

STRING THEORY I

Lecture 3



Chapter 2

Classical relativistic string

Last lecture:

2.1 Classical relativistic point particle ✓

2.2 Classical relativistic string ✓

This lecture

2.3 Classical solutions

{ 2.3.1 EOM and boundary conditions;
solutions for closed & open strings

2.3.2 Conformal maps

2.3.3 Conformal algebra (next lecture)

In summary

- Polyakov action in conformal gauge $\partial_a = \frac{\partial}{\partial \xi^a}$ $\{\xi^a\} = \{\tau, \sigma\}$
$$S_P^{\text{conf. gauge}}[X^\mu] = -\frac{T}{2} \int_{\Sigma} \underbrace{\partial_a X \cdot \partial^a X}_{-\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X} d\tau d\sigma$$

- EOM for γ_{ab}
$$T_{ab} = -\frac{2}{T} \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{ab}} = 0$$

In the conformal gauge $\rightarrow T_{ab} = \partial_a X \cdot \partial_b X - \frac{1}{2} \eta_{ab} \eta^{cd} \partial_c X \cdot \partial_d X$
which in components:

$$T_{\tau\tau} = \frac{1}{2} (\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X)$$

$$T_{\tau\sigma} = \partial_\tau X \cdot \partial_\sigma X$$

$$T_{\sigma\sigma} = \frac{1}{2} (\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X) = T_{\tau\tau}$$

Note: T_{ab} is traceless

• T_{ab} is conserved

$$\eta^{ab} T_{ab} = -T_{\tau\tau} + T_{\sigma\sigma} = 0 \quad \text{due to Weyl inv.}$$

$$\eta^{ab} \partial_a T_{bc} = 0$$

On the transgression of T (Wom BLT):

(off the lecture)

let S be an action

$$S[\gamma, \phi]$$

metric on $\Sigma \rightarrow$

collection of fields ϕ^i on Σ

which is invariant under Weyl transformations, i.e.

$$S[\tilde{\gamma}, \tilde{\phi}] = S[\gamma, \phi]$$

where the Weyl transformation is

$$\gamma_{ab} \rightarrow \tilde{\gamma}_{ab} = e^{2\omega} \gamma_{ab}, \quad \phi^i \rightarrow \tilde{\phi}^i = e^{d_i \omega} \phi^i.$$

Then

$$0 = \delta S = \int d^2 \Sigma \left\{ -2 \frac{\delta S}{\delta \gamma^{ab}} \gamma^{ab} + \sum_i d_i \frac{\delta S}{\delta \phi^i} \phi^i \right\} \delta \omega$$

$$\text{EOM for } \phi^i: \frac{\delta S}{\delta \phi^i} = 0; \quad \text{EOM for } \gamma^{ab}: \frac{\delta S}{\delta \gamma^{ab}} \propto T_{ab} \Rightarrow \gamma^{ab} T_{ab} = 0$$

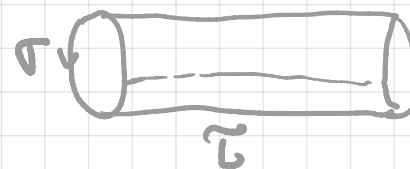
Remark: $\gamma^{ab} T_{ab} = 0$ true without using EOM for ϕ^i iff $d_i = 0$
which is the case of S_p where $\{\phi^i\} = \{X^M\}$

2.3 Classical solutions

We are interested in the equations of motion for the fields X^μ .

$\tau \rightarrow$ time coordinate on Σ

$$-\infty \leq \tau \leq \infty$$



$\sigma \rightarrow$ spatial coordinate on Σ

strings with finite spatial length $\sigma \in [0, \pi]$

2.3.1 Equations of motion and boundary conditions

Writing the action as $S[X^\mu] = \int_{\Sigma} d\tau d\sigma d[\dot{X}^\mu, \partial_a X^\mu]$
a standard computation gives

$$\delta S = \int_{\Sigma} d\tau d\sigma \left\{ \frac{\partial d}{\partial X^\mu} \delta X^\mu + \frac{\partial d}{\partial(\partial_a X^\mu)} \delta \partial_a X^\mu \right\}$$

$$= \int_{\Sigma} d\tau d\sigma \left\{ \underbrace{\partial_a \left(\frac{\partial d}{\partial(\partial_a X^\mu)} \delta X^\mu \right)}_{\text{total derivative}} + \left[\frac{\partial d}{\partial X^\mu} - \partial_a \left(\frac{\partial d}{\partial(\partial_a X^\mu)} \right) \right] \delta X^\mu \right\}$$

$\delta S = 0$: • second terms must vanish $\forall \delta X^\mu$ of the motion

$$\Rightarrow \text{Euler-Lagrange eqs } \frac{\partial d}{\partial X^\mu} - \partial_a \left(\frac{\partial d}{\partial(\partial_a X^\mu)} \right) = 0$$

• 1st term must vanish too

For the Poly action: $S_P^{\text{conf. gauge}}[X^M] = -\frac{T}{2} \int_{\Sigma} \underbrace{\partial_a X \cdot \partial^a X}_{\text{depends only on } \partial_a X \text{ (not on } X^M)} d\tau d\sigma$

• Euler Lagrange eqs for X^M :

$$\boxed{\frac{\delta S}{\delta X^M} = 0} \quad 0 = \frac{\partial}{\partial \xi^a} \left(\frac{\partial S}{\partial (\partial_a X^M)} \right) = \frac{\partial}{\partial \xi^a} \left(-\frac{T}{2} \cdot 2 \partial_a X^M \right)$$

which give

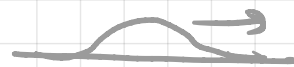
$$\eta^{ab} \partial_a \partial_b X^M = -\partial_\tau^2 X^M + \partial_\sigma^2 X^M = 0$$

two dim wave eq
waves travelling at
 $c=1$

General solution:

$$X^M(\tau, \sigma) = X_R^M(\tau - \sigma) + X_L^M(\tau + \sigma)$$

right moving + left-moving wavefronts



surface terms:

$$\int_{\Sigma} d\tau d\sigma \frac{\partial}{\partial \xi^a} \left(\frac{\partial \mathcal{L}}{\partial (\partial_a X^\mu)} \delta X^\mu \right) = 0$$

$T(\partial_a X^\nu | \eta_{\mu\nu})$

$$0 = -T \int_{\bar{\tau}_i}^{\bar{\tau}_f} d\tau \int_0^\pi d\sigma \left\{ \frac{\partial}{\partial \tau} \left(\eta_{\mu\nu} \partial_\tau X^\mu \delta X^\nu \right) + \frac{\partial}{\partial \sigma} \left(\eta_{\mu\nu} \partial_\sigma X^\mu \delta X^\nu \right) \right\}$$

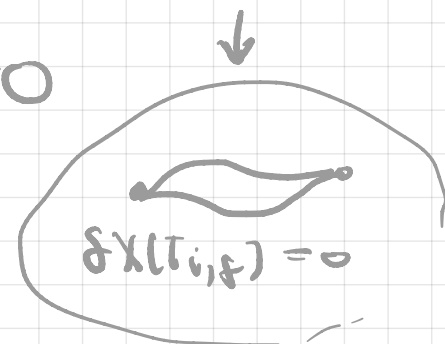
$$\downarrow$$

$$-T \int_0^\pi d\sigma \eta_{\mu\nu} (\partial_\tau X^\mu \delta X^\nu) \Big|_{\bar{\tau}=\bar{\tau}_i}^{\bar{\tau}=\bar{\tau}_f} = 0$$

choose:

$$\delta X^\mu(\bar{\tau}_i, \sigma) = 0, \quad \delta X^\mu(\bar{\tau}_f, \sigma) = 0$$

analogy: particle

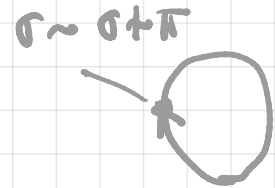


so

$$0 = -T \int_{\bar{\tau}_i}^{\bar{\tau}_f} d\tau \left(\eta_{\mu\nu} \partial_\sigma X^\mu \delta X^\nu \right) \Big|_{\sigma=0}^{\sigma=\pi}$$

surface term: $0 = -T \int_{\tau_i}^{\tau_f} d\tau \left(\eta_{\mu\nu} \partial_\sigma X^\mu \delta X^\nu \right) \Big|_{\sigma=0}^{\sigma=\pi}$

closed strings: periodicity condition
 identifying $\sigma \sim \sigma + \pi$
 (surface term vanishes)



Require

$$\cdot \quad X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \pi)$$

$$\cdot \quad \partial_\sigma X^\mu(\tau, \sigma) = \partial_\sigma X^\mu(\tau, \sigma + \pi)$$

surface term: $0 = -T \int_{\tau_i}^{\tau_f} d\tau \left(\eta_{\mu\nu} \partial_\sigma X^\mu \delta X^\nu \right) \Big|_{\sigma=0}^{\sigma=\pi}$

open strings: boundary conditions on the string endpoints

Newmann: $\partial_\sigma X^\mu(\tau, \pi) = 0, \quad \partial_\sigma X^\mu(\tau, 0) = 0$
ends of the string move freely in M

Dirichlet: $\delta X^\mu \Big|_{\sigma=0, \pi} = 0$ i.e. $X^\mu(\tau, \pi) = c^\mu, \quad X^\mu(\tau, 0) = b^\mu$

↑
end points of the string
are fixed in space-time

the choice of constant space-time vectors
⇒ breaks Poincaré invariance

One can impose Neumann boundary conditions on $d-(p+1)$ coordinates ($d = \dim M$) and Dirichlet conditions on the rest $p+1$ coordinates. In this case the ends of the string are fixed on subspace $\mathcal{Q} \subset M$ of $\dim \mathcal{Q} = p+1$. This subspace is called a Dp-brane with c^μ & b^μ interpreted as the position of the brane. These objects are very important in string theory and we will talk about them later.

Closed strings

general soln

$$X^M(\tau, \sigma) = X_R^M(\tau - \sigma) + X_L^M(\tau + \sigma)$$

Want solns which satisfy the periodicity conditions

$$X^M(\tau, \sigma) = X^M(\tau, \sigma + \pi)$$

$$\partial_\sigma X^M(\tau, \sigma) = \partial_\sigma X^M(\tau, \sigma + \pi)$$

$$\left. \begin{aligned} X_R^M(\tau, \sigma) &= \frac{1}{2} X^M + \frac{1}{2} \ell^2 p^M (\tau - \sigma) + \frac{1}{2} i \ell \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \alpha_n^M e^{-2in(\tau - \sigma)} \\ X_L^M(\tau, \sigma) - X^M(\tau, \sigma - \pi) &= \frac{1}{2} \ell^2 p^M \pi \\ X_L^M(\tau, \sigma) &= \frac{1}{2} X^M + \frac{1}{2} \ell^2 p^M (\tau + \sigma) + \frac{1}{2} i \ell \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \tilde{\alpha}_n^M e^{-2in(\tau + \sigma)} \\ X_L^M(\tau, \sigma + \pi) - X^M(\tau, \sigma) &= \frac{1}{2} \ell^2 p^M \pi \end{aligned} \right\} \begin{aligned} \ell &= (\pi T)^{-1/2} \\ \text{separately} \\ \text{periodic} \\ \text{up to a} \\ \text{zero mode} \end{aligned}$$

where X^M real requires $X^M, p^M \in \mathbb{R}$, $\alpha_{-n}^M = (\alpha_n^M)^*$, $\tilde{\alpha}_{-n}^M = (\tilde{\alpha}_n^M)^*$

Open strings

general soln

$$X^M(\tau, \sigma) = X_R^M(\tau - \sigma) + X_L^M(\tau + \sigma)$$

Want solns which satisfy Newmann boundary conditions

$$\partial_\sigma X^M(\tau, \pi) = 0, \quad \partial_\sigma X^M(\tau, 0) = 0$$

We find

$$X^M(\tau, \sigma) = \underbrace{x^M + \ell^2 p^M \tau}_{\text{av. position}} + i\ell \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \alpha_n^M \omega(n\sigma) e^{-in\tau}$$

av. position $X^M(\tau) = \frac{1}{\pi} \int_0^\pi d\sigma X^M(\tau, \sigma) = \downarrow$

Real X^M : $x^M, p^M \in \mathbb{R}, \quad \alpha_{-n}^M = (\alpha_n^M)^*$

Q. 3. 2

Conserved charges for Poincaré symmetry

Recall Noether's theorem: for ea symmetry in the action, there is a corresponding conserved current

$$J^a = \frac{\delta \mathcal{L}}{\delta(\partial_a \phi)} \delta \phi, \quad \partial_a J^a = 0, \quad \text{for} \quad \mathcal{L} = \mathcal{L}[\phi, \partial_a \phi, \eta_{ab}]$$

$$S[\phi + \delta \phi] = S[\phi]$$

For a space-time translation $\delta X^M = \epsilon^M$

momentum
density

$$J_a^M = T \partial_a X^M(\tau, \sigma)$$

$$(\partial_a J^{aM} = 0)$$

conserved
charge

$$P^M = \int_0^\pi J_0^M(\tau, \sigma) d\sigma = p^M$$

total
momentum

$$= T l^2 \pi p^M$$

$$\partial_\tau X^M = l^2 p^M + \dots$$

↑
true for both
open & closed strings

↖ terms that vanish upon
 σ -integration

For a space-time rotation $\delta X^M = \epsilon^{M\nu} X_\nu$:

$\epsilon^{M\nu}$ antisymmetric matrix

angular
momentum
density

$$J_a^{M\nu}(\sigma, \sigma) = T(X^M \partial_a X^\nu - X^\nu \partial_a X^M)$$

conserved charge

$$M^{M\nu} = \int_0^\pi J_0^{M\nu}(\sigma, \sigma) d\sigma = \begin{cases} L^{M\nu} + E^{M\nu} + \tilde{E}^{M\nu} & \text{closed string} \\ L^{M\nu} + E^{M\nu} & \text{open string} \end{cases}$$

where

$$L^{M\nu} = X^M P^\nu - X^\nu P^M$$

tree mode of angular momentum

$$E^{M\nu} = -i \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} (\alpha_n^M \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^M)$$

oscillator modes
angular momentum

$$\tilde{E}^{M\nu} = -i \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} (\tilde{\alpha}_{-n}^M \tilde{\alpha}_n^\nu - \tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^M)$$

(spin)

Recall: we need to supplement S_p with the constraints from the stress tensor

$$\partial_\tau X \cdot \partial_\tau X - \partial_\sigma X \cdot \partial_\sigma X = 0$$

&

$$\partial_\tau X \cdot \partial_\sigma X = 0$$

Use light cone coordinates: $\sigma^\pm = \tau \pm \sigma$

In these coordinates: $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

metric on Σ : $ds^2 = -d\tau^2 + d\sigma^2 = -d\sigma^+ d\sigma^-$

$$\Rightarrow \eta_{++} = \eta_{--} = 0 \quad \eta_{+-} = \eta_{-+} = -\frac{1}{2}$$

inverse metric

$$\eta^{++} = \eta^{--} = 0$$

$$\eta^{+-} = \eta^{-+} = -2$$

In light cone coordinates,

$$T_{++} = \partial_+ X \cdot \partial_+ X,$$

$$T_{--} = \partial_- X \cdot \partial_- X$$

and $T_{+-} = 2 \partial_+ X \cdot \partial_- X = T_{-+}$

Note T_{ab} is automatically symmetric

$$T_{+-} = T_{-+}$$

The tracelessness condition is: $T_{+-} + T_{-+} = 0$

Then

$$T_{+-} = 0$$

T_{ab} is conserved: $\eta^{ab} \partial_a T_{bc} = 0 \Rightarrow$

$$\begin{aligned} \partial_+ T_{-+} + \partial_- T_{++} &= 0 \\ \partial_+ T_{--} + \partial_- T_{+-} &= 0 \end{aligned}$$

Combining these

$$\boxed{\begin{aligned} T_{+-} &= 0, & \partial_- T_{++} &= 0 \\ & & \partial_+ T_{--} &= 0 \end{aligned}}$$

very powerful

(Lorentzian version of
conds for hol(antihol))

→ next: these imply that there is an infinity of conserved charges

Closed strings

let $f(\sigma^-)$ be an arbitrary function and consider

$$Q_f = \int d\sigma f(\sigma^-) T_{--}(\sigma^-)$$

$\swarrow \partial_+ T_{--} = 0$

$$\begin{aligned} \Rightarrow \frac{\partial}{\partial \tau} Q_f &= \int d\sigma (2\partial_+ - \partial_\sigma)(f(\sigma^-) T_{--}(\sigma^-)) & \partial_+ &= \frac{1}{2}(\partial_\tau + \partial_\sigma) \\ &= - \int d\sigma \partial_\sigma (f(\sigma^-) T_{--}(\sigma^-)) = - \left(f(\sigma^-) T_{--}(\sigma^-) \right) \Big|_{\substack{\sigma=\pi, \tau \text{ fixed} \\ \sigma=0, \tau \text{ fixed}}} \\ &= 0 \quad \text{if } f(\sigma^-) \text{ is periodic} \end{aligned}$$

(Note that $T_{++} = T_{--}$ at end points.)

That is: the current $f T_{--}$ is also conserved!

Since f is arbitrary $\Rightarrow$ there is an infinite set of conserved currents

Similarly: T_{++} is conserved and so is $g T_{++}$, $g = g(\sigma^+)$ periodic

We can get a complete set of conserved charges by taking

$$f(\sigma^-) = e^{2in\sigma^-} \quad n \in \mathbb{Z}$$

$$L_m = \frac{i}{\alpha} \int d\sigma e^{2in\sigma^-} T_{--}(\sigma^-)$$

$$\sigma=0 \rightarrow = \frac{i}{\alpha} \int d\sigma e^{2in\sigma} \partial_- X_\mu \partial_- X_\mu \quad \leftarrow \partial_- X \cdot \partial_- X = \partial_- X_\mu \cdot \partial_- X_\mu$$

$$\Rightarrow L_m = \frac{1}{\alpha} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \quad \text{with} \quad \alpha_0^\mu = \frac{1}{2\ell} p^\mu$$

$$\text{and} \quad L_{-m} = (L_m)^* \quad \text{because } T_{--} \text{ is real}$$

similarly: for $T_{++}(\sigma^+)$, $g(\sigma^+) = e^{2im\sigma^+}$,

$$\tilde{L}_m = \frac{i}{\alpha} \int d\sigma e^{2im\sigma^+} T_{++}(\sigma^+) = \frac{1}{\alpha} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n, \quad \tilde{\alpha}_0^\mu = \frac{\ell}{2} p^\mu$$

Notice that L_m & $\tilde{L}_m$ are the Fourier components of T_{--} & T_{++} respectively. Then setting

$$L_m = 0, \quad \tilde{L}_m = 0 \quad \forall m \in \mathbb{Z}$$

imposes the constraints $T_{++} = 0$ & $T_{--} = 0$.

The vanishing of these charges are equivalent to quadratic constraints on the oscillators α_n^μ & $\tilde{\alpha}_n^\mu$.

Consider these constraints for L_0 & $\tilde{L}_0$

$$L_0 = \frac{1}{\alpha'} \sum_{n \in \mathbb{Z}} \alpha_{-n} \cdot \alpha_n = \frac{\ell^2}{8} p^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = 0$$

$$p^2 = p^\mu p_\mu = p \cdot p$$

$$\tilde{L}_0 = \frac{1}{\alpha'} \sum_{n \in \mathbb{Z}} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = \frac{\ell^2}{8} p^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = 0$$

Then:

$$\frac{e^2}{8} p^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = 0$$

$$\frac{e^2}{8} p^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = 0$$

$$p^2 = p^\mu \cdot p_\mu$$

This implies

$$\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n$$

level matching condition

Recall that p^μ = spacetime momentum $\Rightarrow M^2 = -p^2$

$\therefore$ we have a mass shell condition

$$M^2 = -p^2 = \underbrace{4\pi\alpha'}_{\alpha' = \frac{1}{2\pi\alpha'}} \sum_{n=1}^{\infty} (\alpha_n \cdot \alpha_n + \tilde{\alpha}_n \cdot \tilde{\alpha}_n)$$

Open strings

Let $f(\sigma^+)$ & $g(\sigma^-)$ be arbitrary functions. Then

$$\partial_+(g(\sigma^-) T_{--}) = 0 \quad \& \quad \partial_-(f(\sigma^+) T_{++}) = 0$$

Consider

$$Q_{f,g} = \int d\sigma (f(\sigma^+) T_{++} + g(\sigma^-) T_{--})$$

Then

$$\begin{aligned} \partial_\sigma Q_{f,g} &= \int d\sigma (\partial_\sigma (f(\sigma^+) T_{++}) - \partial_\sigma (g(\sigma^-) T_{--})) \\ &= \left(f(\sigma^+) T_{++} - g(\sigma^-) T_{--} \right) \Big|_0^\pi \end{aligned}$$

$\partial_\sigma = 2\partial_- + \partial_+$ $\partial_\sigma = 2\partial_+ - \partial_-$
 $\sigma^\pm = \tau \pm \sigma$

$\Rightarrow Q$ is conserved if $f(\sigma^+) = g(\sigma^-)$ at $\sigma = 0, \pi$

(i) $\sigma = 0 \Rightarrow f(\tau) = g(\tau)$ so f & g are the same fun

(ii) $\sigma = \pi \Rightarrow f(\tau + \pi) = g(\tau - \pi) = f(\tau - \pi)$

$$\therefore \underline{f(\tau) = f(\tau + 2\pi)}$$

f periodic function
with period 2π

let $f(\sigma^+) = e^{im\sigma^+}$

$g(\sigma^-) = e^{im\sigma^-}$

and consider

$Q_{f,g} \rightarrow L_m = \frac{T}{2} \int_0^\pi d\sigma (e^{im\sigma^+} T_{++} + e^{im\sigma^-} T_{--})$ conserved

$\tau=0 \rightarrow \frac{T}{2} \int_0^\pi d\sigma (e^{im\sigma} T_{++} + \bar{e}^{im\sigma} T_{--})$

$$L_m = \frac{T}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n} \cdot \alpha_n \quad \text{with} \quad \alpha_0^m = \ell p^m$$

L_0 : $L_0 = \frac{T}{2} \ell^2 p^2 + T \sum_{n=1} \alpha_{-n} \cdot \alpha_n$

$L_0 = 0$ is:

$$M^2 = -p^2 = \underbrace{2\pi T}_{\frac{1}{\alpha'}} \sum_{n=1} \alpha_{-n} \cdot \alpha_n$$

mass-shell condition

Next: conformal transformations.