# STRING THEORY I

Lecture 2

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$\boxed{\text{Chapter 2}}$ Classical relativistic string

2.1 Classical relativistic point particle

2.2 Classical relativistic string : Action principle

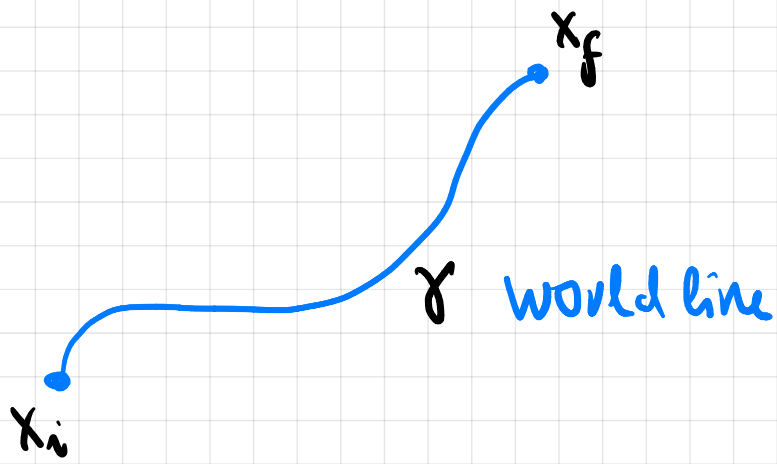
2.3

⋮

# 12.1 Classical relativistic point particle

Action for a relativistic particle of mass $m$ moving in $d$-dim space time $\mathcal{M}$

$$S[\gamma] = -m \int_\gamma ds$$



Treat this as a "0+1" dimensional field theory with

- $\tau$ = local time parameter on the world line
- $\gamma$ considered as a curve embedded in $\mathcal{M}$ $(\gamma \subset \mathcal{M})$ has coordinates $X^\mu(\tau)$, the world line fields

ie
$$\gamma : \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathcal{M} \\ \tau & \longmapsto & X^\mu(\tau) \end{array}$$

In terms of these

$$S[X^\mu(\tau)] = -m \int_{\tau_i}^{\tau_f} \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}\, d\tau = -m \int_{\tau_i}^{\tau_f} d\tau \sqrt{-h}$$

Euler-Lagrange eqs $\Rightarrow$ classical motion of the point particle is along geodesics.

Symmetries of the action:

- The isometry group of $\mathcal{M}$ leaves invariant the line element

If $g_{\mu\nu} = \eta_{\mu\nu}$ (flat Minkowski metric) $\Rightarrow$ spacetime Poincaré invariance

$$X^\mu(\tau) \longrightarrow \Lambda^\mu{}_\nu X^\nu(\tau) + b^\mu \quad \text{where } \Lambda \in SO(1,d-1),\ b \in \mathbb{R}^{1,d-1}$$

More generally, the space-time isometry group is realised as internal symmetries of the world-line theory.

- 1-dimensional reparametrization invariance

$$\tau \longrightarrow \tilde{\tau}(\tau)$$

$$X^\mu(\tau) \longrightarrow \tilde{X}^\mu(\tilde{\tau}) = X^\mu(\tau)$$

True because $S$ is a function of $\gamma \subset \mathcal{M}$ and we do not care about the parametrization of $\gamma$

This is a gauge symmetry (redundancy of the description)

S is a good classical action but
There are two problems with this action:

- it has a square-root $\Rightarrow$ difficult to quantize

- what happens if $m = 0$ ?

Consider instead the action

$$\tilde{S} = \frac{1}{2} \int \left( e^{-1} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - e m^2 \right) d\tau$$

where $e(\tau)$ is an extra field on the worldline

(an "einbein" $e : \mathbb{R}_\tau \longrightarrow \mathbb{R}_{>0}$ on the worldline)

EOM for $e$ are $\quad e^2 = -\frac{1}{m^2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$

Substituting this back into $\tilde{S}$ we get $S$

$\therefore \quad \tilde{S}$ & $S$ are classically equivalent.

## Symmetries of $\tilde{S}$

- spacetime isometries (Poincaré for Minkowski) and $e(\tau)$ invariant
- reparametrization of the world line

$$\tau \longmapsto \tilde{\tau}(\tau)$$

scalars on $\gamma$ $\quad x^\mu(\tau) \longmapsto \tilde{x}^\mu(\tilde{\tau}) = x^\mu(\tau)$

einbein on $\gamma$ $\quad e(\tau) \longmapsto \tilde{e}(\tilde{\tau}) = \frac{d\tau}{d\tilde{\tau}} e(\tau)$

infinitesimal change

$$\tau \mapsto \tau - \xi(\tau)$$

$$x^\mu \mapsto x^\mu + \xi \frac{dx^\mu}{d\tau}$$

$$e \mapsto e + \frac{d}{d\tau}(\xi e)$$

Can use reparametrization to gauge fix.

m ≠ 0: set $e(\tau) = \frac{1}{m}$

$$\tilde{S}_{fixed} = \frac{1}{2} m \int_\gamma \left( g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - 1 \right) d\tau$$

Equations of motion give geodesic equation

$$\frac{dx^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

& from the vielbein equation of motion

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + 1 = 0$$

$\Rightarrow \frac{dx^\mu}{d\tau}$ is the TL 4-velocity with $\tau$=proper time

(TL geodesic)

$m=0$: can gauge fix $e=1$ and we recover the geodesic equation together with

$$g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0 \quad\Rightarrow\quad \text{null geodesic}$$

- $\tilde{S}$ is a good starting point for quantization.

- Could add interactions, build up Feynman diagrams in a first-quantized theory.

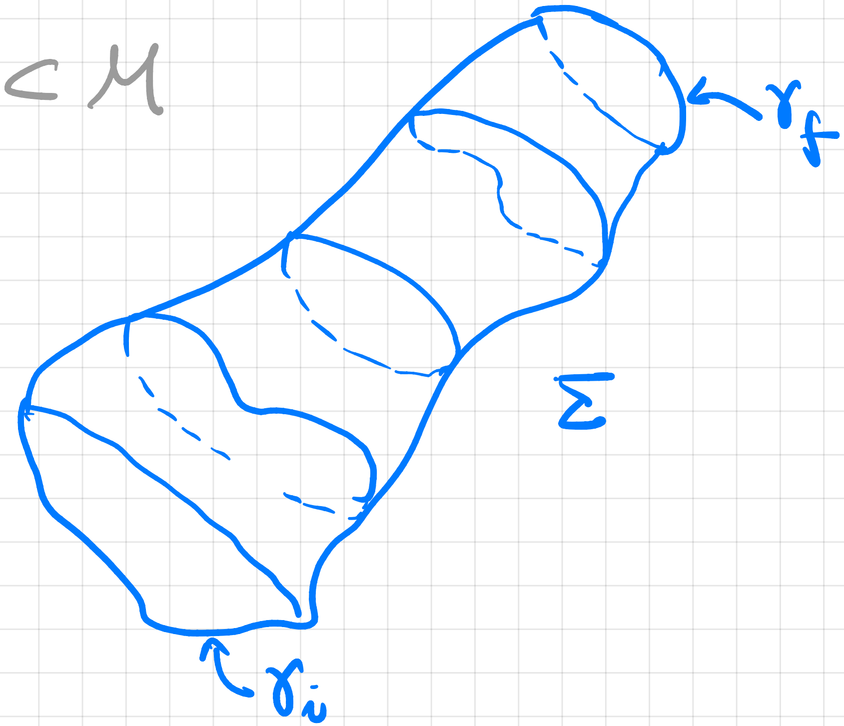
## 12.1 Classical relativistic string

Generalise to a string which sweeps out a two dimensional worldsheet $\Sigma$ in space-time $\mathcal{M}$

target space

analogue of $S[\gamma] = -m \int_\gamma ds$

$\Sigma \subset \mathcal{M}$




$$S[\Sigma] = -T \int_\Sigma \underbrace{dA}$$

area element

tension

$\sim$ mass per unit length

$$\boxed{\text{Nambu-Goto}}$$

Introduce the worldsheet parameters $(\tau, \sigma)$ and

[Treat this as] a "1+1" dimensional field theory with fields

$$X^\mu : \Sigma \longrightarrow \mathcal{M} \longleftarrow \text{"target" space}$$

$$(\tau, \sigma) \longmapsto X^\mu(\sigma, \tau)$$

In terms of these we have:

$$S_{NG}[X^\mu(\tau,\sigma)] = -T\int_\Sigma \left[(-\partial_\tau X\cdot\partial_\tau X)(\partial_\sigma X\cdot\partial_\sigma X)+(\partial_\tau X\cdot\partial_\sigma X)^2\right]^{1/2} d\tau\, d\sigma$$

Nambu-Goto action

$$= -T\int_\Sigma \sqrt{-G}\, d\tau\, d\sigma$$

$G$ = induced (pull back) metric on $\Sigma$

where $u \cdot V = g_{\mu\nu} u^\mu V^\nu$ for space time vectors classical motion of the string along

Euler-Lagrange eqs $\Longrightarrow$ minimal area surfaces

<u>What is $T$?</u> $T$ interpreted as string tension

ie mass of the string per unit length

$S$ is dimensionless ✓

To "see" this set $X^0 = \tau$, $\partial_\tau X^{\mu \neq 0} = 0$ $g_{0\mu} = 0$ $\mu \neq 0$

$\Longrightarrow$ static string in static geometry

$$S = -T \int_\Sigma \sqrt{\partial_\sigma X \cdot \partial_\sigma X}\, d\sigma\, d\tau = -T \int_\tau (\text{string length})\, d\tau$$

$$\overset{\text{OTOH}}{=} \int_\tau (\underbrace{\cancel{\text{kinetic energy}}}_{\to 0} - \text{potential energy})\, d\tau$$

$\frac{1}{2}\partial_t X^i \partial_t X^i$ potential energy $\sim T \times$ length

lecture notes by D. Tong (recommended!) available web

Remarks:

① One can see this also by considering the non-relativistic limit (see Becker+Becker+Schwarz exercise 2.7 with solution!)

② $T = \frac{1}{2\pi\alpha'}$

← (universal) Regge slope parameter

see K Stelle hep-th 1203.4689 sect 2.2.

$[\alpha'] = \text{mass}^{-2}$

$M_s = (\alpha')^{-1/2}$ string mass scale

$l_s = 2\pi\sqrt{\alpha'}$ string length scale

# Symmetries of the NG-action

- The isometry group of $\mathcal{M}$ leaves invariant the area element

If $g_{\mu\nu} = \eta_{\mu\nu}$ (flat Minkowski metric) $\Rightarrow$ spacetime Poincaré invariance

$$X^\mu(\tau,\sigma) \longrightarrow \Lambda^\mu{}_\nu X^\nu(\tau,\sigma) + b^\mu, \text{ where } \Lambda \in SO(1,d-1),\ b \in \mathbb{R}^{1,d-1}$$

More generally, the space-time isometry group is realised as internal symmetries of the world-line theory

- 2-dimensional reparametrization invariance

$$(\tau,\sigma) \longrightarrow (\tilde{\tau}(\tau,\sigma), \tilde{\sigma}(\tau,\sigma))$$

Worldsheet scalars $\rightarrow$ $X^\mu(\tau,\sigma) \longrightarrow \tilde{X}^\mu(\tilde{\tau},\tilde{\sigma}) = X^\mu(\tau,\sigma)$

True because $S$ is a function of $\Sigma \subset \mathcal{M}$ and we do not care about the parametrization of $\Sigma$

Reparametrizations are a gauge symmetry

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$S$ gives a nice classical theory but not clear how to quantize

# The Polyakov action: Consider

$$S_P[\gamma_{ab}, X^\mu] = -\frac{T}{2}\int_\Sigma d\tau\, d\sigma \sqrt{-\gamma}\,\gamma^{ab}\frac{dX^\mu}{d\xi^a}\frac{dX^\nu}{d\xi^b}g_{\mu\nu}$$

$= G_{ab}$ = induced metric on $\Sigma \in \mathcal{M}$

$$\boxed{\xi^a = (\tau, \sigma)}$$

where we have introduced new fields on $\Sigma$

$\gamma_{ab}$ = Lorentzian world-sheet metric (auxiliary field)

$\gamma = \det(\gamma_{ab})$

One can prove that, solving EOM for the worldsheet metric $\gamma_{ab}$ and then using this in $S_P$ one gets that $S_P$ & $S_{NG}$ are classically equivalent.

# Symmetries of the Polyakov action

WS perspective → global symmetry

- spacetime isometries
  (Poincaré invariance when $M$ = Minkowski)
  and $\gamma$ does not transform
- World sheet reparametrization $\xi^a \longmapsto \tilde{\xi}^a(\xi)$ — diffeomorphisms of $\Sigma$

$$\gamma_{ab}(\xi) \longmapsto \tilde{\gamma}_{ab}(\tilde{\xi}) = \gamma_{cd}(\xi) \frac{\partial \xi^c}{\partial \tilde{\xi}^a} \frac{\partial \xi^d}{\partial \tilde{\xi}^b}$$

symmetric 2-tensor

&

$$X^\mu(\xi) \mapsto \tilde{X}^\mu(\tilde{\xi}) = X^\mu(\xi)$$

WS scalars

special to 2dims

- Weyl invariance ie local scale transformations of the 2dim metric on $\Sigma$
  - $\gamma_{ab} \longmapsto e^{2\omega(\tau,\sigma)} \gamma_{ab}$, $X^\mu$ invariant

$[\sqrt{|\gamma|} \mapsto e^{2\omega}\sqrt{\gamma}\ ;\ \gamma^{ab} \mapsto e^{-2\omega}\gamma^{ab}]$

Weyl invariance is also a gauge symmetry

(p+1)-extended object, $\gamma$ metric on WV $\left\{ \gamma^{ab}\sqrt{|\gamma|} \longrightarrow e^{-2\omega}\gamma^{ab} e^{(p+1)\omega}\sqrt{|\gamma|} \right.$ see Becker+Becker+Schwarz

Remark on dimensional analysis:

We have a notion of length scales & units in both the WS $\Sigma$ and space-time $M$

| | $\boxed{\Sigma}$ | $\boxed{M}$ |
|---|---|---|
| $[X^\mu]$ | 0 | -1 |
| $[\frac{d}{d\xi}]$ | 1 | 0 |
| $[\gamma]$ | 0 | 0 |
| $[d\tau d\sigma]$ | -2 | 0 |
| $[T]$ | 0 | 2 |

recall $T = \frac{1}{2\pi\alpha'}$

$\boxed{\text{mass units}}$

length units $\to$ [mass units]$^{-1}$

$(\hbar = c = 1)$

Can one add terms to action which are

and
- compatible with power counting renormalizability
- consistent with the symmetries of the action

$M$ = Minkowski space, no other fields

* $S_1 = \lambda_1 \int_\Sigma \underbrace{\sqrt{-\gamma}}\, d\tau\, d\sigma$ ⇝ cosmological constant term on $\Sigma$

not Weyl invariant

inconsistent EoM (BBS) ⇒ $\lambda_1 = 0$

* $S_{HE} = \frac{\lambda_2}{4\pi} \int_\Sigma \sqrt{-\gamma}\, R^{(2)}(\gamma)\, d\tau\, d\sigma$ ⇝ Hilbert-Einstein term

↖ WS Ricci scalar

$\mathcal{L}' = \mathcal{L} +$ total deriv.

Integrand is (locally) a total derivative

⇒ does not affect the classical equations of motion

$S_{HE}$ is topological (closed strings $S_{HE} = \lambda\, \chi(\Sigma)$)

Ignore for now but it is an important term in string perturbation theory.

# Gauge fixing the Polyakov action

Choose a convenient gauge to simplify the action

reparametrizations: $\gamma_{ab} \rightarrow e^{2\omega(\tau,\sigma)}\eta_{ab}$ conformal gauge

3 indep degrees of freedom

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Weyl: $e^{2\omega(\tau,\sigma)}\eta_{ab} \longrightarrow \eta_{ab}$ unit gauge

($\gamma_{ab}$ has 3 independent components. Using the conformal gauge leaves only one. Finally, using a Weyl transformation one can gauge away the remaining degree of freedom)

Remark: locally one can prove that one can always choose this gauge $\gamma_{ab} = \eta_{ab}$.
However we do not know if this can be done globally on $\Sigma$ !
There are in fact topological obstructions which are better understood in Euclidean signature.
To deal with Lorentzian signatures one does a "Wick rotation" to Euclidean signature

Polyakov action in conformal gauge; M = Minkowski

$$S_P^{CG}[X^\mu] = -\frac{T}{2}\int d\tau d\sigma \left(-\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X\right)$$

$\Rightarrow$ theory of D massless scalar fields in flat (1+1)-dim space (though one term with the wrong sign)

We also have the equation of motion of the WS metric

$$\frac{\delta S_P}{\delta \gamma^{ab}} = 0$$

The 2-dim energy momentum tensor is given by

$$T_{ab} := -\frac{2}{T}\frac{1}{\sqrt{-\gamma}}\frac{\delta S_P}{\delta \gamma^{ab}} = 0$$

EOM constraint

In conformal gauge

$$T_{ab} = \partial_a X \cdot \partial_b X - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X \cdot \partial_d X$$

(one can use this in $S_P$ to get $S_{NG}$ to show that these actions give classically equiv. theories)

Components of $T_{ab}$:

$$T_{\tau\tau} = \frac{1}{2}\partial_\tau X\cdot\partial_\tau X + \frac{1}{2}\partial_\sigma X\cdot\partial_\sigma X$$

$$T_{\tau\sigma} = \partial_\tau X\cdot\partial_\sigma X$$

$$T_{\sigma\sigma} = \frac{1}{2}\partial_\tau X\,\partial_\tau X + \frac{1}{2}\partial_\sigma X\cdot\partial_\sigma X$$

$$\Longrightarrow \quad \gamma^{ab}T_{ab} = T_{\sigma\sigma} - T_{\tau\tau} = 0$$

this is due to Weyl invariance

ie T is traceless identically

$\Longrightarrow$ only two constraints to impose

One can show that imposing these constraints one obtains the Nambu-Goto action (see eg GSW sect 2.1.3)

# Gauge fixing the Nambu-Goto action

One can (also) use reparametrisations to fix the Nambu-Goto action to the conformal gauge

The induced metric $G_{ab}$ on $\Sigma$ (by the fact that $\Sigma \subset M$) is

$$G_{ab} = \partial_a X \cdot \partial_b X$$

$$S_{NG} = -T\int \sqrt{-G}\, d\tau\, d\sigma = -T\int \left[-(\partial_\tau X\cdot\partial_\tau X)(\partial_\sigma X\cdot\partial_\sigma X) - (\partial_\sigma X\cdot\partial_\tau X)^2\right]^{1/2} d\tau\, d\sigma$$

We can fix:

$$\left.\begin{aligned} G_{\tau\sigma} &= \partial_\tau X\cdot\partial_\sigma X = 0 \\ G_{\tau\tau} + G_{\sigma\sigma} &= \partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X = 0 \end{aligned}\right\}\ \text{conformal gauge}$$

Use these constraints in $S_{NG}$

$$S_{NG} = -T\int\left[-\tfrac{1}{2}\,2(\partial_\tau X\cdot\partial_\tau X)\left(\tfrac{1}{2}\right)\cdot 2(\partial_\sigma X\cdot\partial_\sigma X) - \underbrace{\cancel{(\partial_\sigma X\cdot\partial_\tau X)^2}}_{0}\right]^{1/2} d\tau\, d\sigma$$

$$= -T\int\left[-\frac{1}{4}(\partial_\tau X\cdot\partial_\tau X - \partial_\sigma X\cdot\partial_\sigma X)(\partial_\sigma X\cdot\partial_\sigma X - \partial_\tau X\cdot\partial_\tau X)\right]^{1/2} d\tau\, d\sigma$$

$$\Longrightarrow S_{NG} = -\frac{T}{2}\int \left(-\partial_\tau X\,\partial_\tau X + \partial_\sigma X\,\partial_\sigma X\right) d\tau d\sigma$$

- classically equivalent to Polyakov.
- PS 1 : more on $S_{NG}$

## Next

# Chapter 2

2.1 Classical relativistic point particle

2.2 Classical relativistic string → $S_{NG}$, $S_P$ & symmetries

2.3 General classical solutions