

Scientific Programming with Mathematica 2020

Problems 3

Please work through the entire problem set. Only the problems numbered (1) and (2) will be graded, though I will go through the entire sheet in the problem session.

Consider again the space AESZ34, from Problem set 2, for which we have the Picard-Fuchs equation

$$\mathcal{L}\varpi = 0 \quad \text{with} \quad \mathcal{L} = \sum_{j=0}^4 R_j \vartheta^j ; \quad \vartheta = \varphi \frac{\partial}{\partial \varphi} .$$

First, let us seek the four solutions to the differential equation in a neighborhood of $\varphi = 0$. Following Frobenius, we seek a solution of the form

$$\varpi = \sum_{n=0}^{\infty} A_n(\epsilon) \varphi^{n+\epsilon} \quad \text{with} \quad A_0(\epsilon) = 1 .$$

Find a recurrence relation for the $A_n(\epsilon)$ and show that $\epsilon^4 = 0$. By expanding

$$\varpi = \varpi_0(\varphi) + \epsilon \varpi_1(\varphi) + \frac{1}{2!} \epsilon^2 \varpi_2(\varphi) + \frac{1}{3!} \epsilon^3 \varpi_3(\varphi) \quad \text{and} \quad A_n(\epsilon) = a_n + \epsilon b_n + \frac{1}{2!} \epsilon^2 c_n + \frac{1}{3!} \epsilon^3 d_n ,$$

show that there are four solutions of the form

$$\begin{aligned} \varpi_0 &= f_0 \\ \varpi_1 &= f_0 \log \varphi + f_1 \\ \varpi_2 &= f_0 \log^2 \varphi + 2f_1 \log \varphi + f_2 \\ \varpi_3 &= f_0 \log^3 \varphi + 3f_1 \log^2 \varphi + 3f_2 \log \varphi + f_3 , \end{aligned}$$

where the f_j are power series with coefficients a_n , b_n , c_n and d_n . The series f_0 is the series that was found in Problem Set 2.

Write routines that calculate the f_j , and their derivatives $\vartheta^k f_j$, with $0 \leq k \leq 3$, and so also the $\vartheta^k \varpi_j$, both as series to `nmax` terms and numerically to `nmax` terms. Check that $\mathcal{L}\varpi_3$ vanishes to the required order when `nmax` = 100, say.

Now let $\varphi = \psi$ be a regular point of the differential equation, so that there are four solutions that are power series in a neighborhood of ψ . Let us choose a basis of solutions of the form

$$\begin{aligned}\eta_0(\varphi, \psi) &= 1 + \mathcal{O}((\varphi - \psi)^4) \\ \eta_1(\varphi, \psi) &= (\varphi - \psi) + \mathcal{O}((\varphi - \psi)^4) \\ \eta_2(\varphi, \psi) &= (\varphi - \psi)^2 + \mathcal{O}((\varphi - \psi)^4) \\ \eta_3(\varphi, \psi) &= (\varphi - \psi)^3 + \mathcal{O}((\varphi - \psi)^4)\end{aligned}$$

Find a recurrence relation for coefficients A_n, B_n, C_n, D_n , which will depend on ψ , such that $\eta_0 = \sum_{n=0}^{\infty} A_n(\varphi - \psi)^n$, $\eta_1 = \sum_{n=0}^{\infty} B_n(\varphi - \psi)^n$ etc.. Note that the A_n, \dots, D_n all satisfy the same recurrence relation. The difference is in the initial conditions.

Now let $\varpi = (\varpi_j)$ and $\eta = (\eta_j)$, that is we have defined vectors of solutions. If $|\psi| < \frac{1}{25}$, and φ is such that both ϖ and η converge then there will be a matrix M , independent of φ , such that $\eta = M\varpi$. Let W_ϖ and W_η be the Wronskian matrices

$$W_{\varpi jk} = \vartheta^k \varpi_j \quad \text{and} \quad W_{\eta jk} = \vartheta^k \eta_j .$$

Note that

$$W_\eta = MW_\varpi \quad \text{so that} \quad M = W_\eta W_\varpi^{-1} .$$

For the case $\psi = \frac{1}{50}$, calculate M numerically, to machine precision.

A little more analytical work: Let y denote any vector of solutions to the differential equation and let W_y be the corresponding Wronskian. By considering the columns of ϑW_y show that

$$W_y^{-1} \vartheta W_y = \begin{pmatrix} 0 & 0 & 0 & -R_0/R_4 \\ 1 & 0 & 0 & -R_1/R_4 \\ 0 & 1 & 0 & -R_2/R_4 \\ 0 & 0 & 1 & -R_3/R_4 \end{pmatrix} .$$

Now by means of the formulas, valid for a generic matrix A ,

$$\det A = \exp(\text{Tr} \log A) \quad \text{and its consequence} \quad \vartheta \det A = (\det A) \text{Tr}(A^{-1} \vartheta A) ,$$

you can now show that

$$\det W_y = \exp\left(-\int \frac{d\varphi}{\varphi} \frac{R_3}{R_4}\right) = \frac{w_y}{R_4^2(\varphi)} ,$$

where w_y is a constant that depends on the basis y .

Compute w_ϖ and w_η . Note that one can use this relation for $\det W$ to check on the accuracy of a numerical computation of W . Write a function that computes W to a given accuracy, using this check to increase `nmax` as necessary.

1. Now given a sequence of φ values $\{\psi_0 = 0, \psi_1, \psi_2, \dots, \psi_f\}$ such that each ψ_{i+1} lies in a disk such that $\eta(\varphi, \psi_i)$ converges for $\varphi = \psi_{i+1}$. How is $W_{\varpi}(\psi_f)$ related to $W_{\eta(\psi_{f-1})}(\psi_f)$? Write a code that evaluates $W_{\varpi}(\psi_f)$ to a specified accuracy.

One way to choose the ψ_j proceeds as follows. Suppose $\psi_0 = 0$ and that the initial radius of convergence is r ($= 1/25$) and that there are no singularities in $\text{Re } \varphi \leq 0$. Choose $\psi_1 = -r/2$. The new radius of convergence runs as far as the singularity at $\varphi = r$, so has radius $3r/2$. Choose ψ_2 half-way between ψ_1 and the edge of the region of convergence so $\psi_2 = \frac{3\psi_1}{2}$, and so on. In this way we get a sequence of circles, that all pass through $\varphi = 0$ and have radius $(\frac{3}{2})^{n-1}r$, and we can take

$$\psi_n = \left(\frac{3}{2}\right)^{n-1} \psi_1 .$$

A figure is given at the end of Lecture 4.

2. Evaluate $\varpi(-1/7)$ to 100 figures. Evaluate also $L(1)$ and $L(2)$ to the same accuracy, where $L(s)$ is the function introduced in Problem sheet 2, and seek linear relations between each of the quantities $\varpi_j(-1/7)/\pi^j$, $0 \leq j \leq 2$ and the quantities $L(k)/\pi^k$, $k = 1, 2$, using the function **FindIntegerNullVector**. The quantity ϖ_3 is special, and you should seek a linear relation between $\varpi_3(-1/7)/\pi^3$, $\zeta(3)\varpi_0(-1/7)/\pi^3$ and the $L(k)/\pi^k$.