Supersymmetry and Supergravity — Problem Sheet 4

MMathPhys, University of Oxford, HT2021, Dr Federico Bonetti

These problems are due by Saturday before the class on week 8 by 11 am. Links to submit:
TA A. Boido: https://cloud.maths.ox.ac.uk/index.php/s/WP8kazik5pNZjmi
TA J. McGovern: https://cloud.maths.ox.ac.uk/index.php/s/oBKgZcaE9F4bw3z

1 R-symmetry in superspace

To describe $U(1)_R$ symmetry in superspace we introduce the following transformation on the fermionic superspace coordinates $(\theta, \overline{\theta})$,

$$(\theta^\alpha, \overline{\theta}^{\dot{\alpha}}) \mapsto (\theta'^\alpha, \overline{\theta}'^{\dot{\alpha}}) = (e^{it}\,\theta^\alpha, e^{-it}\,\overline{\theta}^{\dot{\alpha}}) \ . \tag{1}$$

The quantity $t \in \mathbb{R}$ is the parameter of the $U(1)_R$ transformation. Let $\mathcal{S}(x, \theta, \overline{\theta})$ be a superfield. We say that $\mathcal{S}$ has definite R-character $R[\mathcal{S}] \in \mathbb{R}$ if $U(1)_R$ acts on $\mathcal{S}$ according to the following transformation law,

$$\mathcal{S}'(x, \theta', \overline{\theta}') = e^{itR[\mathcal{S}]}\,\mathcal{S}(x, \theta, \overline{\theta}) \ , \tag{2}$$

where θ', $\overline{\theta}'$ on the LHS are defined by (1).

1.a Let $\mathcal{S}(x, \theta, \overline{\theta})$ be a generic complex superfield with definite R-character $R[\mathcal{S}]$. Consider the expansion of $\mathcal{S}(x, \theta, \overline{\theta})$ in component fields,

$$\begin{aligned}
\mathcal{S}(x, \theta, \overline{\theta}) = {} & C(x) + i\,\theta^\alpha\,\chi_\alpha(x) - i\,\overline{\theta}_{\dot{\alpha}}\,\overline{\chi}^{\dot{\alpha}}(x) \\
& + \tfrac{1}{2}\,i\,(\theta\,\theta)\,M(x) - \tfrac{1}{2}\,i\,(\overline{\theta}\,\overline{\theta})\,\overline{M}(x) - (\theta\,\sigma^\mu\,\overline{\theta})\,v_\mu(x) \\
& + i\,(\theta\,\theta)\,\overline{\theta}_{\dot{\alpha}}\left[\overline{\lambda}^{\dot{\alpha}}(x) + \tfrac{1}{2}\,i\,(\overline{\sigma}^\mu)^{\dot{\alpha}\beta}\,\partial_\mu\chi_\beta(x)\right] - i\,(\overline{\theta}\,\overline{\theta})\,\theta^\alpha\left[\lambda_\alpha(x) + \tfrac{1}{2}\,i\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\partial_\mu\overline{\chi}^{\dot{\beta}}(x)\right] \\
& + \tfrac{1}{2}\,(\theta\,\theta)\,(\overline{\theta}\,\overline{\theta})\left[D(x) + \tfrac{1}{2}\,\partial^\mu\,\partial_\mu C(x)\right] \ .
\end{aligned} \tag{3}$$

Use (1) and (2) to verify that all component fields have a definite $U(1)_R$ charges. Compute these charges in terms of $R[\mathcal{S}]$. Argue that, if $\mathcal{S}(x, \theta, \overline{\theta})$ is a real superfield with a definite R-character $R[\mathcal{S}]$, then we must have $R[\mathcal{S}] = 0$.

1.b Let $\mathcal{S}$, $\mathcal{T}$ be superfields with definite R-characters $R[\mathcal{S}]$, $R[\mathcal{T}]$. Verify the following claims:

- The superfield $\mathcal{S}\mathcal{T}$ has definite R-character $R[\mathcal{S}] + R[\mathcal{T}]$.
- The superfield $D_\alpha\mathcal{S}$ has definite R-character $R[\mathcal{S}] - 1$.
- The complex conjugate superfield $\overline{\mathcal{S}}(x, \theta, \overline{\theta}) := [\mathcal{S}(x, \theta, \overline{\theta})]^*$ has definite R-character $-R[\mathcal{S}]$.

Notice that the second and third imply that the superfield $\overline{D}_{\dot{\alpha}}\mathcal{S}$ has definite R-character $R[\mathcal{S}]+1$.

1.c Find the transformation laws of the measures $d^2\theta$, $d^2\overline{\theta}$, $d^2\theta\,d^2\overline{\theta}$ under the transformation (1). Hint: keep in mind that we are dealing with Grassmann odd coordinates.

1.d An F-type term in a superspace action is of the form $S_1 = \int d^4x\, d^2\theta\, W + \text{h.c.}$ where W is a chiral superfield. A D-type term in a superspace action is of the form $S_2 = \int d^4x\, d^2\theta\, d^2\overline{\theta}\, K$ where K is a real superfield. Use the results of the previous point to argue that S_1 is invariant under $U(1)_R$ iff W has definite R-character $R[W] = 2$, and that S_2 is invariant under $U(1)_R$ iff K has definite R-character $R[K] = 0$.

1.e Let $\mathcal{W}_\alpha$ be the chiral superfield that encodes the field strength of a vector superfield V (Abelian or non-Abelian). Suppose V has a definite R-character (which must be zero, as you argued above). Use the expression of $\mathcal{W}_\alpha$ in terms of V to argue that $\mathcal{W}_\alpha$ has definite R-character $R[\mathcal{W}_\alpha] = 1$. Consider the SYM term in superspace with arbitrary gauge coupling function f. Schematically, $S_3 = \int d^4x\, d^2\theta\, f\, \mathcal{W}^\alpha\, \mathcal{W}_\alpha$ where f is a chiral superfield. Argue that S_3 is invariant under $U(1)_R$ iff f has definite R-character $R[f] = 0$.

2 Supersymmetric vacua in some Wess-Zumino models

Let us consider some models with chiral superfields, and no vector superfields. In each case we assume a canonical Kähler potential. For each of the following models, describe the (classical) space of supersymmetric vacua.

- Model #1: two chiral multiplets X, Y with superpotential

$$W = \lambda\, X^2\, Y + \mu\, X^2\ , \qquad \lambda, \mu \in \mathbb{C}\ , \quad \lambda, \mu \neq 0\ . \tag{4}$$

- Model #2: three chiral multiplets X, Y, Z with superpotential

$$W = \alpha\, Y + \beta\, Y\, X^2 + \gamma\, X\, Z, \qquad \alpha, \beta, \gamma \in \mathbb{C}\ , \quad \alpha, \beta, \gamma \neq 0\ . \tag{5}$$

- Model #3: one chiral multiplet X with superpotential

$$W = \alpha\, X + \frac{\beta}{X}, \qquad \alpha, \beta \in \mathbb{C}\ , \quad \alpha, \beta \neq 0\ . \tag{6}$$

3 Non-Abelian gauge transformation of vector superfields

The transformation law of a non-Abelian vector superfield is

$$e^{2\,V'} = e^{-i\Lambda^\dagger}\, e^{2V}\, e^{i\Lambda}\ . \tag{7}$$

This is a matrix equation in a representation $\mathbf{R}$ of the gauge group. More precisely, we have $V = V^a\, t_a^{\mathbf{R}}$, $\Lambda = \Lambda^a\, t_a^{\mathbf{R}}$, $\Lambda^\dagger = \overline{\Lambda}^a\, t_a^{\mathbf{R}}$ where $t_a^{\mathbf{R}}$ are Hermitian generators, $a = 1, \ldots, \dim G$ is an adjoint index, V^a are real superfields, Λ^a are chiral superfields.

Our goal is to specialize the above transformation to infinitesimal Λ, but working exactly in V.

3.a The BCH formula can be expressed in the form

$$\log(e^A\, e^B) = A + \left[\int_0^1 dt\, \psi(e^{\mathrm{ad}_A}\, e^{t\,\mathrm{ad}_B})\right] B\ , \qquad \psi(x) := \frac{\log x}{1 - x^{-1}}\ . \tag{8}$$

By definition $\mathrm{ad}_A X = [A, X]$ for any A, X in $\mathrm{Lie}(G)$. Use this formula to verify that

$$\log(e^A\, e^B) = A + \mathrm{ad}_{A/2}\,(1 + \coth \mathrm{ad}_{A/2}) B + (\text{terms of higher order in } B)\ . \tag{9}$$

Use the identity $\log(e^B\, e^A) = -\log(e^{-A}\, e^{-B})$ to prove

$$\log(e^B\, e^A) = A - \mathrm{ad}_{A/2}\,(1 - \coth \mathrm{ad}_{A/2}) B + (\text{terms of higher order in } B)\ . \tag{10}$$

Use (9) and (10) to verify that

$$e^{-i\Lambda^\dagger}\, e^{2V}\, e^{i\Lambda} = \exp\Big[2\,V + i\,\mathrm{ad}_V\,(\Lambda + \Lambda^\dagger) + i\,\mathrm{ad}_V\,\coth \mathrm{ad}_V(\Lambda - \Lambda^\dagger) + \ldots\Big]\ . \tag{11}$$

We conclude that $2\,\delta V = i\,\mathrm{ad}_V\,(\Lambda + \Lambda^\dagger) + i\,\mathrm{ad}_V\,\coth \mathrm{ad}_V(\Lambda - \Lambda^\dagger)$.

3.b Let us consider the expansion the RHS of $2\,\delta V$ in powers of V. Make use of $[t_a^{\mathbf{R}}, t_b^{\mathbf{R}}] = i\, f_{ab}{}^c\, t_c^{\mathbf{R}}$ and of the Taylor expansion of $f(x) = x\,\coth x$ around $x = 0$ to verify that

$$2\,\delta V^a = i\,(\Lambda^a - \overline{\Lambda}^a) - f_{bc}{}^a\, V^b\,(\Lambda^c + \overline{\Lambda}^c) - \tfrac{i}{3}\, f_{bc}{}^a f_{de}{}^c\, V^b\, V^d\,(\Lambda^e - \overline{\Lambda}^e) + \mathcal{O}(V^3)\ . \tag{12}$$