Q_1 = 2\,(-P_0 + P_3) = 2\,(P^0 + P^3)\ \ ,$$
$$Q_2\,(Q_2)^\dagger + (Q_2)^\dagger\,Q_2 = 2\,(-P_0 - P_3) = 2\,(P^0 - P^3)$$
$$Q_1\,(Q_1)^\dagger + (Q_1)^\dagger\,Q_1 + Q_2\,(Q_2)^\dagger + (Q_2)^\dagger\,Q_2 = 4\,P^0$$

2. <u>The vacuum energy is an order parameter for SUSY breaking</u>

   By definition, SUSY is unbroken in the vacuum if

$$Q_\alpha\,|0\rangle = 0\ \ ,\quad \overline{Q}_{\dot{\alpha}}\,|0\rangle = 0$$

   From the above expression for P^0 we have

$$Q_\alpha\,|0\rangle = 0\ \text{ and }\ \overline{Q}_{\dot{\alpha}}\,|0\rangle = 0 \quad \Leftrightarrow \quad \langle 0\,|\,P^0\,|\,0\rangle = 0$$

   Indeed, $\langle 0\,|\,P^0\,|\,0\rangle = \frac{1}{4}\,\|Q_1\,|0\rangle\|^2 + \frac{1}{4}\,\|(Q_1)^\dagger\,|0\rangle\|^2 + \frac{1}{4}\,\|Q_2\,|0\rangle\|^2 + \frac{1}{4}\,\|(Q_2)^\dagger\,|0\rangle\|^2$.

   NB: In ordinary QFT we can shift freely the (renormalized) zero-point of energy. In a SUSY theory the zero-point of energy is physical (compare with GR and cosmological constant).

Supersymmetry and supergravity

Lecture 2

Reminder: Lie algebras

Def. A *Lie algebra* over $\mathbb{R}$ (resp. $\mathbb{C}$) is a vector space V over $\mathbb{R}$ (resp. $\mathbb{C}$) equipped with a *bilinear* operation $[\,\cdot\,,\cdot\,] : V \times V \to V$ such that for all $x, y, z \in V$

$$[x, y] = -[y, x] \qquad \text{(anticommutativity)}$$

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \qquad \text{(Jacobi identity)}$$

Def. An *associative algebra* over $\mathbb{R}$ (resp. $\mathbb{C}$) is a vector space V over $\mathbb{R}$ (resp. $\mathbb{C}$) equipped with a *bilinear* operation $V \times V \to V$ (denoted by juxtaposition) such that for all $x, y, z \in V$

$$x\,(y\,z) = (x\,y)\,z \qquad \text{(associativity)}$$

NB: If V is an associative algebra, we can equip it with a natural Lie bracket using the commutator

$$[x, y] = x\,y - y\,x$$

The Jacobi identity follows from associativity.

Reminder: Lie algebras

Consider a finite-dim. Lie algebra V and choose a basis $\{e_a\}$. The Lie bracket on V is determined by the Lie bracket of basis elements

$$[e_a, e_b] = f_{ab}{}^c e_c \qquad\qquad f_{ab}{}^c = -f_{ba}{}^c \quad \text{structure constants}$$

The Jacobi identity gives a non-linear constraint on the structure constants

$$f_{[ab|}{}^d f_{c]d}{}^e = 0$$

Adjoint representation: action of V on itself via Lie bracket

$$\mathrm{ad}_X Y = [X, Y] \ , \quad (\mathrm{ad}_X)^a{}_b = X^c f_{cb}{}^a$$

Super vector spaces

Def. A $\mathbb{Z}_2$-graded vector space (aka super vector space) over $\mathbb{R}$ (resp. $\mathbb{C}$) is a vector space V over $\mathbb{R}$ (resp. $\mathbb{C}$) together with a decomposition of V of the form

$$V = V_0 \oplus V_1$$

Every element in V is written uniquely as a sum of an element in V_0 and an element in V_1.

Non-zero elements $x \in V_0$ are called “even” or “bosonic”; we assign to them degree $|x| = 0$ mod 2

Non-zero elements $x \in V_1$ are called “odd” or “fermionic”; we assign to them degree $|x| = 1$ mod 2

Example: $\mathbb{C}^{n|m}$ is $\mathbb{C}^{n+m}$ with the standard basis, where the first n basis elements span V_0 and the other m basis elements span V_1. Similar story for $\mathbb{R}^{n|m}$.

Lie superalgebras

Def. A *Lie superalgebra* over $\mathbb{R}$ (resp. $\mathbb{C}$) is a $\mathbb{Z}_2$-graded vector space V over $\mathbb{R}$ (resp. $\mathbb{C}$) equipped with a *bilinear* operation $[\,\cdot\,,\cdot\,\}: V \times V \to V$ (known as the superbracket) such that

- the $\mathbb{Z}_2$-grading is preserved by the superbracket

  if x, y are homogeneous of degrees $|x|, |y|$

  then $[x, y\}$ is homogeneous of degree $|x| + |y|$ mod 2

- the superbracket is super-anticommutative

$$[x, y\} = -(-)^{|x||y|}[y, x\}$$

- the superbracket satisfies the super Jacobi identity

$$(-)^{|x||z|}[x,[y,z\}\} + (-)^{|z||y|}[z,[x,y\}\} + (-)^{|y||x|}[y,[z,x\}\} = 0$$

Lie superalgebras

Def. An *associative superalgebra* over $\mathbb{R}$ (resp. $\mathbb{C}$) is a $\mathbb{Z}_2$-graded vector space V over $\mathbb{R}$ (resp. $\mathbb{C}$) equipped with a *bilinear* operation $V \times V \to V$ (denoted by juxtaposition) such that

- the $\mathbb{Z}_2$-grading is preserved by the product

  if x, y are homogeneous of degrees $|x|, |y|$

  then $x\,y$ is homogeneous of degree $|x| + |y|$ mod 2

- the product is associative

$$x\,(y\,z) = (x\,y)\,z$$

NB: If V is an associative superalgebra, we can equip it with a natural Lie superbracket using the supercommutator

$$[x, y] = x\,y - (-)^{|x||y|}\,y\,x$$

The super Jacobi identity follows from associativity.

Some examples with matrices

- The associative superalgebra $\mathfrak{gl}(n \mid m; \mathbb{C})$

We consider the super vector space $\mathbb{C}^{n|m}$. Any linear map $\mathbb{C}^{n|m} \to \mathbb{C}^{n|m}$ can be decomposed uniquely into a parity-preserving part, and a parity-reversing part. In block-matrix notation:

$$\text{parity-preserving} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad \text{parity-reversing} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

where A is $n \times n$, B is $n \times m$, C is $m \times n$, D is $m \times m$. We declare the parity-preserving transformations to be even/bosonic, the parity reversing transformations to the odd/fermionic. The standard matrix multiplication preserves the odd/even grading. We get the associative superalgebra $\mathfrak{gl}(n \mid m; \mathbb{C})$.

It becomes a Lie superalgebra with the supercommutator.

Given $M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ their supercommutator is

$$\begin{pmatrix} (A_1 A_2 - A_2 A_1) + (B_1 C_2 + B_2 C_1) & (A_1 B_2 - A_2 B_1) + (B_1 D_2 - B_2 D_1) \\ (C_1 A_2 - C_2 A_1) + (D_1 C_2 - D_2 C_1) & (C_1 B_2 + C_2 B_1) + (D_1 D_2 - D_2 D_1) \end{pmatrix}$$

Some examples with matrices

- The Lie superalgebra $\mathfrak{sl}(n\,|\,m;\mathbb{C})$

We start from $\mathfrak{gl}(n\,|\,m;\mathbb{C})$ and we define the supertrace

$$\mathrm{sTr}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mathrm{Tr}\,A - \mathrm{Tr}\,D$$

It does not depend on a choice of basis in the homogeneous subspaces $\mathbb{C}^n$, $\mathbb{C}^m$ of $\mathbb{C}^{n|m}$. One can verify that the super trace of any super commutator is zero. So $\mathfrak{sl}(n\,|\,m;\mathbb{C})$ is also a Lie superalgebra with the supercommutator.

- The Lie superalgebra $\mathfrak{psl}(n\,|\,n;\mathbb{C})$

In the special case $m = n$, the identity matrix $\mathbb{I}$ is an element of $\mathfrak{sl}(n\,|\,n;\mathbb{C})$ (its supertrace is zero). Its supercommutator with anything is zero. We say that $\mathbb{I}$ generates a 1-dim ideal of $\mathfrak{sl}(n\,|\,n;\mathbb{C})$. We can consider the quotient of $\mathfrak{sl}(n\,|\,n;\mathbb{C})$ by this ideal (i.e. we mod out all matrices that are a multiple of $\mathbb{I}$). The resulting superalgebra is denoted $\mathfrak{psl}(n\,|\,n;\mathbb{C})$.

Some remarks

- There is a classification of simple Lie superalgebras over $\mathbb{C}$ (V. Kac 1977) which is similar to the classification of simple Lie algebras
- A nice survey of superalgebras over $\mathbb{C}$ and their real forms can be found eg in in arXiv 9607161
- The SUSY algebra is not a simple Lie superalgebra…
- … but simple Lie superalgebras are found in the study of superconformal theories
- Superconformal algebras were classified by W. Nahm 1978

➡ they only exist in spacetime dimension ≤ 6

Lie superalgebras: structure constants

Let us choose a basis $\{e_a\}$ of V_0 and a basis $\{\varepsilon_i\}$ of V_1.

The super Lie bracket on V is determined by the super Lie bracket of basis elements. It is customary to write the superbracket as $[\,\cdot\,,\,\cdot\,]$ if we consider two even elements or one even and one odd element, and to write it as $\{\,\cdot\,,\,\cdot\,\}$ if we consider two odd elements. The "structure constants" of the super Lie bracket wrt to the bases $\{e_a\}$, $\{\varepsilon_i\}$ can be defined as

$$[e_a, e_b] = f_{ab}{}^c\, e_c \qquad [e_a, \varepsilon_i] = -[\varepsilon_i, e_a] = (M_a)^j{}_i\, \varepsilon_j \qquad \{\varepsilon_i, \varepsilon_j\} = (B^a)_{ij}\, e_a$$

Super anticommutativity:

$$f_{ab}{}^c = -f_{ba}{}^c \qquad (B^a)_{ij} = +(B^a)_{ji}$$

Lie superalgebras: structure constants

$$[e_a, e_b] = f_{ab}{}^c\, e_c \qquad [e_a, \varepsilon_i] = -[\varepsilon_i, e_a] = (M_a)^j{}_i\, \varepsilon_j \qquad \{\varepsilon_i, \varepsilon_j\} = (B^a)_{ij}\, e_a$$

Implications of super Jacobi:

- Three bosonic generators

$$f_{ab}{}^d f_{cd}{}^e + \text{cyclic in } abc = 0$$

i.e. the even subspace V_0 is an ordinary Lie algebra

- One fermionic generator and two bosonic generators

$$(M_a)^i{}_k\,(M_b)^k{}_j - (M_b)^i{}_k\,(M_a)^k{}_j = f_{ab}{}^c\,(M_c)^i{}_j$$

i.e. the matrices M_a form a **representation** $\mathbf{r}$ of the Lie algebra V_0

Lie superalgebras: structure constants

$$[e_a, e_b] = f_{ab}{}^c\, e_c \qquad [e_a, \varepsilon_i] = -[\varepsilon_i, e_a] = (M_a)^j{}_i\, \varepsilon_j \qquad \{\varepsilon_i, \varepsilon_j\} = (B^a)_{ij}\, e_a$$

Implications of super Jacobi:

- Two fermionic generators and one bosonic generator

$$f_{cb}{}^a\, (B^b)_{ij} - (M_c)^k{}_i\, (B^a)_{kj} - (M_c)^k{}_j\, (B^a)_{ik} = 0$$

i.e. the structure constants $(B^a)_{ij} = (B^a)_{ji}$ are **invariant tensors** of the rep.
$\mathrm{ad} \otimes (\mathbf{r}^* \otimes_{\mathrm{sym}} \mathbf{r}^*)$ of the even part of the superalgebra

- Three fermionic generators

$$(B^a)_{ij}\, (M_a)^\ell{}_k + \text{cyclic in } ijk = 0$$

i.e. a non-linear constraint on the M_a, B^a matrices

Supersymmetry and supergravity

Lecture 3

Reminder: reps of Lorentz algebra

The Lorentz algebra:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\,\eta_{\mu\rho}\,J_{\nu\sigma} - i\,\eta_{\nu\rho}\,J_{\mu\sigma} - i\,\eta_{\mu\sigma}\,J_{\nu\rho} + i\,\eta_{\nu\sigma}\,J_{\mu\rho} \quad , \qquad \mu, \nu, \ldots = 0{,}1{,}2{,}3 \ , \qquad \eta_{\mu\nu} = \mathrm{diag}(-1{,}1{,}1{,}1)$$

We can arrange the 6 generators $J_{\mu\nu} = -J_{\nu\mu}$ into two commuting sets of angular momentum generators

$$\mathcal{J}_i = (J^{23}, J^{31}, J^{12}) \quad , \qquad \mathcal{K}_i = (J^{10}, J^{20}, J^{30}) \quad , \qquad i = 1{,}2{,}3$$

$$\mathcal{J}^{\pm}_i = \mathcal{J}_i \pm i\,\mathcal{K}_i \ , \qquad [\mathcal{J}^{\pm}_i, \mathcal{J}^{\pm}_j] = i\,\epsilon_{ijk}\,\mathcal{J}^{\pm}_k \quad , \qquad [\mathcal{J}^{+}_i, \mathcal{J}^{-}_j] = 0$$

We label irreps by a **pair of integer or half-integer spins** (j_1, j_2), $j_1, j_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

Examples:

- $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are positive and negative chirality spinor irreps
- $(\frac{1}{2}, \frac{1}{2})$ is the vector representation
- $(1,0)$ and $(0,1)$ are the imaginary (anti)self dual parts of an antisymm. tensor $T^{\pm}_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} \pm \frac{i}{2}\,\epsilon_{\mu\nu\rho\sigma} T^{\rho\sigma}\right)$

Some notation and conventions

Consider an operator in a representation $\mathbf{r}$ of the Lorentz group with indices $\mathscr{A}, \mathscr{B}$

$$[J_{\mu\nu}, \mathcal{O}^{\mathscr{A}}] = -\,(S^{\mathbf{r}}_{\mu\nu})^{\mathscr{A}}{}_{\mathscr{B}}\,\mathcal{O}^{\mathscr{B}}$$

The matrices $(S^{\mathbf{r}}_{\mu\nu})^{\mathscr{A}}{}_{\mathscr{B}}$ furnish a **representation** of the abstract Lorentz algebra

$$[J_{\mu\nu}, J_{\rho\sigma}] = \big(i\,\eta_{\mu\rho}\,J_{\nu\sigma} - (\mu \leftrightarrow \nu)\big) - (\rho \leftrightarrow \sigma)$$

$$(S^{\mathbf{r}}_{\mu\nu})^{\mathscr{A}}{}_{\mathscr{C}}\,(S^{\mathbf{r}}_{\rho\sigma})^{\mathscr{C}}{}_{\mathscr{B}} - (S^{\mathbf{r}}_{\rho\sigma})^{\mathscr{A}}{}_{\mathscr{C}}\,(S^{\mathbf{r}}_{\mu\nu})^{\mathscr{C}}{}_{\mathscr{B}} = \big(i\,\eta_{\mu\rho}\,(S^{\mathbf{r}}_{\nu\sigma})^{\mathscr{A}}{}_{\mathscr{B}} - (\mu \leftrightarrow \nu)\big) - (\rho \leftrightarrow \sigma)$$

NB: The generators $J_{\mu\nu}$ are hermitian operators $(J_{\mu\nu})^{\dagger} = J_{\mu\nu}$ in the (infinite dim) Hilbert space. The matrices $(S^{\mathbf{r}}_{\mu\nu})^{\mathscr{A}}{}_{\mathscr{B}}$ are finite-dim but not hermitian.

Recall: there are no unitary finite-dim reps of the Lorentz group.

Some notation and conventions

If $\lambda_{\mu\nu} = -\lambda_{\nu\mu}$ are the (real) parameters of an infinitesimal Lorentz transformation, we write

$$\delta_\lambda \mathcal{O}^{\mathscr{A}} = \frac{i}{2} \lambda^{\mu\nu} (S^{\mathbf{r}}_{\mu\nu})^{\mathscr{A}}{}_{\mathscr{B}} \mathcal{O}^{\mathscr{B}}$$

In this language the Lorentz algebra is encoded in the commutator of **two variations**:

$$\delta_{\lambda_1}\delta_{\lambda_2}\mathcal{O}^{\mathscr{A}} - \delta_{\lambda_2}\delta_{\lambda_1}\mathcal{O}^{\mathscr{A}} = \delta_{\lambda_3}\mathcal{O}^{\mathscr{A}} \qquad \text{where} \qquad \lambda_3^{\mu\nu} = \lambda_1^{\mu\rho}\,\lambda_{2\rho}{}^{\nu} - (1 \leftrightarrow 2)$$

Reminder: reps of Lorentz group in 4d

<u>Example</u>: vector representation

$$(S^{\text{vec}}_{\mu\nu})^{\rho}{}_{\sigma} = -\,i\,\delta^{\rho}_{\mu}\,\eta_{\nu\sigma} + i\,\delta^{\rho}_{\nu}\,\eta_{\mu\sigma} \quad , \qquad \frac{i}{2}\,\lambda^{\mu\nu}\,(S^{\text{vec}}_{\mu\nu})^{\rho}{}_{\sigma} = \lambda^{\rho}{}_{\sigma}$$

The JJ and JP commutators can be written as

$$[J_{\mu\nu}, J^{\rho\sigma}] = -\left((S^{\text{vec}}_{\mu\nu})^{\rho}{}_{\tau}\,J^{\tau\sigma} + (S^{\text{vec}}_{\mu\nu})^{\sigma}{}_{\tau}\,J^{\rho\tau}\right) \quad , \qquad [J_{\mu\nu}, P^{\rho}] = -\,(S^{\text{vec}}_{\mu\nu})^{\rho}{}_{\tau}\,P^{\tau}$$

These commutators simply mean that $J^{\rho\sigma}$, P^{ρ} transform under the Lorentz group as expected from their indices:

$$\delta_{\lambda}J^{\rho\sigma} = \frac{i}{2}\lambda^{\mu\nu}\left((S^{\text{vec}}_{\mu\nu})^{\rho}{}_{\tau}\,J^{\tau\sigma} + (S^{\text{vec}}_{\mu\nu})^{\sigma}{}_{\tau}\,J^{\rho\tau}\right) = \lambda^{\rho}{}_{\tau}\,J^{\tau\sigma} + \lambda^{\sigma}{}_{\tau}\,J^{\rho\tau}$$

$$\delta_{\lambda}P^{\rho} = \frac{i}{2}\lambda^{\mu\nu}\,(S^{\text{vec}}_{\mu\nu})^{\rho}{}_{\tau}\,P^{\tau} = \lambda^{\rho}{}_{\tau}\,P^{\tau}$$

Gamma matrices

The basic "building blocks" to construct general representations are obtained via **spinorial representations**.

We start from the **Clifford algebra**

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\,\eta^{\mu\nu}\,\mathbb{I}_4$$

We work with 4d gamma matrices in a "chiral basis":

$$\gamma^\mu = \begin{pmatrix} 0 & i\,\sigma^\mu \\ i\,\bar\sigma^\mu & 0 \end{pmatrix} \quad , \qquad \begin{matrix} \sigma^0 = -\,\mathbb{I}_2 & \sigma^1 = +\,\sigma_x & \sigma^2 = +\,\sigma_y & \sigma^3 = +\,\sigma_z \\ \bar\sigma^0 = -\,\mathbb{I}_2 & \bar\sigma^1 = -\,\sigma_x & \bar\sigma^2 = -\,\sigma_y & \bar\sigma^3 = -\,\sigma_z \end{matrix}$$

where the standard Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

NB: $\gamma^0 = -\,\gamma_0$ is antihermitian; $\gamma^1 = \gamma_1$, $\gamma^2 = \gamma_2$, $\gamma^3 = \gamma_3$ are hermitian.

The **chirality matrix** is

$$\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \;, \qquad \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$$

Lorentz generators

- Using the gamma matrices we construct a representation of the Lorentz algebra:

$$S^{\text{Dirac}}_{\mu\nu} = -\frac{i}{4}\left(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu\right)$$

$$S^{\text{Dirac}}_{\mu\nu} S^{\text{Dirac}}_{\rho\sigma} - S^{\text{Dirac}}_{\rho\sigma} S^{\text{Dirac}}_{\mu\nu} = \left(i\,\eta_{\mu\rho} S^{\text{Dirac}}_{\nu\sigma} - (\mu \leftrightarrow \nu)\right) - (\rho \leftrightarrow \sigma)$$

- The transformation law of a 4-component Dirac spinor is

$$\delta_\lambda \Psi = \frac{i}{2}\lambda^{\mu\nu} S^{\text{Dirac}}_{\mu\nu} \Psi$$

- This representation is reducible. This is manifest in the chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & i\,\sigma^\mu \\ i\,\overline{\sigma}^\mu & 0 \end{pmatrix}, \qquad S^{\text{Dirac}}_{\mu\nu} = \begin{pmatrix} i\,\sigma_{\mu\nu} & 0 \\ 0 & i\,\overline{\sigma}_{\mu\nu} \end{pmatrix}, \qquad \begin{aligned} \sigma^{\mu\nu} &= \tfrac{1}{4}\left(\sigma^\mu \overline{\sigma}^\nu - \sigma^\nu \overline{\sigma}^\mu\right) \\ \overline{\sigma}^{\mu\nu} &= \tfrac{1}{4}\left(\overline{\sigma}^\mu \sigma^\nu - \overline{\sigma}^\nu \sigma^\mu\right) \end{aligned}$$

Lorentz generators

The chiral blocks $i\,\sigma^{\mu\nu}$ and $i\,\bar{\sigma}^{\mu\nu}$ furnish irreps of the Lorentz algebra: we identify them with the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ reps.

Check: earlier we defined

$$\mathcal{J}_i = (J^{23}, J^{31}, J^{12}) \quad , \qquad \mathcal{K}_i = (J^{10}, J^{20}, J^{30}) \quad , \qquad \mathcal{J}_i^{\pm} = \mathcal{J}_i \pm i\,\mathcal{K}_i$$

These expressions imply that

in the rep $J^{\mu\nu} \to i\,\sigma^{\mu\nu}$ we have

$$\mathcal{J}_i^{+} = (\frac{1}{2}\,\sigma_x, \frac{1}{2}\,\sigma_y, \frac{1}{2}\,\sigma_z) \;\; , \quad \mathcal{J}_i^{-} = (0,0,0) \quad : \qquad \text{this is } (\frac{1}{2},0)$$

in the rep $J^{\mu\nu} \to i\,\bar{\sigma}^{\mu\nu}$ we have

$$\mathcal{J}_i^{+} = (0,0,0) \;\; , \quad \mathcal{J}_i^{-} = (\frac{1}{2}\,\sigma_x, \frac{1}{2}\,\sigma_y, \frac{1}{2}\,\sigma_z) \quad : \qquad \text{this is } (0,\frac{1}{2})$$

Spinor indices

We decompose a 4-component Diral spinor Ψ into two chiral Weyl spinors

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \quad , \quad \alpha, \beta = 1,2 \, , \quad \dot{\alpha}, \dot{\beta} = \dot{1}, \dot{2}$$

(**Van der Waerden notation**)

Remarks:

1. Notice the dotted/undotted VS upper/lower arrangement of spinor indices
2. At this point the bar on $\bar{\chi}^{\dot{\alpha}}$ is just part of the name of the Weyl spinor

The index structure on the chiral blocks of the gamma matrices is

$$\gamma^\mu = \begin{pmatrix} 0 & i\,(\sigma^\mu)_{\alpha\dot{\beta}} \\ i\,(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix} \, , \qquad S^{\text{Dirac}}_{\mu\nu} = \begin{pmatrix} i\,(\sigma_{\mu\nu})_\alpha{}^\beta & 0 \\ 0 & i\,(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}$$

The Lorentz variations of $\psi_\alpha, \bar{\chi}^{\dot{\alpha}}$ are

$$\delta_\lambda \psi_\alpha = -\frac{1}{2}\lambda^{\mu\nu}\,(\sigma_{\mu\nu})_\alpha{}^\beta \;\psi_\beta \quad , \qquad \delta_\lambda \bar{\chi}^{\dot{\alpha}} = -\frac{1}{2}\lambda^{\mu\nu}\,(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\;\bar{\chi}^{\dot{\beta}}$$

Spinor indices and $SL(2,\mathbb{C})$

- Define the antisymmetric symbols (we follow the conventions of Wess-Bagger)

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \qquad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \qquad \epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \qquad \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- These symbols are invariant under the action of the Lorentz generators:

$$(\sigma^{\mu\nu})_\alpha{}^\gamma\,\epsilon_{\beta\gamma} + (\sigma^{\mu\nu})_\beta{}^\gamma\,\epsilon_{\gamma\alpha} = 0 \ , \qquad (\sigma^{\mu\nu})_\gamma{}^\alpha\,\epsilon^{\beta\gamma} + (\sigma^{\mu\nu})_\gamma{}^\beta\,\epsilon^{\gamma\alpha} = 0$$

$$(\overline{\sigma}^{\mu\nu})^{\dot\gamma}{}_{\dot\alpha}\,\epsilon_{\dot\gamma\dot\beta} + (\overline{\sigma}^{\mu\nu})^{\dot\gamma}{}_{\dot\beta}\,\epsilon_{\dot\alpha\dot\gamma} = 0 \ , \qquad (\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\gamma}\,\epsilon^{\dot\gamma\dot\beta} + (\overline{\sigma}^{\mu\nu})^{\dot\beta}{}_{\dot\gamma}\,\epsilon^{\dot\alpha\dot\gamma} = 0$$

- The tensors $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$ are the invariant tensors of the group $SL(2,\mathbb{C})$

$$M^\alpha{}_\beta\,M^\gamma{}_\delta\,\epsilon_{\alpha\gamma} = (\det M)\,\epsilon_{\beta\delta} = \epsilon_{\beta\delta} \quad \text{for } M \in SL(2,\mathbb{C})$$

- Interpretation: spinor indices are indices of the fundamental 2-dim representation of $SL(2,\mathbb{C})$

Spinor indices and complex conjugation

- Dotted and undotted indices are exchanged by complex conjugation:
    1. Start from ψ_α transforming as $\delta_\lambda \psi_\alpha = -\frac{1}{2}\lambda^{\mu\nu}(\sigma_{\mu\nu})_\alpha{}^\beta \ \psi_\beta$ and define $\overline{\psi}_{\dot{\alpha}} \equiv (\psi_\alpha)^*$
    2. Using the identity $(\sigma_{\mu\nu})^\dagger = -\,\overline{\sigma}_{\mu\nu}$, we have $\delta_\lambda \overline{\psi}_{\dot{\alpha}} = +\frac{1}{2}\lambda^{\mu\nu}(\overline{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} \ \overline{\psi}_{\dot{\beta}}$
    3. Compare with the transformation property $\delta_\lambda \overline{\chi}^{\dot{\alpha}} = -\frac{1}{2}\lambda^{\mu\nu}(\overline{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \ \overline{\chi}^{\dot{\beta}}$
- The object $\overline{\psi}_{\dot{\alpha}}$ does indeed transform in the dual rep of the object $\overline{\chi}^{\dot{\alpha}}$ (inverse transpose). This confirms that complex conjugation takes a lower/upper undotted index to a lower/upper dotted index
- Convention: alway present a spinor with a dotted index as complex conjugate of a spinor with an undotted index. In this way, spinors with dotted indices have a bar

Raising/lowering spinor indices

We can use the non-degenerate invariant tensors $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$, $\epsilon^{\dot{\alpha}\dot{\beta}}$ to raise/lower van der Waerden indices (just like we use $\eta_{\mu\nu}$, $\eta^{\mu\nu}$ to raise vector indices). Since these invariant tensors are antisymmetric we must be extra careful with minus signs!

We follow the raising/lowering conventions of Wess-Bagger:

$$\psi^\alpha = \epsilon^{\alpha\beta}\,\psi_\beta \;, \qquad \psi_\alpha = \epsilon_{\alpha\beta}\,\psi^\beta \;, \qquad \lambda^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\,\lambda_{\dot{\beta}} \;, \qquad \lambda_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\,\lambda^{\dot{\beta}}$$

Objects with upper/lower α indices transform in dual representations:

$$\delta_\lambda \psi_\alpha = -\frac{1}{2}\,\lambda^{\mu\nu}\,(\sigma_{\mu\nu})_\alpha{}^\beta\,\psi_\beta \;, \qquad \delta_\lambda \psi^\alpha = +\frac{1}{2}\,\lambda^{\mu\nu}\,(\sigma_{\mu\nu})_\beta{}^\alpha\,\psi^\beta$$

$$\delta_\lambda \bar{\chi}^{\dot{\alpha}} = -\frac{1}{2}\,\lambda^{\mu\nu}\,(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\,\bar{\chi}^{\dot{\beta}} \;, \qquad \delta_\lambda \bar{\chi}_{\dot{\alpha}} = +\frac{1}{2}\,\lambda^{\mu\nu}\,(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}\,\bar{\chi}_{\dot{\beta}}$$

For example, we can use $(\sigma^{\mu\nu})_\gamma{}^\alpha\,\epsilon^{\beta\gamma} + (\sigma^{\mu\nu})_\gamma{}^\beta\,\epsilon^{\gamma\alpha} = 0$ and check

$$\delta_\lambda(\epsilon^{\alpha\beta}\psi_\beta) = -\frac{1}{2}\,\epsilon^{\alpha\beta}\,\lambda^{\mu\nu}\,(\sigma_{\mu\nu})_\beta{}^\gamma\,\psi_\gamma = +\frac{1}{2}\,\lambda^{\mu\nu}\,(\sigma_{\mu\nu})_\beta{}^\alpha\,\epsilon^{\beta\gamma}\,\psi_\gamma$$

Raising/lowering spinor indices

- We form Lorentz scalars by contracting upper and lower spinor indices

$$\psi^\alpha \chi_\alpha \ , \qquad \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$$

- Every time that we have a pair of contracted indices arranged as ${}_\alpha{}^\alpha$ we can turn it into ${}^\alpha{}_\alpha$; the price to pay is a minus sign. (Same with dotted indices)

$$\psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha \ , \qquad \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}}$$

- The tensor $(\sigma^\mu)_{\alpha\dot{\beta}}$ with its indices raised gives $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}$:

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\beta} = \epsilon^{\dot{\alpha}\dot{\gamma}} \epsilon^{\beta\delta} (\sigma^\mu)_{\delta\dot{\gamma}}$$

- Invariance of $\epsilon_{\alpha\beta}$ is equivalent to the fact that the Lorentz generators are symmetric in its spinor indices:

$$(\sigma^{\mu\nu})_{\alpha\beta} = +(\sigma^{\mu\nu})_{\beta\alpha} \ , \qquad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} = +(\bar{\sigma}^{\mu\nu})^{\dot{\beta}\dot{\alpha}}$$

Index structure for generic irrep (j_1, j_2)

The irrep (j_1, j_2) has $2j_1$ symmetrized undotted indices and $2j_2$ symmetrized dotted indices. Upper or lower is a matter of taste, because we can unambiguously raise/lower them. If we use all lower indices:

$$\mathcal{O}_{\alpha_1\ldots\alpha_{2j_1}\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2}} = \mathcal{O}_{(\alpha_1\ldots\alpha_{2j_1})(\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2})}$$

$$\begin{aligned}\delta_\lambda \mathcal{O}_{\alpha_1\ldots\alpha_{2j_1}\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2}} = &-\frac{1}{2}\,\lambda^{\mu\nu}\,(\sigma_{\mu\nu})_{\alpha_1}{}^{\beta}\,\mathcal{O}_{\beta\alpha_2\ldots\alpha_{2j_1}\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2}} - \ldots - \frac{1}{2}\,\lambda^{\mu\nu}\,(\sigma_{\mu\nu})_{\alpha_{2j_1}}{}^{\beta}\,\mathcal{O}_{\alpha_1\ldots\alpha_{2j_1-1}\beta\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2}} \\ &+\frac{1}{2}\,\lambda^{\mu\nu}\,(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}_1}\,\mathcal{O}_{\alpha_1\ldots\alpha_{2j_1}\dot{\beta}\dot{\alpha}_2\ldots\dot{\alpha}_{2j_2}} + \ldots + \frac{1}{2}\,\lambda^{\mu\nu}\,(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}_{2j_2}}\,\mathcal{O}_{\alpha_1\ldots\alpha_{2j_1}\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2-1}\dot{\beta}}\end{aligned}$$

Under complex conjugation dotted and undotted indices are exchanged:

$$\left[\mathcal{O}_{\alpha_1\ldots\alpha_{2j_1}\dot{\alpha}_1\ldots\dot{\alpha}_{2j_2}}\right]^\dagger = (\mathcal{O}^\dagger)_{\dot{\alpha}_1\ldots\dot{\alpha}_{2j_1}\alpha_1\ldots\alpha_{2j_2}}$$

An object with an equal number of dotted/undotted indices can be real. Eg: 4-vector rep

$$\mathcal{O}_{\alpha\dot{\beta}} = (\sigma^\mu)_{\alpha\dot{\beta}}\,\mathcal{O}_\mu \qquad (\mathcal{O}_\mu)^\dagger = \mathcal{O}_\mu \quad , \qquad \left[\mathcal{O}_{\alpha\dot{\beta}}\right]^\dagger = \mathcal{O}_{\beta\dot{\alpha}}$$

Majorana VS Weyl spinors

A Dirac spinor has 8 real dof's and is written as

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot\alpha} \end{pmatrix}$$

where ψ_α and $\overline{\chi}^{\dot\alpha}$ are *independent* complex chiral spinors. We can halve the number of dof's in a Lorentz *covariant* fashion by setting $\chi_\alpha = \psi_\alpha$ so that

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \epsilon^{\dot\alpha\dot\beta}\,\overline{\psi}_{\dot\alpha} \end{pmatrix} \qquad \text{where} \qquad \overline{\psi}_{\dot\alpha} := (\psi_\alpha)^*$$

In terms of the 4-component spinor Ψ this reality condition is the **Majorana** condition:

$$\Psi^\dagger\,(-i\,\gamma^0) = \text{Dirac conjugate} = \text{Majorana conjugate} = \Psi^T\,C$$

where C is the charge conjugation matrix

$$C = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot\alpha\dot\beta} \end{pmatrix}, \qquad (\gamma^\mu)^T = -\,C\,\gamma^\mu\,C^{-1} \quad , \qquad C^T = -\,C \quad , \qquad C^\dagger\,C = \mathbb{I}_4$$

One Majorana 4-component spinor is equivalent to one Weyl spinor. They both have 4 real dof's. This is the minimal number of dof's for a spinor in 4d. Using the 4-component or 3-component notation is a matter of taste.

Caveat on notation

- The objects $(\sigma^\mu)_{\alpha\dot{\beta}}$, $(\overline{\sigma}^\mu)^{\dot{\alpha}\beta}$, $(\sigma^{\mu\nu})_\alpha{}^\beta$, $(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}$ and the conventions to raise/lower α, $\dot{\alpha}$ indices are all the same as in Wess-Bagger.
- We also decompose 4-component spinors into 2-component spinors as in Wess-Bagger
- We have slightly different conventions for 4-component spinors and gamma matrices:

$$\gamma^\mu_{\text{here}} = \begin{pmatrix} 0 & i\,\sigma^\mu \\ i\,\overline{\sigma}^\mu & 0 \end{pmatrix} \qquad \text{vs} \qquad \gamma^\mu_{\text{WB}} = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$$

The homomorphism $SL(2,\mathbb{C}) \to SO^+(3,1)$

- Weyl spinors exist in every even spacetime dimensions, but the identification of spinor indices with $SL(2,\mathbb{C})$ indices is special to 4d in Lorentzian signature
- Mathematical statement:

Lie algebra isomorphism $\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})$

- The analogous statement for the Lie groups is that there is a group homomorphism

$$\phi : SL(2,\mathbb{C}) \to SO^+(3,1)$$

- Notation: $SO^+(3,1)$ is the identity connected component of $SO(3,1)$ (proper orthochronous Lorentz transf)

The homomorphism $SL(2,\mathbb{C}) \to SO^+(3,1)$

More on the map $\phi : SL(2,\mathbb{C}) \to SO^+(3,1)$:

1. The set of all 2×2 hermitian matrices is a real vector space of dim 4, hence it is isomorphic to $\mathbb{R}^4$ as a vector space. We can write the isomorphism explicitly using $(\sigma^\mu)_{\alpha\dot{\beta}}$

$$M = \sigma_\mu \, x^\mu \qquad \text{such that} \qquad \det M = \eta_{\mu\nu} \, x^\mu \, x^\nu$$

2. The group $SL(2,\mathbb{C})$ acts on 2×2 hermitian matrices as

$$M' = A \, M A^\dagger \;\; , \qquad A \in SL(2,\mathbb{C}) \;\; , \qquad M' = \sigma_\mu \, x'^\mu$$

3. Since $\det A = 1$, we have $\det M = \det M'$, which means that x^μ and x'^μ must be related by a Lorentz transformation

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu \;\; , \qquad \Lambda^\mu{}_\nu \in O(3,1)$$

4. The map $A \mapsto \Lambda^\mu{}_\nu$ is a group homomorphism; one can prove that its image is $SO^+(3,1)$ and that it is 2-to-1

Supersymmetry and supergravity

Lecture 4

The structure of the SUSY algebra

- Our goal: find a Lie superalgebra that extends Poincaré and is compatible with non-trivial scattering
- The bosonic subalgebra is constrained by Coleman-Mandula:

Poincaré generators: $J_{\mu\nu}$, P_μ ; internal compact symm: T_A

- From the general axioms of a Lie superalgebra, we know that the odd generators are in representations of the even part of the Lie superalgebra
- Odd generators fall into Lorentz irreps (j_1, j_2); their complex conjugates are in the (j_2, j_1) irrep

$$Q^I{}_{\alpha_1\ldots\alpha_{2j_1}\dot{\beta}_1\ldots\dot{\beta}_{2j_2}} \quad , \qquad \overline{Q}^{\bar{I}}{}_{\dot{\alpha}_1\ldots\dot{\alpha}_{2j_1}\beta_1\ldots\beta_{2j_2}} \quad , \qquad I = 1,\ldots,\mathcal{N}$$

here I is an index for some finite-dim rep of the internal symmetry, and bar $\bar{I}$ is the conj. rep.

- What are the viable options for j_1 and j_2?

The structure of the SUSY algebra

- Let us consider the $Q, \overline{Q}$ anticommutator. It must be Lorentz-covariant. From the composition of angular momenta, the $Q, \overline{Q}$ anticommutator contains the irrep $(j_1 + j_2, j_1 + j_2)$, and possibly others.
- We also notice that

$$\{Q^I{}_{1\ldots1i\ldots i}, \overline{Q}^{\bar{I}}{}_{i\ldots i1\ldots1}\} = (Q^I{}_{1\ldots1i\ldots i})(Q^I{}_{1\ldots1i\ldots i})^\dagger + (Q^I{}_{1\ldots1i\ldots i})^\dagger(Q^I{}_{1\ldots1i\ldots i})$$

(no sum over I) is a positive-definite op. in the Hilbert space of the theory. If the generators $Q, \overline{Q}$ are non-zero, this anticommutator cannot be zero

- The bosonic generators we have are P_μ in the $(\frac{1}{2}, \frac{1}{2})$, $J_{\mu\nu}$ in the $(1,0) \oplus (0,1)$, and T_A in the $(0,0)$. We learn that

$$j_1 + j_2 \leq 1$$

- Spin-statistic tells us that the odd generators $Q, \overline{Q}$ should have half-integer spin.

Lesson: odd generators $Q, \overline{Q}$ can only be in the $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ representations

The structure of the SUSY algebra

- We write the odd generators as ${Q^I}_\alpha, {\overline{Q}^{\bar{I}}}_{\dot{\alpha}}$. Their anticommutator must give P and must be written in terms of invariant tensors of Lorentz and internal symmetries

$$\{{Q^I}_\alpha, {\overline{Q}^{\bar{J}}}_{\dot{\beta}}\} = 2\, h^{I\bar{J}}\, (\sigma^\mu)_{\alpha\dot{\beta}}\, P_\mu$$

- Taking $\dagger$ on both sides, we learn that $h^{I\bar{J}}$ is a hermitian matrix. We also know that if we set $\alpha = 1, \dot{\beta} = \dot{1}$ the LHS is a positive-def matrix in its $I, \bar{J}$ indices. Using a unitary redefinition of the ${Q^I}_\alpha, {\overline{Q}^{\bar{I}}}_{\dot{\alpha}}$ we can set

$$h^{I\bar{J}} = \delta^{I\bar{J}}$$

The structure of the SUSY algebra

- We learn that the indices $I, \bar{J}$ are indices of a hermitian rep, because $h^{I\bar{J}} = \delta^{I\bar{J}}$ is an invariant tensor. We can use $\delta^{I\bar{J}}$, $\delta_{I\bar{J}}$ to turn upper barred indices into lower unbarred indices

$$\overline{Q}_{I\dot{\alpha}} := h_{I\bar{J}}\,\overline{Q}^{\bar{J}}{}_{\dot{\alpha}} \quad , \qquad \{Q^I{}_\alpha, \overline{Q}_{J\dot{\beta}}\} = 2\,\delta_{IJ}\,(\sigma^\mu)_{\alpha\dot{\beta}}\,P_\mu$$

- The supercharges are in a unitary representation of the internal symmetry algebra

$$[T_A, Q^I{}_\alpha] = -\,(t_A)^I{}_J\,Q^J{}_\alpha \quad , \qquad [(t_A)^I{}_J]^* = (t_A)^J{}_I \quad , \qquad [T_A, \overline{Q}_{I\dot{\alpha}}] = +\,(t_A)^J{}_J\,\overline{Q}_{I\dot{\alpha}}$$

where the hermitian matrices t_A satisfy the same comm rels as the generators T_A

$$[T_A, T_B] = i f_{AB}{}^C\,T_C\,, \qquad [t_A, t_B] = i f_{AB}{}^C\,t_C$$

The structure of the SUSY algebra

- What about the P, Q commutator? A priori it may yield $(1,\frac{1}{2})$ and $(0,\frac{1}{2})$. There is no generator in the $(1,\frac{1}{2})$ rep, so we can only have $[P,Q] \sim \overline{Q}$ or $[P,Q] = 0$. The super Jacobi identity rules out the first possibility, so

$$[P_\mu, Q^I{}_\alpha] = 0 \ , \qquad [P_\mu, \overline{Q}_{I\dot{\alpha}}] = 0$$

The structure of the SUSY algebra

- The only piece missing is the Q Q anticommutator. It can yield $(0,0)$ or $(1,0)$, so it can only contain $J_{\mu\nu}$ or the internal symm generators T_A
- Super Jacobi and $[P, Q] = 0$ exclude $J_{\mu\nu}$, so we can only write

$$\{Q^I{}_\alpha, Q^J{}_\beta\} = Z^{IJ}\,\epsilon_{\alpha\beta} \quad , \qquad Z^{IJ} = (Z^{IJ})^A\, T_A$$

- Since the anticommutator is symmetric, we must have

$$(Z^{IJ})^A = -\,(Z^{JI})^A$$

- Taking $\dagger$ we get

$$\{\overline{Q}_{I\dot{\alpha}}, \overline{Q}_{J\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}}\,(\bar{Z}_{IJ})^A\, T_A \quad , \quad (\bar{Z}_{IJ})^A := \left[(Z^{IJ})^A\right]^*\, T_A$$

The structure of the SUSY algebra

- We already know that all T_A's commute with P and J, so this is also true for the linear combinations $Z^{IJ} = (Z^{IJ})^A\, T_A$
- Super Jacobi shows that the Z^{IJ}'s also commute with the supercharges

$$[Z, Q] = 0 = [Z, \overline{Q}]$$

- We haven't used the fact that $(Z^{IJ})^A$ must be an invariant tensor of the the internal symm algebra $\mathfrak{g}$ generated by the T_A's
- $(Z^{IJ})^A$ has one adjoint index and two indices in the rep with generators $(t_A)^I{}_J$, so

$$0 = \delta_B (Z^{IJ})^A = (t_B)^K{}_J\, (Z^{KJ})^A + (t_B)^K{}_I\, (Z^{IK})^A + (t_B^{\text{adj}})^A{}_C\, (Z^{IJ})^C$$

  where $(t_B^{\text{adj}})^A{}_C = f_{BC}{}^A$
- The above is the same as

$$[T_A, Z^{IJ}] = -\, (t_A)^I{}_K\, Z^{KJ} - (t_A)^J{}_K\, Z^{IK}$$

The structure of the SUSY algebra

$$[T_A, Z^{IJ}] = -(t_A)^I{}_K Z^{KJ} - (t_A)^J{}_K Z^{IK} \qquad (*)$$

- The above commutator shows that the linear combinations $Z^{IJ} = (Z^{IJ})^A T_A$ generate an *invariant subalgebra* of the internal symm algebra $\mathfrak{g}$ generated by the T_A's
- Coleman-Mandula tells us that $\mathfrak{g}$ consists of simple, non-Abelian, compact factors plus $\mathfrak{u}(1)$ factors. The non-Abelian part cannot accommodate (*) so we learn that

  $Z^{IJ} = (Z^{IJ})^A T_A$ lie inside the Abelian part of the internal symm alg
- Conclusion: the Z^{IJ}'s have zero commutators with everything! They are called **central charges**.

Recap

- The supercharges transform in the $(\frac{1}{2},0)$, $(0,\frac{1}{2})$ reps of the Lorentz group and are translationally invariant
- They can be charged under the internal symmetry algebra $\mathfrak{g} = \mathrm{span}(T_A)$. If they are charged, they transform in a unitary rep

$$[T_A, Q^I{}_\alpha] = -(t_A)^I{}_J\, Q^J{}_\alpha \ , \qquad t_A \text{ hermitian} \ , \qquad I, J = 1,\ldots,\mathcal{N}$$

- The $Q\overline{Q}$ anticommutator is universal

$$\{Q^I{}_\alpha, \overline{Q}_{I\dot\beta}\} = 2\,\delta^I{}_J\,(\sigma^\mu)_{\alpha\dot\beta}\,P_\mu$$

- The QQ anticommutator can yield a linear comb. of generators of $\mathfrak{g}$ ….

$$\{Q^I{}_\alpha, Q^J{}_\beta\} = Z^{IJ}\,\epsilon_{\alpha\beta} \quad , \qquad Z^{IJ} = (Z^{IJ})^A\,T_A \ , \qquad (Z^{IJ})^A = -(Z^{JI})^A$$

- … provided that

$$[Z^{IJ}, \text{everything}] = 0 \quad , \qquad (t_A)^I{}_K\,Z^{KJ} + (t_A)^J{}_K\,Z^{IK} = 0$$

Minimal supersymmetry

- When there is only one supercharge we omit the index I. Since Z^{IJ} is antisymmetric, it vanishes: there cannot be any central charges. We get the minimal SUSY algebra, referred to as 4d $\mathcal{N} = 1$

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\,(\sigma^\mu)_{\alpha\dot\beta}\,P_\mu$$

$$\{Q_\alpha, Q_\beta\} = 0$$

$$\{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0$$

Some remarks

- We have gone through a sketch of the Haag-Lopuszanski-Sohnius theorem
- Our exposition follows Wess-Bagger and Weinberg vol III (with some changes in notation)
- If we only have massless particles, the allowed bosonic Lie algebra of symmetries is conformal algebra + internal symm algebra. It is extended to a superconformal algebra
- Non-conformal SUSY is also referred to as Poincaré SUSY to distinguish it from superconformal symmetry

R-symmetry

- The term R-symmetry refers to any symmetry transformation that acts non-trivially on the supercharges
- In math language: automorphism of the SUSY algebra
- “Inner automorphism”: the action of T_A on $Q^I{}_\alpha$

$$[T_A, Q^I{}_\alpha] = -(t_A)^I{}_J Q^J{}_\alpha \qquad \text{for those } T_A \text{ for which } (t_A)^I{}_J \neq 0$$

- “Outer automorphism”: a redefinition of the supercharges

$$Q'^I{}_\alpha = \mathscr{U}^I{}_J Q^J{}_\alpha$$

that preserves all (anti)commutators

R-symmetry

- Example: the 4d $\mathcal{N}=1$ SUSY algebra is invariant under the U(1) transf

$$Q'_\alpha = e^{i\,s}\,Q_\alpha \quad , \quad \overline{Q}'_{\dot\alpha} = e^{-i\,s}\,\overline{Q}_{\dot\alpha} \quad , \qquad s \in \mathbb{R}$$

- CAVEAT: the R-symmetry of the SUSY algebra is not necessarily a symmetry of the theory!
    1. It can be broken explicitly by interactions in the classical Lagrangian
    2. Even if unbroken classically, it can be broken at one-loop by a quantum anomaly
    3. Even if preserved at the quantum level, it can be spontaneously broken in the vacuum

Supersymmetry and supergravity

Lecture 5

SUSY and 1-particle states

- 1-particle states are unitary irreps of the Poincaré algebra
- Recall that (see eg Weinberg Vol I)
  - 1-particle states are labelled by their 4-momentum, plus "internal" discrete labels. They have definite mass

$$P^\mu \, |p, \sigma\rangle = p^\mu \, |p, \sigma\rangle \ , \qquad m^2 = -\, p_\mu \, p^\mu \ , \qquad m \geq 0$$

  - Using a Lorentz transformation we can map p^μ to a standard representative

$$m > 0 : \quad p^\mu = (m,0,0,0) \ ; \qquad m = 0 : \quad p^\mu = (E,0,0,E)$$

    where $E > 0$ is some reference energy scale
  - The subgroup of the Lorentz group that preserves the reference p^μ is called the **little group**

    massive case: $SO(3)$; massless case: $ISO(2)$
  - In the massive case the label σ is the integer or half-integer **spin** (eigenvalue of J_z)
  - In the massless case it is the integer or half-integer **helicity** (eigenvalue of spin along the direction of 3-momentum)

SUSY and 1-particle states

- The SUSY algebra organizes 1-particle states into supermultiplets
- Since $[P, Q] = 0$, all states in a supermultiplet have the same mass
- Recall: each supermultiplet contains an equal number of bosonic and fermionic dof's
- We study massive and massless cases for minimal and extended SUSY in turn

Minimal SUSY; massive particles

- We use $p^\mu = (m,0,0,0)$ and $\sigma^0 = -\mathbb{I}_2$

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2\,m\,\delta_{\alpha\dot\beta} \ , \quad \{Q_\alpha, Q_\beta\} = 0 \ , \qquad \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0$$

- NB: $\delta_{\alpha\dot\beta}$ is an invariant tensor of the little group $SO(3)$. We can trade lower (upper) dotted indices for upper (lower) undotted indices. The α indices are now fundamental indices of the $SU(2)$ of spatial rotations
- This is the algebra of a set of independent fermionic oscillators

$$a_\alpha = \frac{1}{\sqrt{2m}} Q_\alpha \ , \qquad a^{\dagger\alpha} = (a_\alpha)^\dagger = \frac{1}{\sqrt{2m}} (Q_\alpha)^\dagger$$

$$\{a_\alpha, a^{\dagger\beta}\} = \delta_\alpha{}^\beta \ , \qquad \{a_\alpha, a_\beta\} = 0 \ , \qquad \{a^{\dagger\alpha}, a^{\dagger\beta}\} = 0$$

Minimal SUSY; massive particles

Reps are constructed using a "**Clifford vacuum**" defined by two properties:

1. It is annihilated by all annihilation operators

$$a_\alpha \,|\Omega; s, s_3\rangle = 0$$

2. It transforms in an representation of the little group SO(3) of definite spin

$$J_z \,|\Omega; s, s_3\rangle = s_3 \,|\Omega; s, s_3\rangle \quad , \qquad J^2 \,|\Omega; s, s_3\rangle = s(s+1)\,|\Omega; s, s_3\rangle$$

As usual, s is integer or half-integer and $s_3 = -s, -s+1, \ldots, s-1, s$

We can also denote the Clifford vacuum as a tensor with $2s$ symmetrized α indices $|\Omega\rangle^{\alpha_1\ldots\alpha_{2s}}$.

How do we know that a Clifford vacuum exists? Take any state $|\psi\rangle$ in the supermultiplet. If $a_1 |\psi\rangle = 0$, define $|\psi'\rangle = |\psi\rangle$, otherwise set $|\psi'\rangle = a_1 |\psi\rangle$. In this way we are sure that $a_1 |\psi'\rangle$. Repeat the logic with a_2.

The condition $a_\alpha |\psi''\rangle = 0$ is manifestly rotationally invariant. This is why we can label the Clifford vacuum with s, s_3.

Minimal SUSY; massive particles

We act on the Clifford vacuum with the creation operators. The simplest case is $s = 0$. The states we have are

$\|\Omega\rangle$	scalar of SO(3)
$a^{\dagger\alpha}\|\Omega\rangle$	spinor of SO(3)
$a^{\dagger\alpha}a^{\dagger\beta}\|\Omega\rangle \propto \epsilon^{\alpha\beta}\left(\epsilon_{\gamma\delta}a^{\dagger\gamma}a^{\dagger\delta}\|\Omega\rangle\right)$	scalar of SO(3)

Interpretation: these are the 1-particle states of two real massive scalar fields and one massive Majorana fermion field. It is customary to pair the two real scalar fields into a single complex field. This is usually called a massive **chiral multiplet**

NB: in a QFT where parity P is a symmetry, the two scalar fields have opposite intrinsic parities (one is a scalar, the other a pseudoscalar). For details, see Weinberg vol III

Minimal SUSY; massive particles

The next case is $s = \frac{1}{2}$. When we act with creation op's on the Clifford vacuum, we have to combine their spins. From the familiar rules for combining angular momenta

$$(\tfrac{1}{2}) \otimes (\tfrac{1}{2}) = (0) \oplus (1)$$

Notation: (s) is the irrep of $SU(2)$ with spin s, $2s \in \mathbb{Z}_{\geq 0}$, Casimir $s(s+1)$

The states we have are

$$|\Omega\rangle^{\alpha} \qquad (\tfrac{1}{2})$$

$$\epsilon_{\gamma\delta}\, a^{\dagger\gamma}\,|\Omega\rangle^{\delta} \quad \text{and} \quad a^{\dagger(\alpha}\,|\Omega\rangle^{\beta)} \qquad (0) \oplus (1)$$

$$\epsilon_{\gamma\delta}\, a^{\dagger\gamma}\, a^{\dagger\delta}\,|\Omega\rangle^{\alpha} \qquad (\tfrac{1}{2})$$

Interpretation: these are the 1-particle state of one real massive vector field, one real massive scalar field, and two massive Majorana fields. This is a massive vector multiplet

NB: in a QFT where parity P is a symmetry, the two fermions have opposite intrinsic parities.

Minimal SUSY; massive particles

The general story is similar.

Acting once on the Clifford vacuum: $(\frac{1}{2}) \otimes (s) = (s-\frac{1}{2}) \oplus (s+\frac{1}{2})$

Acting twice on the Clifford vacuum: because the creation operators anticommute, $a^{\dagger\alpha}a^{\dagger\beta} \sim \epsilon^{\alpha\beta}\left(\epsilon_{\gamma\delta}\, a^{\dagger\gamma}a^{\dagger\delta}\right)$. Acting with $\epsilon_{\gamma\delta}\, a^{\dagger\gamma}a^{\dagger\delta}$ does not change the SO(3) spin.

In summary: massive multiplets of minimal SUSY

$s=0$	no $a^\dagger$'s	(0)	1	$s>0$	no $a^\dagger$'s	(s)	$2s+1$
	one $a^\dagger$	$(\frac{1}{2})$	2		one $a^\dagger$	$(s-\frac{1}{2}) \oplus (s+\frac{1}{2})$	$4s+2$
	two $a^\dagger$'s	(0)	1		two $a^\dagger$'s	(s)	$2s+1$

NB: the Clifford vacuum is a boson for integer s, a fermion for half-integer s

We have included the no of states. There is always a balance between bosons and fermions

Minimal SUSY; massless particles

- We use $p^\mu = (E,0,0,E)$ and $\sigma^0 = -\mathbb{I}_2, \sigma^3 = +\sigma_z$

$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 4\,E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \;, \quad \{Q_\alpha, Q_\beta\} = 0 \;, \quad \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0$$

- The supercharge with $\alpha = 1$ and its conjugate give one set of fermionic oscillators

$$a = \frac{1}{2\sqrt{E}} Q_1 \;, \quad a^\dagger = \frac{1}{2\sqrt{E}} (Q_1)^\dagger \;, \quad \{a, a^\dagger\} = 1 \;,$$

- The supercharge with $\alpha = 2$ and its conjugate anticommute

$$Q_2\,(Q_2)^\dagger + (Q_2)^\dagger\,Q_2 = 0 \quad \Rightarrow \quad \begin{matrix} \|Q_2\,|\psi\rangle\|^2 = 0 \\ \|Q_2^\dagger\,|\psi\rangle\|^2 = 0 \end{matrix}$$ for any state $|\psi\rangle$

- The inner product in the 1-particle Hilbert space is positive definite, so $Q_2 = 0 = Q_2^\dagger$

Minimal SUSY; massless particles

- The relevant quantum number in the massless case is helicity. With our choice $p^\mu = (E,0,0,E)$ of reference 4-momentum, helicity is the eigenvalue λ of J_z
- From the JQ commutators we find

$$[J_z, Q_1] = -\frac{1}{2}Q_1, \qquad [J_z, Q_1^\dagger] = \frac{1}{2}Q_1^\dagger$$

- We learn that the annihilation operator a lowers the helicity by 1/2, and $a^\dagger$ raises it by 1/2
- The Clifford vacuum is annihilated by a by definition, so it has the lowest helicity in the supermultiplet

$$a\,|\Omega;\lambda\rangle = 0 \ , \qquad J_z\,|\Omega;\lambda\rangle = \lambda\,|\Omega;\lambda\rangle$$

- Acting once with $a^\dagger$ we get a state of helicity $\lambda+\frac{1}{2}$

$$J_z\,a^\dagger\,|\Omega;\lambda\rangle = \left(\lambda+\frac{1}{2}\right)a^\dagger\,|\Omega;\lambda\rangle$$

- Lesson: the general massless SUSY multiplet has two states of helicities λ, $\lambda + 1/2$. λ can be any integer or half-integer

Minimal SUSY; massless particles

CPT must be a good symmetry in any relativistic QFT. To preserve CPT we need to consider a pair of massless SUSY multiplets:

no $a^\dagger$	λ	together with	no $a^\dagger$	$-\lambda - \frac{1}{2}$
one $a^\dagger$	$\lambda + \frac{1}{2}$		one $a^\dagger$	$-\lambda$

Relevant examples:

- $\lambda = 0:$ the 1-particle states correspond to two real massless scalars and one massless Majorana fermion (equiv. one massless Weyl fermion). This is a massless **chiral multiplet**
- $\lambda = 1/2:$ the 1-particle states correspond to one real vector boson and one massless Majorana fermion (equiv. one massless Weyl fermion). This is a massless **vector multiplet**

NB: There is a SUSY version of the Higgs mechanism, in which a massless vector multiplet "eats" a massless chiral multiplet to give a massive vector multiplet

Supersymmetry and supergravity

Lecture 6

Extended SUSY; massless particles

Let us now turn to extended SUSY; we study the massless case first: $p^\mu = (E,0,0,E)$

$$\{Q^I{}_\alpha, \overline{Q}_{J\dot\beta}\} = 4\,E\,\delta^I{}_J \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot\beta} \qquad I, J = 1,\ldots,\mathcal{N}$$

$$\{Q^I{}_\alpha, Q^J{}_\beta\} = \epsilon_{\alpha\beta}\, Z^{IJ} \;, \qquad \{\overline{Q}_{I\dot\alpha}, \overline{Q}_{J\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\, \overline{Z}_{IJ}$$

The first anticommutator gives

$$(Q^I{}_2)(Q^I{}_2)^\dagger + (Q^I{}_2)^\dagger(Q^I{}_2) = 0 \qquad \text{no sum on } I$$

so it is still true that $Q^I{}_2$ is represented by zero. From the relation

$$Z^{IJ} = -\{Q^I{}_1, Q^J{}_2\} = -\,Q^I{}_1\, Q^J{}_2 - Q^J{}_2\, Q^I{}_1$$

we conclude that the central charges Z^{IJ} must also be represented by zero.

Extended SUSY; massless particles

We have a collection of fermionic creation/annihilation operators

$$a^I = \frac{1}{2\sqrt{E}} Q^I{}_1 \quad , \qquad \{a^I, a_J^\dagger\} = \delta^I{}_J \quad , \quad \{a^I, a^J\} = 0 \quad , \quad \{a_I^\dagger, a_J^\dagger\} = 0$$

All the a's and $a^\dagger$'s have definite helicity

$$[J_z, a^I] = -\frac{1}{2}\, a^I \quad , \qquad [J_z, a_I^\dagger] = +\frac{1}{2}\, a_I^\dagger$$

The Clifford vacuum is annihilated by all a's and thus has minimal helicity

$$a^I\, |\, \Omega; \lambda\rangle = 0 \quad , \qquad J_z\, |\, \Omega; \lambda\rangle = \lambda\, |\, \Omega; \lambda\rangle$$

We act on the Clifford vacuum with the $a^\dagger$'s in all possible ways.

Extended SUSY; massless particles

The $a^\dagger$'s anticommute, so we get states that are totally antisymmetric in their I indices. The latter are indices of an $U(\mathcal{N})$ R-symmetry.

state	helicity
$\|\Omega;\lambda\rangle$	λ
$a^\dagger_{I_1}\|\Omega;\lambda\rangle$	$\lambda+\frac{1}{2}$
$a^\dagger_{[I_1}a^\dagger_{I_2]}\|\Omega;\lambda\rangle$	$\lambda+1$
$\vdots$	
$a^\dagger_{[I_1}\dots a^\dagger_{I_p]}\|\Omega;\lambda\rangle$	$\lambda+\frac{1}{2}p$
$\vdots$	
$a^\dagger_{[I_1}\dots a^\dagger_{I_\mathcal{N}]}\|\Omega;\lambda\rangle$	$\lambda+\frac{1}{2}\mathcal{N}$

- For generic values of $\mathcal{N}$ and λ we have include by hand the CPT conjugate multiplet
- The degeneracies ensure the balance between bosonic and fermionic states

Extended SUSY; massless particles

The interactions of massless particles of helicity $|\lambda| \geq 1$ are very constrained:

- a massless boson of helicity ± 1 must be a gauge boson in a gauge theory, which couples to an internal symmetry current
- a massless fermion of helicity $\pm 3/2$ must be the so-called gravitino, the field that couples to the supersymmetry current (supergravity)
- a massless boson of helicity ± 2 must be the (unique) graviton, which couples to the (unique) stress tensor of the QFT

It is not known how to write interactions for massless particles of higher helicity in 4d Minkowski.

These physical arguments give us a bound on helicity, hence a bound in the number of SUSYs:

- If we want a QFT without dynamical gravity

$$|\lambda| \leq 1 \qquad \text{hence} \qquad \mathcal{N} \leq 4$$

- If we allow dynamical gravity (supergravity)

$$|\lambda| \leq 2 \qquad \text{hence} \qquad \mathcal{N} \leq 8$$

Extended SUSY; massless particles

$\mathcal{N} = 2$ vector multiplet

$$\begin{array}{ccc} |\Omega\rangle & -1 & 1 \\ a^\dagger|\Omega\rangle & -\frac{1}{2} & 2 \\ a^\dagger a^\dagger|\Omega\rangle & 0 & 1 \end{array} \qquad \text{and the CPT conj.} \qquad \begin{array}{ccc} |\Omega\rangle & 0 & 1 \\ a^\dagger|\Omega\rangle & \frac{1}{2} & 2 \\ a^\dagger a^\dagger|\Omega\rangle & 1 & 1 \end{array}$$

real vector A_μ + complex scalar Φ + Weyl fermions ${\lambda^I}_\alpha$

NB: $I \equiv$ fund index of $\mathfrak{su}(2)_R$.

They all transform in the adjoint rep of the gauge group.

Extended SUSY; massless particles

$\mathcal{N} = 2$ hypermultiplet

$$\begin{array}{ccc} |\Omega\rangle & -\frac{1}{2} & 1 \\ a^\dagger\,|\Omega\rangle & 0 & 2 \\ a^\dagger a^\dagger\,|\Omega\rangle & +\frac{1}{2} & 1 \end{array} \quad + \quad \text{its CPT conjugate}$$

4 real scalars + two Weyl fermions

In SUSY gauge theories, hypermultiplets are the matter fields. Their 4 scalars and two fermions transform generically as $\mathbf{R} \oplus \overline{\mathbf{R}}$ where $\mathbf{R}$ is any representation of the gauge group and $\overline{\mathbf{R}}$ is the conjugate rep

Extended SUSY; massless particles

$\mathcal{N} = 4$ vector multiplet

$\|\Omega\rangle$	-1	1
$a^\dagger\|\Omega\rangle$	$-\frac{1}{2}$	4
$a^\dagger a^\dagger\|\Omega\rangle$	0	6
$a^\dagger a^\dagger a^\dagger\|\Omega\rangle$	$\frac{1}{2}$	4
$a^\dagger a^\dagger a^\dagger a^\dagger\|\Omega\rangle$	1	1

(CPT self conj.) = real vector A_μ + 6 real scalars + 4 Weyl fermions

The R-symm algebra is $\mathfrak{su}(4)_R \cong \mathfrak{so}(6)_R$. The fermions are in the fundamental of $\mathfrak{su}(4)_R$.
The 6 real scalars are in the vec rep of $\mathfrak{so}(6)_R$. (This is a real rep, that's why this multiplet can be self CPT conj.)
All fields transform in the adjoint rep of the gauge group.

Extended SUSY; massless particles

What about $\mathcal{N} = 3$?

It turns out that a $\mathcal{N} = 3$ vector multiplet (plus its CPT conj) is the same as a $\mathcal{N} = 4$ vector multiplet. Any Lagrangian theory with $\mathcal{N} = 3$ SUSY is also automatically $\mathcal{N} = 4$.

This does NOT hold for theories without a Lagrangian descriptions. Interesting superconformal 4d $\mathcal{N} = 3$ theories (that are not $\mathcal{N} = 4$) have been constructed recently, but are strongly-coupled

Extended SUSY; massless particles

NB: extended SUSY does not allow for chiral reps of the gauge group

- For $\mathcal{N} = 4$ we only have vector multiplets, and all fields are in the adjoint representation, which is a real representation
- For $\mathcal{N} = 2$ we can have vector multiplets in the adjoint or hypermultiplet in the any rep $\mathbf{R} \oplus \overline{\mathbf{R}}$, but these are real representations

The positive-chirality and negative-chirality fermions in the standard model transform in different representations of the gauge group.

For applications to pheno, 4d $\mathcal{N} = 1$ is the most promising

Extended SUSY; massive particles

In the massive case $p^\mu = (m,0,0,0)$ and the little group is $SO(3)$

$$\{Q^I{}_\alpha, \overline{Q}_{J\dot\beta}\} = 2\,m\,\delta^I{}_J\,\delta_{\alpha\dot\beta} \qquad I,J = 1,\ldots,\mathcal{N}$$

$$\{Q^I{}_\alpha, Q^J{}_\beta\} = \epsilon_{\alpha\beta}\,Z^{IJ}\ , \qquad \{\overline{Q}_{I\dot\alpha}, \overline{Q}_{J\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\,\overline{Z}_{IJ}$$

Since the operators Z^{IJ} commute with everything, all states in a supermultiplet have the same eigenvalue of Z^{IJ} (which we denote by the same symbol).

For definiteness we analyze $\mathcal{N} = 2$. Similar story for $\mathcal{N} > 2$

Extended SUSY; massive particles

By a unitary redef of the Qs we can cast Z^{IJ} in the standard form

$$Z^{IJ} = \begin{pmatrix} 0 & Z \\ -Z & 0 \end{pmatrix} = \epsilon^{IJ} Z \qquad \text{with} \quad Z \ \text{real and positive}$$

The SUSY algebra becomes

$$\{Q^I{}_\alpha, \overline{Q}_{J\dot\beta}\} = 2\,m\,\delta^I{}_J\,\delta_{\alpha\dot\beta} \ , \qquad \{Q^I{}_\alpha, Q^J{}_\beta\} = \epsilon_{\alpha\beta}\,\epsilon^{IJ}\,Z \ , \qquad \{\overline{Q}_{I\dot\alpha}, \overline{Q}_{J\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\,\epsilon_{IJ}\,\overline{Z}$$

Useful redefinition:

$$a_\alpha = \frac{1}{\sqrt{2}}\left(Q^1{}_\alpha + \epsilon_{\alpha\beta}\,\delta^{\beta\dot\beta}\,\overline{Q}_{2\dot\beta}\right) \ , \quad b_\alpha = \frac{1}{\sqrt{2}}\left(Q^1{}_\alpha - \epsilon_{\alpha\beta}\,\delta^{\beta\dot\beta}\,\overline{Q}_{2\dot\beta}\right)$$

The algebra becomes

$$\{a_\alpha, a^{\dagger\beta}\} = (2\,m + Z)\,\delta_\alpha{}^\beta \ , \quad \{b_\alpha, b^{\dagger\beta}\} = (2\,m - Z)\,\delta_\alpha{}^\beta$$

$$\{a_\alpha, a_\beta\} = \{a_\alpha, b_\alpha\} = \{b_\alpha, b_\beta\} = 0$$

Extended SUSY; massive particles

$$\{a_\alpha, a^{\dagger\beta}\} = (2\,m + Z)\,\delta_\alpha{}^\beta \quad , \qquad \{b_\alpha, b^{\dagger\beta}\} = (2\,m - Z)\,\delta_\alpha{}^\beta$$
$$\{a_\alpha, a_\beta\} = \{a_\alpha, b_\alpha\} = \{b_\alpha, b_\beta\} = 0$$

We can prove the "**BPS bound**"

$$Z \leq 2\,m$$

Proof: The $bb^\dagger$ anticommutator implies in particular

$$2m - Z = b_1\,(b_1)^\dagger + (b_1)^\dagger\,b_1$$

Let us take the VEV of the above on any state $|\psi\rangle$ in the supermultiplet

$$(2m - Z)\,\||\psi\rangle\|^2 = \|b_1\,|\psi\rangle\|^2 + \|(b_1)^\dagger\,|\psi\rangle\|^2$$

We conclude that the c-number $2m - Z$ is non-negative.

BPS stands for Bogomol'ny-Prasad-Sommerfield. These authors derived a similar bound between mass and charge of monopoles in non-Abelian gauge theories.

Extended SUSY; massive particles

We have two **qualitatively** different cases whether the BPS bound is saturated, or not

- Generic $Z < 2m$:   “long multiplets”

After a suitable rescaling, both the a_α’s and the b_α’s are standard fermionic oscillators. The Clifford vacuum is annihilated by all a_α’s and b_α’s. We act on it with $a^{\dagger\alpha}$’s and $b^{\dagger\alpha}$’s in all possible ways.

For example: if the Clifford vacuum is a scalar

$\vert\Omega\rangle$	1
$a^\dagger\vert\Omega\rangle,\quad b^\dagger\vert\Omega\rangle$	4
$a^\dagger a^\dagger\vert\Omega\rangle\,,\quad b^\dagger b^\dagger\vert\Omega\rangle\,, a^\dagger b^\dagger\vert\Omega\rangle$	6
$a^\dagger a^\dagger b^\dagger\vert\Omega\rangle\quad,\; a^\dagger b^\dagger b^\dagger\vert\Omega\rangle$	4
$a^\dagger a^\dagger b^\dagger b^\dagger\vert\Omega\rangle$	1

Total of 8 + 8 states

Extended SUSY; massive particles

We have two **qualitatively** different cases whether the BPS bound is saturated, or not

- Special $Z = 2m$:   "short multiplets"

In this case the b_α's and $b^{\dagger\alpha}$'s are represented by 0. We only have one set of fermionic oscillators. In other words, all states in the multiplet are annihilated by half of the supercharges: "1/2-BPS condition"

For example: if the Clifford vacuum is a scalar

$\lvert\Omega\rangle$	(0)	1
$a^{\dagger\alpha}\lvert\Omega\rangle$	$(\frac{1}{2})$	2
$\epsilon_{\gamma\delta}\, a^{\dagger\gamma} a^{\dagger\delta}\lvert\Omega\rangle$	(0)	1

Total of 2 + 2 states (cfr with 8 + 8 in the long multiplet)

Quantum corrections cannot change the size of a multiplet. The relation $Z = 2m$ must remain **exact** at the full quantum level. This includes non-perturbative corrections!

Supersymmetry and supergravity

Lecture 7

SUSY action on fields

- We have studied the action of the SUSY algebra on 1-particle states
- We know that particles are quanta of fields. To write SUSY Lagrangians, we need to study the action of the SUSY algebra on fields
- Useful language: SUSY variations of fields in the Lagrangian

A bosonic example

Purely bosonic analogy: a set of complex scalar fields X^i that transform in the fundamental rep of an $SU(n)$ internal symmetry

Abstract $SU(n)$ algebra: $\quad [T_a, T_b] = i f_{ab}{}^c\, T_c \qquad a = 1, \ldots, n^2 - 1$

$SU(n)$ variation of fields: $\quad \delta_\lambda X^i = i\, \lambda^a\, (t_a)^i{}_j\, X^j \qquad i = 1, \ldots, n$

λ^a: real param's of infinitesimal transf.; $\quad (t_a)^i{}_j$: generators of $SU(n)$ in the fund rep

The commutator of two variations encodes the abstract algebra:

$$\delta_{\lambda_1}\delta_{\lambda_2} X^i = \delta_{\lambda_1}\left[i\, \lambda_2^a\, (t_a)^i{}_j\, X^j\right] = i\, \lambda_2^a\, (t_a)^i{}_j\, \delta_{\lambda_1} X^j = i\, \lambda_2^a\, (t_a)^i{}_j\, i\, \lambda_1^b\, (t_b)^j{}_k\, X^k$$

$$\delta_{\lambda_1}\delta_{\lambda_2} X^i - \delta_{\lambda_2}\delta_{\lambda_1} X^i = -\, \lambda_2^a\, \lambda_1^b\, [t_a, t_b]^i{}_j\, X^j = +\, i\, \lambda_1^a\, \lambda_2^b\, f_{ab}{}^c\, (t_c)^i{}_j\, X^j$$

$$\delta_{\lambda_1}\delta_{\lambda_2} X^i - \delta_{\lambda_2}\delta_{\lambda_1} X^i = \delta_{\lambda_3} X^i \qquad \lambda_3^c = f_{ab}{}^c\, \lambda_1^a\, \lambda_2^b$$

Anticommuting parameters

- Since the supercharges are anticommuting, it is convenient to take the SUSY parameters to be anticommuting c-numbers (aka Grassmann c-numbers)
- For 4d $\mathcal{N} = 1$ the supercharges are Q_α, $\overline{Q}_{\dot\alpha} = (Q_\alpha)^\dagger$ so the SUSY parameters are ξ_α, $\bar\xi_{\dot\alpha} = (\xi_\alpha)^*$. All their components anticommute with all other components

$$\xi_\alpha \xi_\beta = -\xi_\beta \xi_\alpha \ , \qquad \xi_\alpha \bar\xi_{\dot\beta} = -\bar\xi_{\dot\beta} \xi_\alpha \ , \qquad \bar\xi_{\dot\alpha} \bar\xi_{\dot\beta} = -\bar\xi_{\dot\beta} \bar\xi_{\dot\alpha}$$

- If X stands for any field, we use the notation

$$\delta_{\xi,\bar\xi} X = \delta_\xi X + \delta_{\bar\xi} X \ , \qquad \delta_\xi X = \xi^\alpha (Q_\alpha \cdot X) \ , \qquad \delta_{\bar\xi} X = \bar\xi_{\dot\alpha} (\overline{Q}^{\dot\alpha} \cdot X)$$

  Here $(Q_\alpha \cdot X)$ and $(\overline{Q}^{\dot\alpha} \cdot X)$ denote the fields into which X is transformed by SUSY. They have opposite statistics to X. The SUSY params anticommute with the operations $(Q_\alpha \cdot)$, $(\overline{Q}^{\dot\alpha} \cdot)$

Anticommuting fermionic fields

- The fermionic fields in the classical Lagrangian are also anticommuting c-numbers
- In the quantum theory they are promoted to operators that satisfy non-trivial anticommmutation relations
- The components of a (classical or quantum) fermionic field ψ_α, $\overline{\psi}_{\dot\alpha} = (\psi_\alpha)^*$ anticommute with the SUSY parameters

$$\xi_\alpha \psi_\beta = -\psi_\beta \xi_\alpha \quad , \qquad \xi_\alpha \overline{\psi}_{\dot\beta} = -\overline{\psi}_{\dot\beta} \xi_\alpha \quad , \qquad \text{etc.}$$

SUSY action on fields

- The anticommutator of two supercharges is encoded in the commutator of two SUSY variations. For example, if X is any field

$$\delta_{\xi_1}\delta_{\bar\xi_2}X = \xi_1^\alpha Q_\alpha \cdot \left[\bar\xi_{2\dot\beta}(\overline{Q}^{\dot\beta}\cdot X)\right] = -\,\xi_1^\alpha\,\bar\xi_{2\dot\beta}\;Q_\alpha\cdot(\overline{Q}^{\dot\beta}\cdot X)$$

$$\delta_{\bar\xi_2}\delta_{\xi_1}X = \bar\xi_{2\dot\beta}\,\overline{Q}^{\dot\beta}\cdot\left[\xi_1^\alpha(Q_\alpha\cdot X)\right] = -\,\bar\xi_{2\dot\beta}\,\xi_1^\alpha\;\overline{Q}^{\dot\beta}\cdot(Q_\alpha\cdot X) = +\,\xi_1^\alpha\,\bar\xi_{2\dot\beta}\;\overline{Q}^{\dot\beta}\cdot(Q_\alpha\cdot X)$$

$$\begin{aligned}\delta_{\xi_1}\delta_{\bar\xi_2}X - \delta_{\bar\xi_2}\delta_{\xi_1}X &= -\,\xi_1^\alpha\,\bar\xi_{2\dot\beta}\left[Q_\alpha\cdot(\overline{Q}^{\dot\beta}\cdot X) + \overline{Q}^{\dot\beta}\cdot(Q_\alpha\cdot X)\right]\\ &= -\,\xi_1^\alpha\,\bar\xi_{2\dot\beta}\;\;\{Q_\alpha,\overline{Q}^{\dot\beta}\}\cdot X\\ &= +\,\xi_1^\alpha\,\bar\xi_2{}^{\dot\beta}\;\;\{Q_\alpha,\overline{Q}_{\dot\beta}\}\cdot X\end{aligned}$$

SUSY action on fields

$$\delta_{\xi_1}\,\delta_{\bar{\xi}_2}X - \delta_{\bar{\xi}_2}\,\delta_{\xi_1}X = \xi_1^{\alpha}\,\bar{\xi}_2{}^{\dot{\beta}}\quad \{Q_\alpha, \overline{Q}_{\dot{\beta}}\}\cdot X$$

- The SUSY variations of fields must furnish a rep of the abstract SUSY anticommutator $\{Q_\alpha, \overline{Q}_{\dot{\beta}}\} = 2\,(\sigma^\mu)_{\alpha\dot{\beta}}\,P_\mu$. Similar story with the other anticommutators.
- We introduce the compact notation

$$\delta_1 \equiv \delta_{\xi_1,\bar{\xi}_1} \quad , \quad \delta_2 \equiv \delta_{\xi_2,\bar{\xi}_2}$$

- We must have

$$\delta_1\,\delta_2 X - \delta_2\,\delta_1 X = 2\,(\xi_1\,\sigma^\mu\,\bar{\xi}_2 - \xi_2\,\sigma^\mu\,\bar{\xi}_1)\,P_\mu\cdot X \qquad , \qquad P_\mu\cdot X = -\,i\,\partial_\mu X$$

- These conditions are usually referred to as “closure of the SUSY algebra”
- The param of a translation has mass dim = -1. We learn that the SUSY params must have mass dim = $-1/2$

Intermezzo: more spinor technology

- Recall that spinor indices α, $\dot\alpha$ are raised/lowered with $\epsilon_{\alpha\beta}$, $\epsilon^{\alpha\beta}$, $\epsilon_{\dot\alpha\dot\beta}$, $\epsilon^{\dot\alpha\dot\beta}$

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \quad , \quad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \quad , \quad \epsilon_{12} = -1, \quad \epsilon^{12} = +1$$

  and idem for dotted indices (Wess-Bagger conventions)

- In every expression where we find a pair of contracted indices as ${}^\alpha{}_\alpha$ we can always flip them to the configuration ${}_\alpha{}^\alpha$; the price to pay is a minus sign
- If the objects ψ_α, λ_α are anticommuting we can write

$$\psi^\alpha\lambda_\alpha = -\psi_\alpha\lambda^\alpha = +\lambda^\alpha\psi_\alpha \quad , \qquad \bar\psi_{\dot\alpha}\bar\lambda^{\dot\alpha} = -\bar\psi^{\dot\alpha}\bar\lambda_{\dot\alpha} = +\bar\lambda_{\dot\alpha}\bar\psi^{\dot\alpha}$$

- Index-free notation for bilinears: when we omit undotted indices, they are understood to be in the ${}^\alpha{}_\alpha$ configuration; for dotted indices, the implicit position is instead ${}_{\dot\alpha}{}^{\dot\alpha}$. Eg:

$$\psi\lambda = \psi^\alpha\lambda_\alpha \quad , \qquad \bar\psi\bar\lambda = \bar\psi_{\dot\alpha}\bar\lambda^{\dot\alpha}$$

Intermezzo: more spinor technology

- Vector bilinears:

$$\psi^\alpha\,(\sigma^\mu)_{\alpha\dot\beta}\,\bar\chi^{\dot\beta} = \psi\,\sigma^\mu\,\bar\chi\ , \qquad \bar\chi_{\dot\alpha}\,(\bar\sigma^\mu)^{\dot\alpha\beta}\,\psi_\beta = \bar\chi\,\bar\sigma^\mu\,\psi$$

- Tensor bilinears:

$$\psi^\alpha\,(\sigma^{\mu\nu})_\alpha{}^\beta\,\chi_\beta = \psi\,\sigma^{\mu\nu}\,\chi\ , \qquad \bar\psi_{\dot\alpha}\,(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,\bar\chi^{\dot\beta} = \bar\psi\,\bar\sigma^{\mu\nu}\,\bar\chi$$

- There are various “flip identities” for bilinears. For example, we know that $(\bar\sigma^\mu)^{\dot\alpha\beta} = (\sigma^\mu)^{\beta\dot\alpha}$ and this implies

$$\begin{aligned}\bar\chi\,\bar\sigma^\mu\,\psi &= \bar\chi_{\dot\alpha}\,(\bar\sigma^\mu)^{\dot\alpha\beta}\,\psi_\beta = -\,\psi_\beta\,(\bar\sigma^\mu)^{\dot\alpha\beta}\,\bar\chi_{\dot\alpha} \\ &= -\,\psi_\beta\,(\sigma^\mu)^{\beta\dot\alpha}\,\bar\chi_{\dot\alpha} = -\,\psi^\beta\,(\sigma^\mu)_{\beta\dot\alpha}\,\bar\chi^{\dot\alpha} = -\,\psi\,\sigma^\mu\,\bar\chi\end{aligned}$$

- NB: all spinors are assumed to be anticommuting

Intermezzo: more spinor technology

- To check the reality properties of bilinears we need the fact that complex conjugation reverses the order of a product (convention inspired by † on operators)

$$(\psi_\alpha \, \chi_\beta)^* = (\chi_\beta)^* \, (\psi_\alpha)^* = \bar{\chi}_{\dot{\beta}} \, \bar{\psi}_{\dot{\alpha}}$$

- For example:

$$(\bar{\chi} \, \bar{\sigma}^\mu \, \psi)^* = \left[\bar{\chi}_{\dot{\alpha}} \, (\bar{\sigma}^\mu)^{\dot{\alpha}\beta} \, \psi_\beta\right]^* = \bar{\psi}_{\dot{\beta}} \, [(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}]^* \, \chi_\alpha$$

$$= \bar{\psi}_{\dot{\beta}} \, (\bar{\sigma}^\mu)^{\dot{\beta}\alpha} \, \chi_\alpha = \bar{\psi} \, \bar{\sigma}^\mu \, \chi$$

Intermezzo: more spinor technology

- Expressions with 3 fermions can be manipulated using "**Fierz identities**"
- A simple example: the quantity $\psi_\alpha \lambda_\beta - \psi_\beta \lambda_\alpha$ is antisymmetric in $\alpha\beta$ and hence must be proportional to $\epsilon_{\alpha\beta}$. We can fix the prop. constant by taking a trace:

$$\psi_\alpha \lambda_\beta - \psi_\beta \lambda_\alpha = (\psi \lambda)\, \epsilon_{\alpha\beta}$$

- Contract both sides with χ^β and rearrange:

$$(\psi \lambda)\, \chi_\alpha + (\chi \psi)\, \lambda_\alpha + (\lambda \chi)\, \psi_\alpha = 0$$

Free massless chiral multiplet

- The 1-particle states are two states of helicity 0 and a pair of states of helicities $\pm 1/2$. These states corresponds to a massless complex scalar X and a massless Weyl fermion ψ_α.
- We start considering a free theory. The Lagrangian contains only the canonical kinetic terms. The canonical mass dimensions of X, ψ_α are 1 and 3/2, respectively.

$$S = \int d^4x\, \mathscr{L} \qquad\qquad \mathscr{L} = -\,\partial^\mu \overline{X}\, \partial_\mu X + i\, \partial_\mu \overline{\psi}_{\dot\alpha}\, (\overline{\sigma}^\mu)^{\dot\alpha\beta}\, \psi_\beta$$

- NB: the fermion kinetic term is real up to a total spacetime derivative. $\overline{X}$ is the complex conjugate of X.
- Claim: the SUSY variations are ($\delta \equiv \delta_{\xi,\overline{\xi}}$)

$$\delta X = \sqrt{2}\, \xi^\alpha \psi_\alpha \qquad\qquad \delta\psi_\alpha = i\sqrt{2}\, (\sigma^\mu)_{\alpha\dot\beta}\, \overline{\xi}^{\dot\beta}\, \partial_\mu X$$

Free massless chiral multiplet

$$\mathscr{L} = -\,\partial^\mu \overline{X}\,\partial_\mu X + i\,\partial_\mu \overline{\psi}_{\dot\alpha}\,(\overline{\sigma}^\mu)^{\dot\alpha\beta}\,\psi_\beta$$

$$\delta X = \sqrt{2}\,\xi^\alpha\,\psi_\alpha \qquad\qquad \delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu)_{\alpha\dot\beta}\,\overline{\xi}^{\dot\beta}\,\partial_\mu X$$

This proposal for the SUSY variations passes two basic sanity checks:

- They are manifestly Lorentz covariant
- The mass dimensions of both sides are equal (recall $[\xi] = -\,1/2$)

How do we know this is correct?

1. The Lagrangian must be invariant under a SUSY variation, up to a total derivative (so that the action is invariant)
2. The SUSY algebra must close on both X and ψ_α

The first requirement turns out to be satisfied

SUSY commutators

$$\delta X = \sqrt{2}\, \xi^{\alpha}\, \psi_{\alpha} \qquad \delta \psi_{\alpha} = i\sqrt{2}\, (\sigma^{\mu})_{\alpha\dot{\beta}}\, \bar{\xi}^{\dot{\beta}}\, \partial_{\mu} X$$

The commutators of SUSY variations on the scalar X:

$$\delta_1 \delta_2 X = \delta_1(\sqrt{2}\, \xi_2\, \psi) = \sqrt{2}\, \xi_2^{\alpha}\, (\delta_1 \psi)_{\alpha}$$

$$= \sqrt{2}\, \xi_2^{\alpha}\, i\sqrt{2}\, (\sigma^{\mu}\, \bar{\xi}_1)_{\alpha}\, \partial_{\mu} X = 2\, i\, (\xi_2\, \sigma^{\mu}\, \bar{\xi}_1)\, \partial_{\mu} X$$

$$\delta_1 \delta_2 X - \delta_2 \delta_1 X = -\, 2\, i\, (\xi_1\, \sigma^{\mu}\, \bar{\xi}_2 - \xi_2\, \sigma^{\mu}\, \bar{\xi}_2)\, \partial_{\mu} X$$

It works!

SUSY commutators

$$\delta X = \sqrt{2}\,\xi^\alpha\,\psi_\alpha \qquad\qquad \delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu)_{\alpha\dot\beta}\,\bar\xi^{\dot\beta}\,\partial_\mu X$$

The commutators of SUSY variations on the fermion:

$$\delta_1\delta_2\psi_\alpha = \delta_1(i\sqrt{2}\,(\sigma^\mu\,\bar\xi_2)_\alpha\,\partial_\mu X) = i\sqrt{2}\,(\sigma^\mu\,\bar\xi_2)_\alpha\,\partial_\mu\delta_1 X = 2\,i\,(\sigma^\mu\,\bar\xi_2)_\alpha\,(\xi_1\,\partial_\mu\psi)$$

To proceed, we use the Fierz identity

$$(\chi_1\,\chi_2)\,\chi_{3\alpha} + (\chi_1\,\chi_3)\,\chi_{2\alpha} + (\chi_2\,\chi_3)\,\chi_{1\alpha} = 0 \;\text{ with }\; \chi_1 = \xi_1,\;\; \chi_2 = \partial_\mu\psi,\;\; \chi_3 = \sigma^\mu\,\bar\xi_2$$

which reads

$$(\xi_1\,\partial_\mu\psi)\,(\sigma^\mu\,\bar\xi_2)_\alpha + (\xi_1\,\sigma^\mu\,\bar\xi_2)\,\partial_\mu\psi_\alpha + (\partial_\mu\psi\,\sigma^\mu\,\bar\xi_2)\,\xi_{1\alpha} = 0$$

We get (using $\partial_\mu\psi\,\sigma^\mu\,\bar\xi_2 = -\,\bar\xi_2\,\bar\sigma^\mu\,\partial_\mu\psi$)

$$\begin{aligned}\delta_1\delta_2\psi_\alpha - \delta_2\delta_1\psi_\alpha = &-2\,i\,(\xi_1\,\sigma^\mu\,\bar\xi_2 - \xi_2\,\sigma^\mu\,\bar\xi_1)\,\partial_\mu\psi_\alpha \\ &-2\,i\,(\bar\xi_1\,\bar\sigma^\mu\,\partial_\mu\psi)\,\xi_{2\alpha} + 2\,i\,(\bar\xi_2\,\bar\sigma^\mu\,\partial_\mu\psi)\,\xi_{1\alpha}\end{aligned}$$

SUSY commutators

$$\delta_1\delta_2\psi_\alpha - \delta_2\delta_1\psi_\alpha = -\,2\,i\,(\xi_1\,\sigma^\mu\,\bar{\xi}_2 - \xi_2\,\sigma^\mu\,\bar{\xi}_1)\,\partial_\mu\psi_\alpha$$
$$-\,2\,i\,(\bar{\xi}_1\,\bar{\sigma}^\mu\,\partial_\mu\psi)\,\xi_{2\alpha} + 2\,i\,(\bar{\xi}_2\,\bar{\sigma}^\mu\,\partial_\mu\psi)\,\xi_{1\alpha}$$

- The SUSY algebra does not close, because of the terms on the second line…
- … but they vanish if we impose the fermion EOM

$$\bar{\sigma}^\mu\,\partial_\mu\psi = 0$$

- We say that the SUSY algebra closes “**on-shell**”

On-shell SUSY actions and variations

- In the on-shell formalism for SUSY, the action and the SUSY variations of all fields have to be tuned together, to ensure that the SUSY algebra closes up the EOMs that follow from the action
- When we change the action, we must change the SUSY variations!
- For example, we can include a mass term for a chiral multiplet…

$$\mathscr{L} = -\,\partial^\mu \overline{X}\,\partial_\mu X + i\,\partial_\mu \overline{\psi}_{\dot\alpha}\,(\overline{\sigma}^\mu)^{\dot\alpha\beta}\,\psi_\beta - m\,\overline{X}\,X - \frac{1}{2}\,m\,\psi^\alpha\,\psi_\alpha - \frac{1}{2}\,m\,\overline{\psi}_{\dot\alpha}\,\overline{\psi}^{\dot\alpha}$$

… but we must modify the SUSY variation of the fermion

$$\delta X = \sqrt{2}\,\xi^\alpha\,\psi_\alpha \qquad\qquad \delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu)_{\alpha\dot\beta}\,\overline{\xi}^{\dot\beta}\,\partial_\mu X - \sqrt{2}\,m\,\overline{X}\,\xi_\alpha$$

- This way of constructing SUSY actions and variations is referred to as “Noether method”. One has to find all the correct terms in the action and variation by trial and error.
- Luckily, for 4d $\mathcal{N} = 1$ supersymmetry an **off-shell formalism** exists, in which the SUSY variations are universal, independent of the action

Supersymmetry and supergravity

Lecture 8

Off-shell SUSY for a chiral multiplet

- A complex scalar and a Weyl fermion have the same number of on-shell dof’s, but a different number of off-shell dof’s
- To restore the balance we use an **auxiliary field**: a complex scalar F
- The field F carries no on-shell dof’s because it is fixed in terms of the other fields by its EOM

	off-shell real dof's	on-shell real dof's
X	2	2
ψ_α	4	2
F	2	0

Off-shell SUSY for a chiral multiplet

The SUSY variations are

$$\delta X = \sqrt{2}\, \xi \psi \quad ,$$

$$\delta \psi_\alpha = i \sqrt{2}\, (\sigma^\mu \bar{\xi})_\alpha\, \partial_\mu X + \sqrt{2}\, F\, \xi_\alpha \quad ,$$

$$\delta F = i \sqrt{2}\, \bar{\xi}\, \bar{\sigma}^\mu\, \partial_\mu \psi$$

Remarks:

- The auxiliary field F has mass dimension 2; it is the field with the highest dimension in the multiplet
- The SUSY variation of F is a total spacetime derivative

Off-shell closure of the algebra

$$\delta X = \sqrt{2}\,\xi\,\psi \quad , \quad \delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\,\partial_\mu X + \sqrt{2}\,F\,\xi_\alpha \quad , \quad \delta F = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi$$

SUSY commutator on the dynamical scalar:

$$\delta_1\delta_2 X = \sqrt{2}\,\xi_2^\alpha\,(\delta_1\psi)_\alpha = 2\,i\,(\xi_2\,\sigma^\mu\,\bar{\xi}_1)\,\partial_\mu X + 2\,F\,\xi_2\,\xi_1$$

The additional term with F drops away when we consider the commutators:

$$\delta_1\delta_2 X - \delta_2\delta_1 X = -\,2\,i\,(\xi_1\,\sigma^\mu\,\bar{\xi}_2 - \xi_2\,\sigma^\mu\,\bar{\xi}_1)\,\partial_\mu X$$

Off-shell closure of the algebra

$$\delta X = \sqrt{2}\,\xi\,\psi \quad , \quad \delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\,\partial_\mu X + \sqrt{2}\,F\,\xi_\alpha \quad , \quad \delta F = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi$$

SUSY commutator on the auxiliary scalar:

$$\delta_1\delta_2 F = i\sqrt{2}\,(\bar{\xi}_2\,\bar{\sigma}^\mu)^\alpha\,\partial_\mu(\delta_1\psi)_\alpha = -\,2\,(\bar{\xi}_2\,\bar{\sigma}^\mu\,\sigma^\nu\,\bar{\xi}_1)\,\partial_\mu\partial_\nu X + 2\,i\,(\bar{\xi}_2\,\bar{\sigma}^\mu\,\xi_1)\,\partial_\mu F$$

Because of the two partial derivatives, we project onto the part symmetric in $\mu\nu$. We can use the Clifford algebra relation

$$\bar{\sigma}^\mu\,\sigma^\nu + \bar{\sigma}^\nu\,\sigma^\mu = -\,2\,\eta^{\mu\nu}$$

$$\delta_1\delta_2 F = i\sqrt{2}\,(\bar{\xi}_2\,\bar{\sigma}^\mu)^\alpha\,\partial_\mu(\delta_1\psi)_\alpha = 2\,(\bar{\xi}_2\,\bar{\xi}_1)\,\partial^\mu\partial_\mu X + 2\,i\,(\bar{\xi}_2\,\bar{\sigma}^\mu\,\xi_1)\,\partial_\mu F$$

The term with X drops away taking the commutator:

$$\delta_1\delta_2 F - \delta_2\delta_1 F = -\,2\,i\,(\xi_1\,\sigma^\mu\,\bar{\xi}_2 - \xi_2\,\sigma^\mu\,\bar{\xi}_1)\,\partial_\mu F$$

Off-shell closure of the algebra

$$\delta X = \sqrt{2}\,\xi\,\psi \quad , \quad \delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\,\partial_\mu X + \sqrt{2}\,F\,\xi_\alpha \quad , \quad \delta F = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi$$

SUSY commutator on the fermion:

$$\delta_1\delta_2\psi_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi}_2)_\alpha\,\partial_\mu(\delta_1 X) + \sqrt{2}\,\xi_{2\alpha}\,\delta_1 F$$

$$= 2\,i\,(\xi_1\,\partial_\mu\psi)\,(\sigma^\mu\,\bar{\xi}_2)_\alpha + 2\,i\,(\bar{\xi}_1\,\bar{\sigma}^\mu\,\partial_\mu\psi)\,\xi_{2\alpha}$$

To proceed, we use the Fierz identity

$$(\chi_1\,\chi_2)\,\chi_{3\alpha} + (\chi_1\,\chi_3)\,\chi_{2\alpha} + (\chi_2\,\chi_3)\,\chi_{1\alpha} = 0 \quad \text{with} \quad \chi_1 = \xi_1,\ \ \chi_2 = \partial_\mu\psi,\ \ \chi_3 = \sigma^\mu\,\bar{\xi}_2$$

which reads

$$(\xi_1\,\partial_\mu\psi)\,(\sigma^\mu\,\bar{\xi}_2)_\alpha + (\xi_1\,\sigma^\mu\,\bar{\xi}_2)\,\partial_\mu\psi_\alpha + (\partial_\mu\psi\,\sigma^\mu\,\bar{\xi}_2)\,\xi_{1\alpha} = 0$$

The terms from δF cancel against the “bad” terms and we find

$$\delta_1\delta_2\psi_\alpha - \delta_2\delta_1\psi_\alpha = -\,2\,i\,(\xi_1\,\sigma^\mu\,\bar{\xi}_2 - \xi_2\,\sigma^\mu\,\bar{\xi}_1)\,\partial_\mu\psi_\alpha$$

SUSY kinetic terms

The kinetic terms for the dynamical fields and quadratic for the auxiliary field are collected in

$$\mathscr{L}_{\text{kin}} = -\,\partial^\mu \overline{X}\,\partial_\mu X + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi + \overline{F}\,F$$

This Lagrangian varies into a total spacetime derivative under a SUSY variation. Let us perform this check. NB:

- We always assume that fields fall off at infinity sufficiently rapidly that we can discard any spacetime total derivative in the Lagrangian
- Even though we want to check rigid SUSY (the parameters $\xi, \overline{\xi}$ are constant spinors), it is useful to compute the variation under general spacetime-dependent $\xi, \overline{\xi}$ (this will help us find the supercurrent later)

Intermezzo: complex conj. of the SUSY variations

The SUSY variations are

$$\delta X = \sqrt{2}\,\xi\,\psi \quad , \qquad \delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\,\partial_\mu X + \sqrt{2}\,F\,\xi_\alpha \quad , \qquad \delta F = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi$$

Let us compute their complex conjugates. We need:

$$(\xi\,\psi)^* = (\xi^\alpha\,\psi_\alpha)^* = (\psi_\alpha)^*\,(\xi^\alpha)^* = \bar{\psi}_{\dot\alpha}\,\bar{\xi}^{\dot\alpha} = \bar{\psi}\,\bar{\xi} = \bar{\xi}\,\bar{\psi}$$

and also

$$[(\sigma^\mu)_{\alpha\dot\beta}\,\bar{\xi}^{\dot\beta}]^* = (\bar{\xi}^{\dot\beta})^*\,[(\sigma^\mu)_{\alpha\dot\beta}]^* = \xi^\beta\,(\sigma^\mu)_{\beta\dot\alpha} = (\xi\,\sigma^\mu)_{\dot\alpha}$$

$$[\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi]^* = [\bar{\xi}_{\dot\alpha}\,(\bar{\sigma}^\mu)^{\dot\alpha\beta}\,\partial_\mu\psi_\beta]^* = \partial_\mu\bar{\psi}_{\dot\beta}\,[(\bar{\sigma}^\mu)^{\dot\alpha\beta}]^*\,\xi_\alpha = \partial_\mu\bar{\psi}_{\dot\beta}\,(\bar{\sigma}^\mu)^{\dot\beta\alpha}\,\xi_\alpha = \partial_\mu\bar{\psi}\,\bar{\sigma}^\mu\,\xi$$

The complex conjugate variations are

$$\delta\bar{X} = \sqrt{2}\,\bar{\xi}\,\bar{\psi} \quad , \qquad \delta\bar{\psi}_\alpha = -\,i\sqrt{2}\,(\xi\,\sigma^\mu)_{\dot\alpha}\,\partial_\mu\bar{X} + \sqrt{2}\,\bar{F}\,\bar{\xi}_{\dot\alpha} \quad , \qquad \delta\bar{F} = -\,i\sqrt{2}\,\partial_\mu\bar{\psi}\,\bar{\sigma}^\mu\,\xi$$

Check of SUSY for kinetic terms

$$\delta X = \sqrt{2}\, \xi\, \psi \quad , \ \delta\psi_\alpha = i\sqrt{2}\, (\sigma^\mu\, \overline{\xi})_\alpha\, \partial_\mu X + \sqrt{2}\, F\, \xi_\alpha \quad , \quad \delta F = i\sqrt{2}\, \overline{\xi}\, \overline{\sigma}^\mu\, \partial_\mu \psi$$

$$\delta \overline{X} = \sqrt{2}\, \overline{\xi}\, \overline{\psi} \ , \quad \delta\overline{\psi}_\alpha = -\, i\sqrt{2}\, (\xi\, \sigma^\mu)_{\dot\alpha}\, \partial_\mu \overline{X} + \sqrt{2}\, \overline{F}\, \overline{\xi}_{\dot\alpha} \quad , \quad \delta \overline{F} = -\, i\sqrt{2}\, \partial_\mu \overline{\psi}\, \overline{\sigma}^\mu\, \xi$$

We start with the kinetic term for X:

$$\delta(-\partial^\mu \overline{X}\, \partial_\mu X) = -\, \partial^\mu\, \delta\overline{X}\, \partial_\mu X - \partial^\mu \overline{X}\, \partial_\mu \delta X = \delta\overline{X}\, \partial^\mu \partial_\mu X + \delta X\, \partial^\mu \partial_\mu \overline{X} + \partial_\mu(\ldots)$$

$$= \sqrt{2}\, (\overline{\xi}\, \overline{\psi})\, \partial^\mu \partial_\mu X + \sqrt{2}\, (\xi\, \psi)\, \partial^\mu \partial_\mu \overline{X} + \partial_\mu(\ldots)$$

Check of SUSY for kinetic terms

$$\delta X = \sqrt{2}\,\xi\,\psi \quad , \;\; \delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\,\partial_\mu X + \sqrt{2}\,F\,\xi_\alpha \quad , \quad \delta F = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi$$

$$\delta\bar{X} = \sqrt{2}\,\bar{\xi}\,\bar{\psi} \;\; , \quad \delta\bar{\psi}_\alpha = -\,i\sqrt{2}\,(\xi\,\sigma^\mu)_{\dot{\alpha}}\,\partial_\mu\bar{X} + \sqrt{2}\,\bar{F}\,\bar{\xi}_{\dot{\alpha}} \quad , \quad \delta\bar{F} = -\,i\sqrt{2}\,\partial_\mu\bar{\psi}\,\bar{\sigma}^\mu\,\xi$$

Next the quadratic term for the auxiliary field:

$$\delta(\bar{F}\,F) = \bar{F}\,\delta F + F\,\delta\bar{F} = i\sqrt{2}\,\bar{F}\,(\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi) - i\sqrt{2}\,F\,(\partial_\mu\bar{\psi}\,\bar{\sigma}^\mu\,\xi)$$

Check of SUSY for kinetic terms

$$\delta X = \sqrt{2}\,\xi\,\psi \quad , \quad \delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\,\partial_\mu X + \sqrt{2}\,F\,\xi_\alpha \quad , \quad \delta F = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi$$

$$\delta\bar{X} = \sqrt{2}\,\bar{\xi}\,\bar{\psi} \quad , \quad \delta\bar{\psi}_\alpha = -\,i\sqrt{2}\,(\xi\,\sigma^\mu)_{\dot{\alpha}}\,\partial_\mu\bar{X} + \sqrt{2}\,\bar{F}\,\bar{\xi}_{\dot{\alpha}} \quad , \quad \delta\bar{F} = -\,i\sqrt{2}\,\partial_\mu\bar{\psi}\,\bar{\sigma}^\mu\,\xi$$

The fermion kinetic term:

$$\delta(i\,\partial_\mu\bar{\psi}\,\bar{\sigma}^\mu\,\psi) = i\,\partial_\mu\delta\bar{\psi}\,\bar{\sigma}^\mu\,\psi + i\,\partial_\mu\bar{\psi}\,\bar{\sigma}^\mu\,\delta\psi = -\,i\,\delta\bar{\psi}\,\bar{\sigma}^\mu\,\partial_\mu\psi + i\,\partial_\mu\bar{\psi}\,\bar{\sigma}^\mu\,\delta\psi + \partial_\mu(\ldots)$$

We look at the two terms in turn:

$$-i\,\delta\bar{\psi}\,\bar{\sigma}^\mu\,\partial_\mu\psi = -\sqrt{2}\,(\xi\,\sigma^\nu\,\bar{\sigma}^\mu\,\partial_\mu\psi)\,\partial_\nu\bar{X} - i\sqrt{2}\,\bar{F}\,(\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi)$$

$$= \sqrt{2}\,(\xi\,\sigma^\nu\,\bar{\sigma}^\mu\,\psi)\,\partial_\mu\partial_\nu\bar{X} + \sqrt{2}\,(\partial_\mu\xi\,\sigma^\nu\,\bar{\sigma}^\mu\,\psi)\,\partial_\nu\bar{X} - i\sqrt{2}\,\bar{F}\,(\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi) + \partial_\mu(\ldots)$$

In the first term we project onto the part symmetric in $\mu\nu$ so we can use $\sigma^\mu\,\bar{\sigma}^\nu + \sigma^\nu\,\bar{\sigma}^\mu = -\,2\,\eta^{\mu\nu}$

$$-i\,\delta\bar{\psi}\,\bar{\sigma}^\mu\,\partial_\mu\psi = -\sqrt{2}\,(\xi\,\psi)\,\partial^\mu\partial_\mu\bar{X} + \sqrt{2}\,(\partial_\mu\xi\,\sigma^\nu\,\bar{\sigma}^\mu\,\psi)\,\partial_\nu\bar{X} - i\sqrt{2}\,\bar{F}\,(\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi) + \partial_\mu(\ldots)$$

The term $i\,\partial_\mu\bar{\psi}\,\bar{\sigma}^\mu\,\delta\psi$ is treated in a similar way

Check of SUSY for kinetic terms

In conclusion:

$$\begin{aligned}\delta\mathscr{L}_{\text{kin}} = &\sqrt{2}\,(\bar{\xi}\,\overline{\psi})\,\partial^\mu\partial_\mu X + \sqrt{2}\,(\xi\,\psi)\,\partial^\mu\partial_\mu\overline{X} \\ &+ i\sqrt{2}\,\overline{F}\,(\bar{\xi}\,\overline{\sigma}^\mu\,\partial_\mu\psi) - i\sqrt{2}\,F\,(\partial_\mu\overline{\psi}\,\overline{\sigma}^\mu\,\xi) \\ &- \sqrt{2}\,(\xi\,\psi)\,\partial^\mu\partial_\mu\overline{X} + \sqrt{2}\,(\partial_\mu\xi\,\sigma^\nu\,\overline{\sigma}^\mu\,\psi)\,\partial_\nu\overline{X} - i\sqrt{2}\,\overline{F}\,(\bar{\xi}\,\overline{\sigma}^\mu\,\partial_\mu\psi) \\ &- \sqrt{2}\,(\overline{\psi}\,\bar{\xi})\,\partial^\mu\partial_\mu X + \sqrt{2}\,(\overline{\psi}\,\overline{\sigma}^\mu\,\sigma^\nu\,\partial_\mu\bar{\xi})\,\partial_\nu X + i\sqrt{2}\,F\,(\partial_\mu\overline{\psi}\,\overline{\sigma}^\mu\,\xi) + \partial_\mu(\ldots)\end{aligned}$$

All terms without derivatives of the SUSY params drop away: we have checked rigid SUSY!

If we allow the SUSY params to be spacetime-dependent,

$$\delta S_{\text{kin}} = \int d^4x\,\left[\sqrt{2}\,(\partial_\mu\xi\,\sigma^\nu\,\overline{\sigma}^\mu\,\psi)\,\partial_\nu\overline{X} + \sqrt{2}\,(\partial_\mu\bar{\xi}\,\overline{\sigma}^\nu\,\sigma^\mu\,\overline{\psi})\,\partial_\nu X\right]$$

SUSY current

We re-write the variation of the action as

$$\delta S_{\text{kin}} = \int d^4x \left[\partial_\mu \xi^\alpha \, J^\mu_\alpha + \partial_\mu \bar{\xi}_{\dot{\alpha}} \, \bar{J}^{\mu\dot{\alpha}} \right]$$

$$J^\mu_\alpha = \sqrt{2}\, (\sigma^\nu \, \bar{\sigma}^\mu \, \psi)_\alpha \, \partial_\nu \bar{X} \ , \qquad \bar{J}^{\mu\dot{\alpha}} = \sqrt{2}\, (\bar{\sigma}^\nu \, \sigma^\mu \, \bar{\psi})^{\dot{\alpha}} \, \partial_\nu X$$

Suppose we go on-shell. The action must be stationary under <u>arbitrary</u> variations of all fields, $\delta S_{\text{kin}} = 0$. We conclude that the quantities J^μ_α, $\bar{J}^{\mu\dot{\alpha}}$ are **conserved on-shell**

$$\partial_\mu J^\mu_\alpha = 0 \ , \qquad \partial_\mu \bar{J}^{\mu\dot{\alpha}} = 0$$

This is Noether's theorem for SUSY. The SUSY current (aka supercurrent) is a vector-spinor

Supercharges from the SUSY current

- A conserved current yields conserved charges by integrating over a spatial slice

$$Q_\alpha \propto \int_{t=0} d^3x\, J_\alpha^{\mu=0} \quad , \qquad \overline{Q}_{\dot\alpha} \propto \int_{t=0} d^3x\, \bar{J}_{\dot\alpha}^{\mu=0}$$

- The operators Q_α, $\overline{Q}_\alpha$ satisfy $\frac{d}{dt}Q_\alpha = 0$, $\frac{d}{dt}\overline{Q}_{\dot\alpha} = 0$
- They are indeed the generators of SUSY variations, in the sense that

$$[\xi^\alpha Q_\alpha + \bar{\xi}_{\dot\alpha}\,\overline{Q}^{\dot\alpha}, \Phi] = \delta_{\xi,\bar{\xi}}\Phi \qquad \text{for any field } \Phi \quad \text{(on-shell)}$$

- In a given theory, the LHS can be computed using the expression of the supercurrent and the equal-time canonical (anti)commutators of the fundamental dynamical fields in the Lagrangian

Supersymmetry and supergravity

Lecture 9

Wess-Zumino model with a single chiral

$$\delta X = \sqrt{2}\,\xi\,\psi \quad , \quad \delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\,\partial_\mu X + \sqrt{2}\,F\,\xi_\alpha \quad , \quad \delta F = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi$$

This is the simplest interacting model. The total Lagrangian is the sum of

$$\mathscr{L}_{\rm kin} = -\,\partial^\mu \bar{X}\,\partial_\mu X + i\,\partial_\mu\bar{\psi}\,\bar{\sigma}^\mu\,\psi + \bar{F}\,F$$

$$\mathscr{L}_m = m\,X\,F - \frac{1}{2}\,m\,\psi\,\psi + \bar{m}\,\bar{X}\,\bar{F} - \frac{1}{2}\,\bar{m}\,\bar{\psi}\,\bar{\psi}$$

$$\mathscr{L}_g = g\,X^2\,F - g\,X\,(\psi\,\psi) + \bar{g}\,\bar{X}^2\,\bar{F} - \bar{g}\,\bar{X}\,(\bar{\psi}\,\bar{\psi})$$

The parameters m and g are arbitrary complex constants. $\mathscr{L}_{\rm kin}$, $\mathscr{L}_m$, $\mathscr{L}_g$ are separately invariant under SUSY variations up to tot. der. We will see the derivation of this fact for $\mathscr{L}_m$, $\mathscr{L}_g$ later

Wess-Zumino model with a single chiral

$$\mathscr{L}_{\text{kin}} = -\,\partial^\mu \overline{X}\,\partial_\mu X + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi + \overline{F}\,F$$

$$\mathscr{L}_m = m\,X\,F - \frac{1}{2}\,m\,\psi\,\psi + \overline{m}\,\overline{X}\,\overline{F} - \frac{1}{2}\,\overline{m}\,\overline{\psi}\,\overline{\psi}$$

$$\mathscr{L}_g = g\,X^2\,F - g\,X\,(\psi\,\psi) + \overline{g}\,\overline{X}^2\,\overline{F} - \overline{g}\,\overline{X}\,(\overline{\psi}\,\overline{\psi})$$

The auxiliary field F enter the action:

- algebraically (no derivatives)
- quadratically

This means that we can integrate out the field F exactly using its equation of motion:

- in the classical theory: we can derive all the EOMs, and plug the EOM for F into the other EOMs; or equivalently compute the EOM for F, solve it for F, $\overline{F}$, ang plug them back in the action
- in the quantum theory: the path integral over F, $\overline{F}$ is Gaussian and is computed exactly

Eliminating the auxiliary field

$$\mathscr{L}_{\text{kin}} = -\,\partial^\mu \overline{X}\,\partial_\mu X + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi + \overline{F}\,F$$

$$\mathscr{L}_m = m\,X\,F - \frac{1}{2}\,m\,\psi\,\psi + \overline{m}\,\overline{X}\,\overline{F} - \frac{1}{2}\,\overline{m}\,\overline{\psi}\,\overline{\psi}$$

$$\mathscr{L}_g = g\,X^2\,F - g\,X\,(\psi\,\psi) + \overline{g}\,\overline{X}^2\,\overline{F} - \overline{g}\,\overline{X}\,(\overline{\psi}\,\overline{\psi})$$

Variation wrt $\overline{F}$, F gives respectively

$$F + \overline{m}\,\overline{X} + \overline{g}\,\overline{X}^2 = 0 \quad , \qquad \overline{F} + m\,X + g\,X^2 = 0$$

Plugging these back in the Lagrangian we find

$$\begin{aligned}\mathscr{L} = &-\,\partial^\mu \overline{X}\,\partial_\mu X + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi \\ &-|m|^2\,\overline{X}\,X - \frac{1}{2}\,m\,\psi\,\psi - \frac{1}{2}\,\overline{m}\,\overline{\psi}\,\overline{\psi} \\ &-g\,X\,\psi\,\psi - \overline{g}\,\overline{X}\,\overline{\psi}\,\overline{\psi} - m\,\overline{g}\,X\,\overline{X}^2 - \overline{m}\,g\,\overline{X}\,X^2 - |g|^2\,|X^2|^2\end{aligned}$$

Action for dynamical fields

$$\mathscr{L} = -\,\partial^\mu \overline{X}\,\partial_\mu X + i\,\partial_\mu \overline{\psi}\,\overline{\sigma}^\mu\,\psi$$
$$-\,|m|^2\,\overline{X}\,X - \frac{1}{2}\,m\,\psi\,\psi - \frac{1}{2}\,\overline{m}\,\overline{\psi}\,\overline{\psi}$$
$$-\,g\,X\psi\,\psi - \overline{g}\,\overline{X}\,\overline{\psi}\,\overline{\psi} - m\,\overline{g}\,X\,\overline{X}^2 - \overline{m}\,g\,\overline{X}\,X^2 - |g|^2\,|X^2|^2$$

Remarks:

- This is a renormalizable model with mass terms, Yukawa interactions, and a cubic and quartic scalar couplings
- SUSY dictates that different couplings (eg Yukawa and $|X^2|^2$) are related in a prescribed way, because they are determined by the same parameter
- The total scalar potential of the model can be written as

$$V(X,\overline{X}) = \overline{F}(X)\,F(\overline{X}) \quad , \qquad F + \overline{m}\,\overline{X} + \overline{g}\,\overline{X}^2 = 0 \quad , \qquad \overline{F} + m\,X + g\,X^2 = 0$$

Scalar potential and vacua

$$V(X,\overline{X}) = \overline{F}(X)\,F(\overline{X}) \quad , \qquad F + \overline{m}\,\overline{X} + \overline{g}\,\overline{X}^2 = 0 \quad , \qquad \overline{F} + m\,X + g\,X^2 = 0$$

- The scalar potential is non-negative: this is a consequence of the positivity of energy in SUSY theories (H is a sum of squares of Qs)
- To preserve Lorentz symmetry in the vacuum, the VEV of the scalar X must be constant. The scalar EOM dictates that the value of X must be a critical point of V, i.e. $\partial_X V = 0 = \partial_{\overline{X}} V$:

$$(m + 2\,g\,X)\,(\overline{m}\,\overline{X} + \overline{g}\,\overline{X}^2) = 0 \qquad\qquad (m\,X + g\,X^2)\,(\overline{m} + 2\,\overline{g}\,\overline{X}) = 0$$

- The stationary points are $X = -\,2\,g\,m^{-1}$, $X = 0$, $X = -\,m\,g^{-1}$
- The point $X = -\,2\,g\,m^{-1}$ is a saddle point, while $X = 0$, $X = -\,m\,g^{-1}$ are absolute minima of V. Indeed, these points are the solutions to $\overline{F}(X) = 0$ and

$$F = 0 \qquad \Rightarrow \qquad \nabla V = 0 = V$$

SUSY is unbroken in the vacuum

We know from the SUSY algebra that the vacuum energy can be regarded as an order parameter for SUSY. Since our vacua satisfy $V = 0$, SUSY is unbroken. We can check it explicitly by looking at the SUSY variations:

- $\delta X = \sqrt{2}\,\xi\,\psi = 0$ and $\delta F = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^{\mu}\,\partial_{\mu}\psi = 0$

  because the VEV of ψ is zero

- $\delta\psi_{\alpha} = i\sqrt{2}\,(\sigma^{\mu}\,\bar{\xi})_{\alpha}\,\partial_{\mu}X + \sqrt{2}\,F\,\xi_{\alpha} = 0$

  because the VEV of X is a constant such that $F = 0$

Bosons and fermions have the same mass

Let us expand around the vacuum at $X = 0$. To find the propagators of the fields we only need the quadratic terms in the Lagrangian:

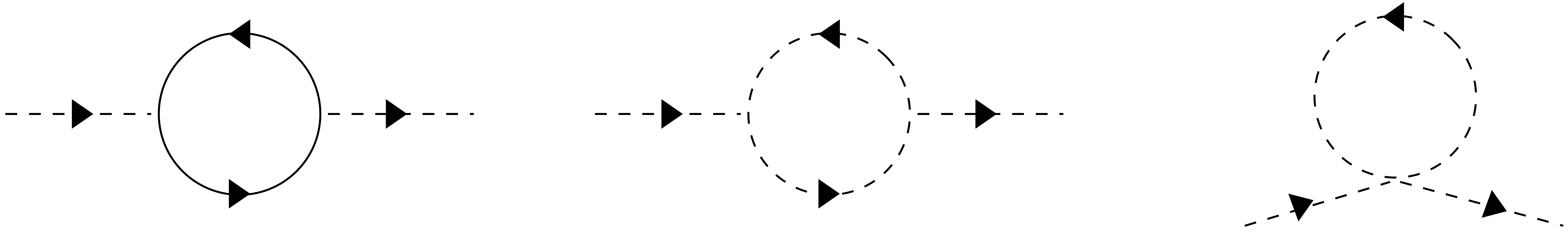
$$\mathscr{L} = -\partial^\mu \overline{X}\, \partial_\mu X + i\, \partial_\mu \overline{\psi}\, \overline{\sigma}^\mu\, \psi$$
$$- |m|^2\, \overline{X}\, X - \frac{1}{2}\, m\, \psi\, \psi - \frac{1}{2}\, \overline{m}\, \overline{\psi}\, \overline{\psi}$$

The physical mass of both the boson and the fermion is $|m|$, as required by unbroken SUSY.

What about the vacuum $X = -m\, g^{-1}$? Expanding around it, one finds again that both the boson and the fermion have a physical mass $|m|$.

The model has cancellations at 1-loop

Let us consider 1-loop corrections to the mass of quanta of X (expanded around the vacuum $X = 0$). Three kinds of contributions:



Each individual diagram has quadratic and log divergences

- To cancel quadratic divergences we need a relation of the form (quartic coupl.) $\sim |\text{Yukawa}|^2$
- To cancel log divergences we need $|\text{cubic coupl.}|^2 \sim |m|^2$ (quartic coupl.)

Both these conditions are guaranteed by SUSY!

$$\mathscr{L} \supset -g X \psi \psi - \bar{g} \bar{X} \bar{\psi} \bar{\psi} - m \bar{g} X \bar{X}^2 - \bar{m} g \bar{X} X^2 - |g|^2 |X^2|^2$$

R-symmetry?

The SUSY variations of the fields in a chiral multiplet are compatible with assigning definite $U(1)_R$ charges to X, ψ, F, and the SUSY parameter ξ

$$\delta X = \sqrt{2}\,\xi\,\psi \quad , \quad \delta\psi_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\,\partial_\mu X + \sqrt{2}\,F\,\xi_\alpha \quad , \quad \delta F = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi$$

field/param	X	ψ	F	ξ
charge	R_X	$R_X - 1$	$R_X - 2$	1

Barred fields/params have opposite charges

If this is a symmetry, it is an R-symmetry, because the SUSY parameter has a non-zero charge (and therefore the Qs are charged, too)

Is there any value of R_X for which this $U(1)_R$ action is a symmetry?

- For generic m, g: no. For instance $g\,X\,\psi\,\psi$ requires $R_X = 2/3$, but then $m\,\bar{g}\,X\,\bar{X}^2$ is not invariant
- For $g = 0$ we can take $R_X = 1$. For $m = 0$ we can take $R_X = 2/3$

Supersymmetry and supergravity

Lecture 10

Interacting models with chiral multiplets

We consider a collection of chiral multiplets $(X^i, \psi^i_\alpha, F^i)$ with label i. Their SUSY variations are the usual expressions

$$\delta X^i = \sqrt{2}\,\xi\,\psi^i \quad , \;\; \delta\psi^i_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\,\partial_\mu X^i + \sqrt{2}\,F^i\,\xi_\alpha \;\; , \;\; \delta F^i = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi^i$$

The conjugate fields carry a lower label i.

We want to study general interacting SUSY models. For the time being, we do not modify the kinetic terms, keeping them canonical:

$$\mathscr{L}_{\rm kin} = -\,\partial^\mu \overline{X}_i\,\partial_\mu X^i + i\,\partial_\mu\overline{\psi}_i\,\bar{\sigma}^\mu\,\psi^i + \overline{F}_i\,F^i$$

This is a sum over i of kinetic terms that are SUSY invariant (we have checked it explicitly earlier)

The superpotential

Fact of life: Consider any **holomorphic** function $W = W(X^i)$, known as the **superpotential**. The following Lagrangian is SUSY invariant

$$\mathscr{L}_W = F^i W_i(X) - \frac{1}{2} W_{ij}(X)\, \psi^i \psi^j + \overline{F}_i\, \overline{W}^i(\overline{X}) - \frac{1}{2} \overline{W}^{ij}(\overline{X})\, \overline{\psi}_i \overline{\psi}_j$$

where we have introduced the notation

$$W_i = \frac{\partial W}{\partial X^i}\,, \qquad W_{ij} = \frac{\partial^2 W}{\partial X^i\, \partial X^j}\,, \qquad \overline{W}^i = \frac{\partial \overline{W}}{\partial \overline{X}_i}\,, \qquad \overline{W}^{ij} = \frac{\partial^2 \overline{W}}{\partial \overline{X}_i\, \partial \overline{X}_j}$$

and $\overline{W}(\overline{X}_i)$ is the complex conjugate of $W(X^i)$ (hence antiholomorphic)

Example: the simplest Wess-Zumino model is recovered with

$$W(X) = \frac{1}{2} m X^2 + \frac{1}{3} g X^2$$

Some SUSY checks

- One can check by a brute-force computation that

$$\mathscr{L}_W = F^i\, W_i(X) - \frac{1}{2}\, W_{ij}(X)\, \psi^i\, \psi^j + \overline{F}_i\, \overline{W}^i(\overline{X}) - \frac{1}{2}\, \overline{W}^{ij}(\overline{X})\, \overline{\psi}_i\, \overline{\psi}_j$$

  is SUSY invariant. We will see a more clever approach using superspace.
- We do not perform the full SUSY check, but only highlight some parts of the computation

Some SUSY checks

- Let's be agnostic and parametrize the wanna-be SUSY interaction Lagrangian as
$$\mathscr{L} = F^i f_i(X,\overline{X}) - \frac{1}{2} f_{ij}(X,\overline{X})\,\psi^i \psi^j + U(X,\overline{X}) + \mathrm{h.c.}$$
for some unknown (not necessarily holomorphic) functions $f_i,\ f_{ij},\ U$. f_{ij} is symmetric. This Ansatz is general enough to cover all renormalizable interactions (mass dim of ≤ 4)
- The SUSY variation of the U term is
$$\sqrt{2}\,\frac{\partial U}{\partial X^i}\,\xi\,\psi^i + \sqrt{2}\,\frac{\partial U}{\partial \overline{X}_i}\,\bar{\xi}\,\overline{\psi}_i$$
- It is linear in the fermions and has no F: it cannot be canceled by any other term. We must set $U \equiv 0$

Some SUSY checks

$$\mathscr{L} = F^i f_i(X, \overline{X}) - \tfrac{1}{2} f_{ij}(X, \overline{X})\, \psi^i \psi^j + \mathrm{h.c.}$$

- Let us now collect all the 4-Fermi terms that originate from the SUSY variation. They can only come from the $f\,\psi\,\psi$ terms

$$\delta\mathscr{L} \supset -\frac{1}{2}\frac{\partial f_{ij}}{\partial X^k}\sqrt{2}\,(\xi\,\psi^k)\,(\psi^i\,\psi^j) - \frac{1}{2}\frac{\partial f_{ij}}{\partial \overline{X}_k}\sqrt{2}\,(\bar{\xi}\,\overline{\psi}_k)\,(\psi^i\,\psi^j) + \mathrm{h.c.}$$

- For the terms $(\xi\,\psi^k)\,(\psi^i\,\psi^j)$ we can use a Fierz identity $(\xi\,\psi^k)\,(\psi^i\,\psi^j)$ + cyclic $= 0$ so $\dfrac{\partial f_{ij}}{\partial X^k}$ can be non-zero, provided it is totally symmetric in ijk. We don't have a similar mechanism for $(\bar{\xi}\,\overline{\psi}_k)\,(\psi^i\,\psi^j)$, so $\dfrac{\partial f_{ij}}{\partial \overline{X}_k} = 0$
- We write f_{ij} with this property as a the second derivative of a holomorphic function

$$f_{ij} = \frac{\partial^2 W}{\partial X^i\,\partial X^j} \quad , \quad W = W(X^i)$$

Some SUSY checks

$$\delta X^i = \sqrt{2}\,\xi\,\psi^i \quad , \ \delta\psi^i_\alpha = i\sqrt{2}\,(\sigma^\mu\,\bar{\xi})_\alpha\,\partial_\mu X^i + \sqrt{2}\,F^i\,\xi_\alpha \ , \ \delta F^i = i\sqrt{2}\,\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi^i$$

$$\mathcal{L} = F^i f_i(X,\bar{X}) - \frac{1}{2}\,f_{ij}(X,\bar{X})\,\psi^i\,\psi^j + \mathrm{h.c.}$$

- Let us now consider terms with one derivative:

$$\delta\mathcal{L} \supset i\sqrt{2}\,(\bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\psi^i)f_i - i\sqrt{2}\,f_{ij}\,(\psi^i\,\sigma^\mu\,\bar{\xi})\,\partial_\mu X^j + \mathrm{h.c.}$$

- They must recollect into a total derivative, which requires

$$f_{ij}\,\partial_\mu X^j = \frac{\partial f_i}{\partial X^j}\,\partial_\mu X^j + \frac{\partial f_i}{\partial \bar{X}_j}\,\partial_\mu \bar{X}_j \qquad \text{hence} \quad f_{ij} = \frac{\partial f_i}{\partial X^j}\,, \qquad \frac{\partial f_i}{\partial \bar{X}_j} = 0$$

- Using the fact that $f_{ij} = \partial_i\partial_j W$ we learn $\partial_i\partial_j W = \partial_j f_i$, which gives us $f_i = \partial_i W$

SUSY current

The terms $\mathscr{L}_{\text{kin}}$ and $\mathscr{L}_W$ both contribute to the conserved SUSY current of the model. It can be derived with the trick of spacetime-dependent SUSY parameters

$$J^\mu_\alpha = \sqrt{2}\,(\sigma^\nu\,\overline{\sigma}^\mu\,\psi^i)_\alpha\,\partial_\nu \overline{X}_i - i\sqrt{2}\,\overline{W}^i(\overline{X})\,(\sigma^\mu\,\overline{\psi}_i)_\alpha$$

$$\overline{J}^{\mu\dot{\alpha}} = \sqrt{2}\,(\overline{\sigma}^\nu\,\sigma^\mu\,\overline{\psi}_i)^{\dot{\alpha}}\,\partial_\nu X^i - i\sqrt{2}\,W_i(X)\,(\overline{\sigma}^\mu\,\psi^i)^{\dot{\alpha}}$$

The SUSY currents are conserved on-shell.

Integrating out the auxiliary fields

$$\mathscr{L}_{\rm kin} = -\,\partial^\mu \overline{X}_i\,\partial_\mu X^i + i\,\partial_\mu \overline{\psi}_i\,\overline{\sigma}^\mu\,\psi^i + \overline{F}_i\,F^i$$

$$\mathscr{L}_W = F^i\,W_i(X) - \frac{1}{2}\,W_{ij}(X)\,\psi^i\,\psi^j + \overline{F}_i\,\overline{W}^i(\overline{X}) - \frac{1}{2}\,\overline{W}^{ij}(\overline{X})\,\overline{\psi}_i\,\overline{\psi}_j$$

The EOMs of the auxiliary fields are

$$F^i = -\,\overline{W}^i(\overline{X})\ \ ,\qquad \overline{F}_i = -\,W_i(X)$$

Plugging these relations back in the Lagrangian we find

$$\mathscr{L} = -\,\partial^\mu \overline{X}_i\,\partial_\mu X^i + i\,\partial_\mu \overline{\psi}_i\,\overline{\sigma}^\mu\,\psi^i - \frac{1}{2}\,W_{ij}(X)\,\psi^i\,\psi^j - \frac{1}{2}\,\overline{W}^{ij}(\overline{X})\,\overline{\psi}_i\,\overline{\psi}_j - V$$

where the scalar potential is

$$V(X,\overline{X}) = F^i(\overline{X})\,\overline{F}_i(X)$$

It is non-negative, as expected from SUSY on general grounds.

Renormalizable models

Supersymmetry of $\mathscr{L}_W$ holds for any holomorphic function $W(X^i)$. If we want a renormalizable theory, however, $W(X^i)$ must be a polynomial of degree at most 3. We see this from

$$\mathscr{L} = -\,\partial^\mu \overline{X}_i\,\partial_\mu X^i + i\,\partial_\mu \overline{\psi}_i\,\overline{\sigma}^\mu\,\psi^i - \frac{1}{2}\,W_{ij}(X)\,\psi^i\,\psi^j - \frac{1}{2}\,\overline{W}^{ij}(\overline{X})\,\overline{\psi}_i\,\overline{\psi}_j - V$$

$$V(X,\overline{X}) = F^i(\overline{X})\,\overline{F}_i(X)$$

The most general renormalizable model of this class has

$$W = E_i\,X^i + \frac{1}{2}\,m_{ij}\,X^i\,X^j + \frac{1}{3}\,g_{ijk}\,X^i\,X^j\,X^k$$

Note:

- The constants E_i, m_{ij}, g_{ijk} are complex; m_{ij}, g_{ijk} are totally symmetric
- Since the Lagrangian only depends on derivatives of W we do not need a constant term

SUSY vacua

- The scalar VEVs are constants and must be stationary points of the scalar potential

vacua: $$\partial_\mu X^i = 0 \ , \qquad \partial_{X^i} V = 0 = \partial_{\overline{X}^i} V$$

- If we want a SUSY vacuum, we need a stronger condition:

SUSY vacua: $$\partial_\mu X^i = 0 \ , \qquad F^i = 0$$

- This works because values of X^i for which $F^i = 0$ give automatically $\partial_{X^i} V = 0 = \partial_{\overline{X}^i} V$ and also $V = 0$. The condition $F^i = 0$ can also be seen from the SUSY variations

$$\delta X^i = \sqrt{2}\, \xi\, \psi^i \ , \ \delta\psi^i_\alpha = i\sqrt{2}\, (\sigma^\mu\, \bar{\xi})_\alpha\, \partial_\mu X^i + \sqrt{2}\, F^i\, \xi_\alpha \ , \ \delta F^i = i\sqrt{2}\, \bar{\xi}\, \bar{\sigma}^\mu\, \partial_\mu \psi^i$$

- SUSY vacua might not exist! It depends on the form of W. If they don't exist, we have spontaneous SUSY breaking

Mass matrices

Let's consider for simplicity $E_i = 0$, $W \sim mX^2 + gX^3$, so that $X = 0$ is a SUSY vacuum. The propagators for the fermions and the fluctuations of X^i around 0 are read off from

$$\mathscr{L} = -\,\partial^\mu \overline{X}_i\,\partial_\mu X^i + i\,\partial_\mu \overline{\psi}_i\,\overline{\sigma}^\mu\,\psi^i - \frac{1}{2}\,m_{ij}\,\psi^i\,\psi^j - \frac{1}{2}\,\overline{m}^{ij}\,\overline{\psi}_i\,\overline{\psi}_j - m_{ij}\,\overline{m}^{ik}\,X^j\,\overline{X}_k$$

The free EOMs are

$$\partial^\mu \partial_\mu X^i = -\,\overline{m}^{ij}\,m_{jk}\,X^k \quad , \quad i\,\overline{\sigma}^\mu\,\partial_\mu \psi^i = \overline{m}^{ij}\,\psi_j \;, \quad i\,\sigma^\mu\,\partial_\mu \overline{\psi}_i = m_{ij}\,\overline{\psi}^j$$

To compare bosons and fermions we hit the Dirac equations with an extra derivative, to obtain the Klein-Gordon equation,

$$\partial^\mu \partial_\mu \psi^i = -\,\overline{m}^{ij}\,m_{jk}\,\psi^k \quad , \quad \partial^\mu \partial_\mu \overline{\psi}_i = -\,\overline{\psi}_k\,\overline{m}^{kj}\,m_{ji}$$

We see that the same mass matrix $\mathscr{M}^i{}_k = \overline{m}^{ij}\,m_{jk}$ governs the mass eigenvalues of both the bosons and the fermions.

1-loop cancellations

There are cancellations in the 1-loop corrections to the $X_i \bar{X}^j$ 2-pt function, similar to the simplest Wess-Zumino model.

- Quadratic divergences: they are cancelled because of the interplay between

$$\mathscr{L} \supset -\, g_{ijk}\, \bar{g}^{ij'k'}\, X^j\, X^j\, \bar{X}_{j'}\, \bar{X}_{k'} \quad \text{and} \quad \mathscr{L} \supset -\, g_{ijk}\, X^i\, \psi^j\, \psi^k + \mathrm{h.c.}$$

- Log divergences: they are cancelled because of the interplay between

$$\mathscr{L} \supset -\, g_{ijk}\, \bar{g}^{ij'k'}\, X^j\, X^j\, \bar{X}_{j'}\, \bar{X}_{k'} \quad \text{and} \quad \mathscr{L} \supset -\, m_{ij}\, \bar{g}^{ik\ell}\, X^i\, \bar{X}_k\, \bar{X}_\ell + \mathrm{h.c.}$$

R-symmetry?

Recall the structure of a possible $U(1)_R$ symmetry of the model:

field/param	X^i	ψ^i_α	F^i	ξ
charge	$R[X^i]$	$R[X^i]-1$	$R[X^i]-2$	1

An equivalent statement is that we have $U(1)_R$ variations $\delta_R X^i = i\,R[X^i]\,\Lambda\,X^i$ etc.

The kinetic Lagrangian is always invariant, but

$$\mathscr{L}_W = F^i\,W_i(X) - \frac{1}{2}\,W_{ij}(X)\,\psi^i\,\psi^j + \overline{F}_i\,\overline{W}^i(\overline{X}) - \frac{1}{2}\,\overline{W}^{ij}(\overline{X})\,\overline{\psi}_i\,\overline{\psi}_j$$

may or may not be invariant. We have

$$\delta_R(F^i\,W_i) = i\,\Lambda\,\sum_i (R[X^i]-2)\,F^i\,W_i + i\,\Lambda\,\sum_{i,j} R[X^j]\,F^i\,W_{ij}\,X^j$$

This must vanish for any F^i, so we learn

$$R[X^i]\,W_i + \sum_j R[X^j]\,W_{ij}\,X^j = 2\,W_i \qquad \frac{\partial}{\partial X^i}\Big(\sum_j R[X^j]\,W_j\,X^j\Big) = 2\,\frac{\partial}{\partial X^i}W$$

R-symmetry?

$$R[X^i]\, W_i + \sum_j R[X^j]\, W_{ij}\, X^j = 2\, W_i \qquad \frac{\partial}{\partial X^i}\Big(\sum_j R[X^j]\, W_j\, X^j \Big) = 2\, \frac{\partial}{\partial X^i} W$$

Recall that we can always shift W by a constant. Without loss of generality we integrate the above relation as

$$\sum_j R[X^j]\, X^j\, \frac{\partial W}{\partial X^j} = 2\, W$$

This has a simple interpretation: the charges $R[X^i]$ must be chosen in such a way that all terms in W have the same charge, and that charge is 2:

$$R[W] = 2$$

One can verify that if this is satisfied, all terms in $\mathscr{L}_W$ are invariant. Note:

- Some models do not have any R-symmetry
- Some models have more that one choice of R-symmetry

Supersymmetry and supergravity

Lecture 11

Vector supermultiplet

- Remember the structure of the massless vector mutliplet of minimal SUSY

no $a^\dagger$	$\lambda = 1/2$	and the CPT conjugate	no $a^\dagger$	$\lambda = -1$
one $a^\dagger$	$\lambda = 1$		one $a^\dagger$	$\lambda = -1/2$

- A massless particle of helicity ± 1 is described by a gauge field A_μ subject to a gauge redundancy $A_\mu \to A_\mu + \partial_\mu \Lambda$
- 4 real components − 1 gauge redundancy = 3 off-shell real dof's
- The free EOM is removes one further dof:

$$\partial_\mu F^{\mu\nu} = 0 \qquad \text{where} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- The fermionic states are described by a Weyl spinor λ_α. It is usually called the **gaugino**

Off-shell and on-shell counting

	off-shell	on-shell
A_μ	3	2
λ_α	4	2
D	1	0

From this table we see the natural candidate for the auxiliary field D in a vector multiplet. It is a real scalar field. When we construct SUSY actions, we have to make sure that D does not describe any propagating dof's on-shell

SUSY variations in the off-shell formalism

- The SUSY variations for an off-shell vector multiplet can be found by trial-and-error. We know that they are linear in the SUSY params. We know the mass dimensions of fields and params:

$$[\xi] = -1/2 \ , \qquad [A_\mu] = 1 \, , \qquad [\lambda] = 3/2 \, , \qquad [D] = 2$$

- NB: we anticipate $[D] = 2$ because we want the auxiliary field to enter the free vector multiplet Lagrangian algebraically and quadratically
- We know that A_μ and D are real
- We also know that λ_α , $F_{\mu\nu}$, and D are gauge-invariant, while A_μ is not
- A more systematic way of deriving the SUSY variations if from superspace

SUSY variations in the off-shell formalism

At the end, this is the correct set of variations:

$$\delta A_\mu = i\,\bar\xi\,\bar\sigma_\mu\,\lambda - i\,\bar\lambda\,\bar\sigma_\mu\,\xi$$

$$\delta\lambda_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F_{\mu\nu} + i\,D\,\xi_\alpha$$

$$\delta D = \bar\xi\,\bar\sigma^\mu\,\partial_\mu\lambda + \partial_\mu\bar\lambda\,\bar\sigma^\mu\,\xi$$

Zeroth order sanity checks: reality; mass dimensions; gauge-invariance. Also: $\delta D = 0$ if we use the Dirac equation for the gaugino.

Recall the conjugation of flip identities

$$[\bar\chi_1\,\bar\sigma^\mu\,\chi_2]^* = +\,\bar\chi_2\,\bar\sigma^\mu\,\chi_1 \quad , \qquad \bar\chi_1\,\bar\sigma^\mu\,\chi_2 = -\,\chi_2\,\sigma^\mu\,\bar\chi_1$$

To verify that these variations are correct we have to check the off-shell closure of the SUSY algebra. First: let's compute the SUSY variation of $F_{\mu\nu}$

$$\delta(\partial_\mu A_\nu) = \partial_\mu\delta A_\nu = i\,\bar\xi\,\bar\sigma_\nu\,\partial_\mu\lambda - i\,\partial_\mu\bar\lambda\,\bar\sigma_\nu\,\xi$$

$$\delta F_{\mu\nu} = i\,\bar\xi\,\bar\sigma_\nu\partial_\mu\lambda - i\,\partial_\mu\bar\lambda\,\bar\sigma_\nu\,\xi - (\mu \leftrightarrow \nu)$$

Closure of the SUSY algebra

$$\delta A_\mu = i\,\bar\xi\,\bar\sigma_\mu\,\lambda - i\,\bar\lambda\,\bar\sigma_\mu\,\xi \ \ , \quad \delta\lambda_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F_{\mu\nu} + i\,D\,\xi_\alpha \ \ , \quad \delta D = \bar\xi\,\bar\sigma^\mu\,\partial_\mu\lambda + \partial_\mu\bar\lambda\,\bar\sigma^\mu\,\xi$$

We start with the auxiliary field:

$$\delta_1\delta_2 D = \bar\xi_2\,\bar\sigma^\mu\,\partial_\mu\delta_1\lambda + \mathrm{h.c.} = (\bar\xi_2\,\bar\sigma^\mu\,\sigma^{\rho\sigma}\,\xi_1)\,\partial_\mu F_{\rho\sigma} + i\,\partial_\mu D\,(\bar\xi_2\,\bar\sigma^\mu\,\xi_1) + \mathrm{h.c.}$$

The second term has the correct structure. In order to kill the first term, we need $\sigma^{\rho\sigma} = \frac{1}{4}(\sigma^\rho\,\bar\sigma^\sigma - \sigma^\sigma\,\bar\sigma^\rho)$ and the identity

$$\bar\sigma^\mu\,\sigma^\rho\,\bar\sigma^\sigma = \eta^{\mu\sigma}\,\bar\sigma^\rho - \eta^{\rho\sigma}\,\bar\sigma^\mu - \eta^{\mu\rho}\,\bar\sigma^\sigma - i\,\epsilon^{\mu\rho\sigma\tau}\,\bar\sigma_\tau$$

We have

$$(\bar\xi_2\,\bar\sigma^\mu\,\sigma^{\rho\sigma}\,\xi_1)\,\partial_\mu F_{\rho\sigma} = (\bar\xi_2\,\bar\sigma^\sigma\,\xi_1)\,\partial^\mu F_{\mu\sigma} - \frac{i}{2}\,\epsilon^{\mu\rho\sigma\tau}\,\partial_\mu F_{\rho\sigma}\,(\bar\xi_2\,\bar\sigma_\tau\,\xi_1)$$

The first term goes away when we sum h.c. and we subtract $(1 \leftrightarrow 2)$. The second term is zero thanks to the Bianchi identity for $F_{\mu\nu}$. At the end we verify the expected relation

$$\delta_1\delta_2 D - \delta_2\delta_1 D = -2\,i\,(\xi_1\,\sigma^\mu\,\bar\xi_2 - \xi_2\,\sigma^\mu\,\bar\xi_1)\,\partial_\mu D$$

Closure of the SUSY algebra

$$\delta A_\mu = i\,\bar\xi\,\bar\sigma_\mu\,\lambda - i\,\bar\lambda\,\bar\sigma_\mu\,\xi \ , \quad \delta\lambda_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F_{\mu\nu} + i\,D\,\xi_\alpha \ , \quad \delta D = \bar\xi\,\bar\sigma^\mu\,\partial_\mu\lambda + \partial_\mu\bar\lambda\,\bar\sigma^\mu\,\xi$$

We now analyze the gaugino:

$$\delta_1\delta_2\lambda_\alpha = (\sigma^{\mu\nu}\,\xi_2)_\alpha\,(2\,i\,\bar\xi_1\,\bar\sigma_\nu\,\partial_\mu\lambda - 2\,i\,\partial_\mu\bar\lambda\,\bar\sigma_\nu\,\xi_1) + i\,(\bar\xi_1\,\bar\sigma^\mu\,\partial_\mu\lambda + \partial_\mu\bar\lambda\,\bar\sigma^\mu\,\xi_1)\,\xi_{2\alpha}$$

Some useful identities:

$$(\sigma^\mu)_{\alpha\dot\beta}\,(\bar\sigma_\mu)^{\dot\gamma\delta} = -\,2\,\delta_\alpha{}^\delta\,\delta^{\dot\gamma}{}_{\dot\beta}\,, \qquad (\sigma^\mu)_{\alpha\dot\beta}\,(\sigma_\mu)_{\gamma\dot\delta} = -\,2\,\epsilon_{\alpha\gamma}\,\epsilon_{\dot\gamma\dot\delta} \ , \qquad (\bar\sigma^\mu)^{\dot\alpha\beta}\,(\bar\sigma_\mu)^{\dot\gamma\delta} = -\,2\,\epsilon^{\dot\alpha\dot\gamma}\,\epsilon^{\gamma\delta}$$

The second and third follow from the first recalling that $(\bar\sigma^\mu)^{\dot\alpha\beta} = \epsilon^{\dot\alpha\dot\gamma}\,\epsilon^{\beta\delta}\,(\sigma^\mu)_{\delta\dot\gamma}$. Recall the definition $\sigma^{\mu\nu} = \frac{1}{4}\,(\sigma^\mu\,\bar\sigma^\nu - \sigma^\nu\,\bar\sigma^\mu)$. We can simplify all terms where $\sigma^{\mu\nu}$ and $\bar\sigma_\nu$ appear together:

$$(\bar\xi_1\,\bar\sigma_\nu\,\partial_\mu\lambda)\,(\sigma^\mu\,\bar\sigma^\nu\,\xi_2)_\alpha \propto (\sigma^\mu\,\bar\xi_1)_\alpha\,(\xi_2\,\partial_\mu\lambda)\,, \qquad (\bar\xi_1\,\bar\sigma_\nu\,\partial_\mu\lambda)\,(\sigma^\nu\,\bar\sigma^\mu\,\xi_2)_\alpha \propto (\bar\xi_1\,\bar\sigma^\mu\,\xi_2)\,\partial_\mu\lambda_\alpha \ , \text{ etc}$$

All remaining terms have a sigma matrix contracted with $\partial_\mu\lambda$ or $\partial_\mu\bar\lambda$. After some Fierz rearrangements based on $(\chi_1)^\beta\,(\chi_2)_\beta\,(\chi_3)_\alpha + \text{cyclic} = 0$ we verify the closure of the algebra:

$$\delta_1\delta_2\lambda_\alpha - \delta_2\delta_1\lambda_\alpha = -\,2\,i\,(\xi_1\,\sigma^\mu\,\bar\xi_2 - \xi_2\,\sigma^\mu\,\bar\xi_1)\,\partial_\mu\lambda_\alpha$$

Closure of the SUSY algebra

$$\delta A_\mu = i\,\bar\xi\,\bar\sigma_\mu\,\lambda - i\,\bar\lambda\,\bar\sigma_\mu\,\xi \;\;, \quad \delta\lambda_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F_{\mu\nu} + i\,D\,\xi_\alpha \;\;, \quad \delta D = \bar\xi\,\bar\sigma^\mu\,\partial_\mu\lambda + \partial_\mu\bar\lambda\,\bar\sigma^\mu\,\xi$$

Finally we turn to the gauge field:

$$\delta_1\delta_2 A_\mu = i\,\bar\xi_2\,\bar\sigma_\mu\,\delta_1\lambda + \mathrm{h.c.} = i\,(\bar\xi_2\,\bar\sigma_\mu\,\sigma^{\rho\sigma}\,\xi_1)\,F_{\rho\sigma} - D\,(\bar\xi_2\,\bar\sigma_\mu\,\xi_1) + \mathrm{h.c.}$$

The term with D goes away when we subtract $(1 \leftrightarrow 2)$. To proceed we use

$$(\bar\xi_2\,\bar\sigma_\mu\,\sigma^{\rho\sigma}\,\xi_1)\,F_{\rho\sigma} = \frac{1}{2}(\bar\xi_2\,\bar\sigma_\mu\,\sigma^\rho\,\bar\sigma^\sigma\,\xi_1)\,F_{\rho\sigma}$$

and the identity

$$\bar\sigma^\mu\,\sigma^\rho\,\bar\sigma^\sigma = \eta^{\mu\sigma}\,\bar\sigma^\rho - \eta^{\rho\sigma}\,\bar\sigma^\mu - \eta^{\mu\rho}\,\bar\sigma^\sigma - i\,\epsilon^{\mu\rho\sigma\tau}\,\bar\sigma_\tau$$

We arrive at

$$i\,(\bar\xi_2\,\bar\sigma_\mu\,\sigma^{\rho\sigma}\,\xi_1)\,F_{\rho\sigma} = \frac{1}{2}\,\epsilon^{\mu\rho\sigma\tau}\,F_{\rho\sigma}\,(\bar\xi_2\,\bar\sigma_\tau\,\xi_1) + i\,(\bar\xi_2\,\bar\sigma^\nu\,\xi_1)\,F_{\nu\mu}$$

The term with epsilon goes away when we add the hc and subtract $(1 \leftrightarrow 2)$. We are left with

$$\begin{aligned}
\delta_1\delta_2 A_\mu - \delta_2\delta_1 A_\mu &= -\,2\,i\,(\xi_1\,\sigma^\nu\,\bar\xi_2 - \xi_2\,\sigma^\nu\,\xi_1)\,F_{\nu\mu} \\
&= -\,2\,i\,(\xi_1\,\sigma^\nu\,\bar\xi_2 - \xi_2\,\sigma^\nu\,\xi_1)\,\partial_\nu A_\mu + \partial_\mu\left[2\,i\,(\xi_1\,\sigma^\nu\,\bar\xi_2 - \xi_2\,\sigma^\nu\,\xi_1)\,A_\nu\right]
\end{aligned}$$

Closure of the SUSY algebra

$$\delta_1\delta_2 A_\mu - \delta_2\delta_1 A_\mu = -\,2\,i\,(\xi_1\,\sigma^\nu\,\bar{\xi}_2 - \xi_2\,\sigma^\nu\,\xi_1)\,F_{\nu\mu}$$
$$= -\,2\,i\,(\xi_1\,\sigma^\nu\,\bar{\xi}_2 - \xi_2\,\sigma^\nu\,\xi_1)\,\partial_\nu A_\mu + \partial_\mu\left[2\,i\,(\xi_1\,\sigma^\nu\,\bar{\xi}_2 - \xi_2\,\sigma^\nu\,\xi_1)\,A_\nu\right]$$

- We find a new feature of SUSY algebras in the presence of gauge symmetries: the off-shell SUSY algebra closes on gauge fields only up to a gauge transformation.
- The difference of two $U(1)$ gauge fields is invariant under gauge transformations. We can think of the SUSY variation δA_μ as an infinitesimal difference. The RHS of $\delta_1\delta_2 A_\mu - \delta_2\delta_1 A_\mu$ should therefore be gauge invariant, and this is exactly what we find
- In the present example we see that the parameter of the gauge transformation is

$$\Lambda = 2\,i\,(\xi_1\,\sigma^\nu\,\bar{\xi}_2 - \xi_2\,\sigma^\nu\,\xi_1)\,A_\nu$$

It is constructed with the constant SUSY params, but it also includes a field. These field-dependent gauge parameters are a standard feature. This pattern is also find in supergravity.

SUSY Lagrangian

A simple Lagrangian that is invariant under the SUSY variations

$$\delta A_\mu = i\,\bar{\xi}\,\bar{\sigma}_\mu\,\lambda - i\,\bar{\lambda}\,\bar{\sigma}_\mu\,\xi \ , \quad \delta\lambda_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F_{\mu\nu} + i\,D\,\xi_\alpha \ , \quad \delta D = \bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\lambda + \partial_\mu\bar{\lambda}\,\bar{\sigma}^\mu\,\xi$$

is given by

$$\mathscr{L} = -\frac{1}{4}\,F_{\mu\nu}\,F^{\mu\nu} - i\,\lambda\,\sigma^\mu\,\partial_\mu\bar{\lambda} + \frac{1}{2}\,D^2$$

The EOM for the auxiliary field simply sets it to zero. This is a free theory of:

- a $U(1)$ gauge field (a “photon”)
- a massless and neutral Weyl fermion

One can verify that this theory admits a conserved SUSY current,

$$J^\mu_\alpha = -\frac{1}{2\sqrt{2}}\,(\sigma^\nu\,\bar{\sigma}^\rho\,\sigma^\mu\,\bar{\lambda})_\alpha\,F_{\nu\rho}$$

Non-Abelian models

The SUSY variations for a $U(1)$ vector multiplet are easily generalized to a vector multiplet in the adjoint representation of a simple non-Abelian gauge group G

$$(A^a_\mu, \lambda^a_\alpha, D^a) \ , \qquad a = 1, \ldots, \dim G$$

The adjoint indices a, b are raised/lowered with the Cartan metric; we choose a basis where it is just δ_{ab}. The generators are T_a and satisfy

$$[T_a, T_b] = i f_{ab}{}^c\, T_c$$

The structure constants are real and if we lower the 3rd index they are totally antisymmetric.

Our conventions for covariant derivatives and field strengths are

$$F^{ab} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc}\, A_{b\mu}\, A_{c\nu} \quad , \qquad D_\mu D^a = \partial_\mu D^a - g\, f^{abc}\, A_{b\mu}\, D_c$$

where g is the gauge coupling constant

$$\delta A_\mu = i\, \bar{\xi}\, \bar{\sigma}_\mu\, \lambda - i\, \bar{\lambda}\, \bar{\sigma}_\mu\, \xi \ , \quad \delta\lambda_\alpha = (\sigma^{\mu\nu}\, \xi)_\alpha\, F_{\mu\nu} + i\, D\, \xi_\alpha \ , \quad \delta D = \bar{\xi}\, \bar{\sigma}^\mu\, \partial_\mu \lambda + \partial_\mu \bar{\lambda}\, \bar{\sigma}^\mu\, \xi$$

Non-Abelian models

The Abelian SUSY variations

$$\delta A_\mu = i\,\bar{\xi}\,\bar{\sigma}_\mu\,\lambda - i\,\bar{\lambda}\,\bar{\sigma}_\mu\,\xi \ \ , \quad \delta\lambda_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F_{\mu\nu} + i\,D\,\xi_\alpha \ \ , \quad \delta D = \bar{\xi}\,\bar{\sigma}^\mu\,\partial_\mu\lambda + \partial_\mu\bar{\lambda}\,\bar{\sigma}^\mu\,\xi$$

are generalized to

$$\delta A^a_\mu = i\,\bar{\xi}\,\bar{\sigma}_\mu\,\lambda^a - i\,\bar{\lambda}^a\,\bar{\sigma}_\mu\,\xi$$

$$\delta\lambda^a_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F^a_{\mu\nu} + i\,D^a\,\xi_\alpha$$

$$\delta D^a = \bar{\xi}\,\bar{\sigma}^\mu\,D_\mu\lambda^a + D_\mu\bar{\lambda}^a\,\bar{\sigma}^\mu\,\xi$$

Remarks:

- The variation of the gauge field is a difference of connections; it transforms in the adjoint rep. The expression for δA^a_μ has this property
- The variations of λ^a_α, D^a must be gauge-covariant: this explains the appearance of the covariant field strength and the covariant derivative

Gauge-covariant closure of the SUSY algebra

$$\delta A^a_\mu = i\,\bar{\xi}\,\bar{\sigma}_\mu\,\lambda^a - i\,\bar{\lambda}^a\,\bar{\sigma}_\mu\,\xi$$

$$\delta\lambda^a_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F^a_{\mu\nu} + i\,D^a\,\xi_\alpha$$

$$\delta D^a = \bar{\xi}\,\bar{\sigma}^\mu\,D_\mu\lambda^a + D_\mu\bar{\lambda}^a\,\bar{\sigma}^\mu\,\xi$$

The closure of the SUSY algebra is not on ordinary translations, but rather on their gauge-covariant extensions:

$$\delta_1\delta_2 A^a_\mu - \delta_2\delta_1 A^a_\mu = -\,2\,i\,(\xi_1\,\sigma^\nu\,\bar{\xi}_2 - \xi_2\,\sigma^\nu\,\bar{\xi}_1)\,F^a_{\nu\mu}$$

$$\delta_1\delta_2\lambda^a_\alpha - \delta_2\delta_1\lambda^a_\alpha = -\,2\,i\,(\xi_1\,\sigma^\nu\,\bar{\xi}_2 - \xi_2\,\sigma^\nu\,\bar{\xi}_1)\,D_\nu\lambda^a_\alpha$$

$$\delta_1\delta_2 D^a - \delta_2\delta_1 D^a = -\,2\,i\,(\xi_1\,\sigma^\nu\,\bar{\xi}_2 - \xi_2\,\sigma^\nu\,\bar{\xi}_1)\,D_\nu D^a$$

In the Abelian case λ, D are gauge invariant, that's why we found ordinary translations

SUSY Lagrangian

$$\delta A^a_\mu = i\,\bar{\xi}\,\bar{\sigma}_\mu\,\lambda^a - i\,\bar{\lambda}^a\,\bar{\sigma}_\mu\,\xi$$

$$\delta\lambda^a_\alpha = (\sigma^{\mu\nu}\,\xi)_\alpha\,F^a_{\mu\nu} + i\,D^a\,\xi_\alpha$$

$$\delta D^a = \bar{\xi}\,\bar{\sigma}^\mu\,D_\mu\lambda^a + D_\mu\bar{\lambda}^a\,\bar{\sigma}^\mu\,\xi$$

The SUSY Lagrangian that extends the standard YM theory is

$$\mathscr{L} = -\frac{1}{4}\,F^a{}_{\mu\nu}\,F_a{}^{\mu\nu} - i\,\lambda^a\,\sigma^\mu\,D_\mu\bar{\lambda}_a + \frac{1}{2}\,D^a\,D_a$$

The auxiliary field is again zero on-shell. We now have an interacting SUSY theory. Its SUSY current takes the form

$$J^\mu_\alpha = -\frac{1}{2\sqrt{2}}\,(\sigma^\nu\,\bar{\sigma}^\rho\,\sigma^\mu\,\bar{\lambda}^a)_\alpha\,F_{a\nu\rho}$$

It is conserved on-shell and gauge-invariant.