# STRING THEORY I

## Lecture 5

# $\boxed{\text{Chapter } \cancel{4}}$ $\boxed{3}$ Old covariant quantization

Classical theory: field theory on the 2dim world-sheet

$$S_P = -\frac{T}{2}\int_{\Sigma} d\tau\, d\sigma \left(-\partial_\tau X \cdot \partial_\tau X + \partial_\sigma X \cdot \partial_\sigma X\right)$$

in the conformal unit gauge $\gamma_{ab} = \eta_{ab}$.

This is supplemented by the constraints

$$T_{++} = 0 \quad \& \quad T_{--} = 0 .$$

The OCQ approach consists on promoting the canonical fields $X^\mu$ & their conjugate momenta $\Pi^\mu_\tau = T\partial_\tau X^\mu$ to operators and the Poisson brackets $\{\cdot,\cdot\}_{PB}$ to commutators of operators

$$\{\cdot,\cdot\}_{PB} \rightsquigarrow i[\cdot,\cdot]$$

We get the canonical equal time commutation relations ($\tau$)

$$[\Pi^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = -i\,\delta(\sigma-\sigma')\,\eta^{\mu\nu}$$

(with $[X^\mu(\sigma), X^\nu(\sigma')] = 0$ , $[P^\mu(\sigma), P^\nu(\sigma')] = 0$ )

The operators $X^\mu$ & $\Pi^\mu$ are hermitian

$$X^\mu = (X^\mu)^\dagger, \quad \Pi^\mu = (\Pi^\mu)^\dagger$$

The commutation relations for the oscillator modes follow immediately from this:

$$[\hat{p}^\mu, \hat{x}^\nu] = -i\eta^{\mu\nu} \qquad \text{(Heisenberg algebra)}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

$$[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu}$$

The hermiticity of the operators $X^\mu$ & $\Pi^\mu$ imply that the two mode operators $\hat{p}^\mu, \hat{x}^\nu$ are Hermitian while

$$\alpha^\mu_{-m} = (\alpha^\mu_m)^\dagger$$

$$\tilde{\alpha}^\mu_{-m} = (\tilde{\alpha}^\mu_m)^\dagger$$

The set $\{\alpha_m^\mu, \tilde{\alpha}_m^\mu\}$ is an infinite set of harmonic oscillators raising & lowering operators.

We think of

$$\alpha_m^\mu \quad (\tilde{\alpha}_m^\mu) \quad m \geq 0 \text{ as } \quad \text{annihilation ops}$$

$$\alpha_{-m}^\mu \quad (\tilde{\alpha}_{-m}^\mu) \quad m \geq 0 \text{ as } \quad \text{creation ops}$$

Now we can construct the <u>Hilbert space</u>

# Hilbert space:

(1) Define the oscillator vacuum state $|0\rangle_{vac}$

$$\alpha^\mu_m |0\rangle_{vac} = \tilde{\alpha}^\mu_m |0\rangle_{vac} = 0 \quad \forall m \geq 1$$

(2) Next, we build the oscillator Fock spaces:

$$\mathcal{H}^{Fock}_{open} = \text{span}\Big\{ \overbrace{\prod_{i=1}^{k} \alpha^{\mu_i}_{-n_i} |0\rangle_{vac}}^{|0\rangle,\ \alpha^\mu_{-n}|0\rangle,\ \ldots} \Big\}_{n_i \geq 1}$$

$$\mathcal{H}^{Fock}_{closed} = \text{span}\Big\{ \prod_{i=1}^{k} \alpha^{\mu_i}_{-n_i} \prod_{j=1}^{l} \tilde{\alpha}^{\mu_j}_{-n_j} |0\rangle_{vac} \Big\}_{n_i, n_j \geq 1} = \underset{\mathcal{H}_{open}}{\underset{\cong}{\mathcal{H}^{Fock}_R}} \otimes \underset{\mathcal{H}_{open}}{\underset{\cong}{\mathcal{H}^{Fock}_L}}$$

Introduce oscillator number operators

$$N = \sum_{k>0} \alpha_{-k}\cdot\alpha_k \qquad\qquad \tilde{N} = \sum_{k>0} \tilde{\alpha}_{-k}\cdot\tilde{\alpha}_k$$

which count oscillators

$$N\left(\prod_{i=1}^{k} \alpha_{-n_i}^{\mu_i}\,|0\rangle_{vac}\right) = \left(\sum_{i=1}^{k} n_i\right)\left(\prod_{i=1}^{k} \alpha_{-n_i}^{k}\,|0\rangle_{vac}\right)$$

$$N\left(\prod_{i=1}^{k} \tilde{\alpha}_{-n_i}^{\mu_i}\,|0\rangle_{vac}\right) = \left(\sum_{i=1}^{k} n_i\right)\left(\prod_{i=1}^{k} \tilde{\alpha}_{-n_i}^{k}\,|0\rangle_{vac}\right)$$

$$N\,\alpha_{-n}^{\mu}|0\rangle = \sum_{m\geq 1}\left(\alpha_{-n}\cdot\alpha_m\right)\alpha_{-n}^{\mu}|0\rangle =$$

$$= \sum_{m\geq 1} \alpha_{-m}\cdot\Big(\underbrace{[\alpha_m, \alpha_{-n}^{\mu}]}_{n\,\delta_{m-n,0}} + \cancel{\alpha_{-n}^{\mu}\alpha_m}\Big)|0\rangle = n\,\alpha_{-n}^{\mu}|0\rangle$$

One can organize the oscillator states into levels, ie states with a given $N$ ($\hat{N}$) eigenvalue.

For open strings (or R-movers, closed string)

- $N=0$ $\quad |0\rangle_{vac}$
- $N=1$ $\quad \alpha_{-1}^{\mu}|0\rangle_{vac}$
- $N=2$ $\quad \alpha_{-2}^{\mu}|0\rangle_{vac}\,,\quad \alpha_{-1}^{\mu_1}\alpha_{-1}^{\mu_2}|0\rangle_{vac}$

$N=3$ $\quad \alpha_{-3}^{\mu}|0\rangle_{vac}\,,\quad \alpha_{-2}^{\mu_1}\alpha_{-1}^{\mu_2}|0\rangle_{vac}\,,\quad \alpha_{-1}^{\mu_1}\alpha_{-1}^{\mu_2}\alpha_{-1}^{\mu_3}|0\rangle_{vac}$

$\vdots$

## 3 zero-modes

But we have not yet completely specified the states as we still have the zero-mode.

The ground state is in fact $\Psi(x) |0\rangle_{vac}$

is a spatial wave function

In momentum space we can choose states to be eigenstates of the center of mass momentum operator $\hat{p}^\mu$

$$\hat{p}^\mu |K\rangle = K^\mu |K\rangle , \quad K^\mu \in \mathbb{R}^{1,D-1}$$

obeying

$$\langle K' | K \rangle = \delta^{(D)}(K-K')$$

$$\therefore \quad \mathcal{H}_{\text{zero-mode}} \simeq L^2(\mathbb{R}^{1,D-1})$$

The Hilbert space is then

$$\mathcal{H}_{\text{open}} = \mathcal{H}^{\text{Fock}}_{\text{open}} \otimes L^2(\mathbb{R}^{1,D-1})$$

$$\mathcal{H}_{\text{closed}} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}^{\text{Fock}}_{L} \otimes \mathcal{H}^{\text{Fock}}_{R}$$

state $\prod_{i=1}^{n} \alpha^{h}_{-n_i} |0,p\rangle$ $N = \sum_{i=1}^{n} n_i$

momentum eivalue $p^\mu$

closed string $\prod \alpha \cdot \prod \tilde{\alpha} |0,p\rangle$

Problem (or not?) : Consider the state

$$|\varphi\rangle = \alpha^0_{-1}|0;K\rangle$$

Then

$$\langle\varphi|\varphi\rangle = \langle 0;K|\alpha^0_{+1}\,\alpha^0_{-1}|0;K'\rangle = \eta^{00}\,\delta(K-K')$$

$[\alpha^0_{+1},\alpha^0_{-1}]$

$$= -\delta(K-K')$$

Wrong sign! There are negative norm states

ghosts!

However: we have not imposed the constraints

# Normal ordering:

Recall the Witt-generators

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \,, \qquad L_m^\dagger = L_{-m} \quad (m \neq 0)$$

$$\tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \,, \qquad \tilde{L}_m^\dagger = \tilde{L}_{-m} \quad (m \neq 0)$$

are quadratic in the oscillators.

The operators $\alpha_{m-n}$ & $\alpha_n$ commute $\forall n$ unless $m=0$ so the only "problematic" operator is $L_0$ (similar for $\tilde{L}_0$).

Suppose we choose naively

$$L_0 = \frac{1}{2} \sum_{k=-\infty}^{\infty} \alpha_{-k} \cdot \alpha_k = \frac{1}{2} \alpha_0 \cdot \alpha_0 + \frac{1}{2} \sum_{k \geq 1} \alpha_{-k} \cdot \alpha_k + \frac{1}{2} \sum_{k \geq 1} \alpha_k \cdot \alpha_{-k}$$

Then (open string)

$\alpha_0^\mu = \cancel{\frac{p^\mu \ell}{2}}$

$$L_0 |0;p\rangle = \left( \frac{\ell^2 p^2}{2} + \frac{1}{2} \sum_{k \geq 1} \alpha_k \cdot (\alpha_{-k} |0,p\rangle \right)$$

$[\alpha_k^\mu, \alpha_{-k}^\nu] \eta_{\mu\nu}$

$[\alpha_k^\mu, \alpha_{-k}^\nu] = k \eta^{\mu\nu}$

Note corrections!

$$= \left( \frac{\ell^2 p^2}{2} + \frac{1}{2} \sum_{k \geq 1} k D \right) |0,p\rangle$$

$$= \left( \frac{\ell^2 p^2}{2} + \frac{1}{2} D \sum_{k \geq 1} k \right) |0,p\rangle$$

Zeta function regularization

$$\sum_{k \geq 1} k = \zeta(-1) = -\frac{1}{12}$$

$$= \left( \frac{\ell^2 p^2}{2} - \frac{1}{24} D \right) |0,p\rangle$$

so maybe not!

To account properly for the normal ordering one can try to define the quantum operator as

$$L_0 = \frac{1}{2} \sum_{n \in \mathbb{Z}} \underbrace{:\alpha_{-n} \cdot \alpha_n:}$$

normal ordered product

(lowering ops to the right)

by

closed strings
$$\begin{cases} L_0 \equiv \frac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{n\geq 1} \alpha_{-n} \cdot \alpha_n = \frac{\ell^2 p^2}{8} + N \\ \tilde{L}_0 \equiv \frac{1}{2}\tilde{\alpha}_0 \cdot \tilde{\alpha}_0 + \sum_{n\geq 1} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n = \frac{\ell^2 p^2}{8} + \tilde{N} \end{cases}$$

$\Rightarrow L_0 |0;p\rangle = \frac{\ell^2 p^2}{8}|0;p\rangle$

open strings
$$L_0 = \frac{\ell^2 p^2}{2} + N$$

Virasoro algebra We need to check the commutator algebra of the operators $L_m$ ($\tilde{L}_m$)

PS 2: a direct computation gives

$$\boxed{[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12}(m^3 - m)\,\delta_{m+n,0}}$$

(similar for $[\tilde{L}_m, \tilde{L}_n]$ for closed strings)

This is the Virasoro algebra with central charge D.

The Virasoro algebra is a central extension of the Witt-algebra

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathrm{Vir} & \longrightarrow & \mathrm{Witt} & \longrightarrow & 0 \\
 & & \nearrow & & \Uparrow & & \Uparrow & & \\
 & \hat{c}\,;\ [\hat{c}, L_n] = 0 & & & \hat{c} = 0 & & \mathrm{Vir}/\langle \hat{c} \rangle & &
\end{array}
$$

The central extension is related to an anomaly of the Weyl invariance (more later).

The global $sl(2)$ algebra generated by $\{L_0, L_1, L_{-1}\}$ (or $\{\tilde{L}_0, \tilde{L}_1, \tilde{L}_{-1}\}$) is not anomalous.

Remark: An alternative way to deal with the normal ordering is to replace the naive operator $L_0$ above with

$$L_0 - a$$

where $a$ = constant. This also gives the central charge term, see GSW.

# Constraints

Impose the constraints in the quantum theory to identify the physical states

Another problem arises due to appearance of the central charge $\boxed{\text{if}}$ we define the physical as the states $|\psi\rangle$ which satisfy

$$L_m |\psi\rangle = 0 \qquad \forall m$$

In fact, there is a contradiction.

Consider $[L_m, L_n]|\psi\rangle$ ← physical state $L_m|\psi\rangle = 0 \ \forall m$

(a) $[L_m, L_n]|\psi\rangle = (L_m L_n - L_n L_m)|\psi\rangle = 0 \quad \forall m, n$

(b) $[L_m, L_n]|\psi\rangle = \Big[\underbrace{(m-n)\cancel{L_{m+n}} + \frac{D}{12}(m^3-m)\delta_{m+n,0}}\Big]|\psi\rangle$ $\neq 0 \ m = -n$

But for $n = -m$

$[L_m, L_{-m}]|\psi\rangle = \frac{D}{12}(m^3-m)|\psi\rangle$

Imposing all constraints leads to a $\boxed{\text{trivial}}$ Hilbert space when $\boxed{D \neq 0}$ !

Instead we define physical states $\phi, \psi \in \mathcal{H}_{phys}$ by the constraints $\langle \psi | L_m | \phi \rangle = 0 \quad \forall m \neq 0$

<u>Definition</u>: a state $|\phi\rangle$ is physical if

$\{L_m, m \geq 1\}$ subalgebra →

- $L_m |\phi\rangle = 0 \quad \forall m \geq 1$
- $(L_0 - a)|\phi\rangle = 0$ for fixed $a$

normal ordering constant.

Virasoro constraints or physical state conditions

Remark: $\langle \psi | L_m | \phi \rangle = 0 \quad \forall m \geq 1$

$\langle \psi | L_m | \phi \rangle^* = \langle \phi | L_{-m} | \psi \rangle = 0 \quad \forall m \leq -1$

$\langle \phi | (L_0 - a) | \psi \rangle = 0$

Important remark: the generators of D-dim spacetime Poincaré symmetries have no normal ordering ambiguities as can be seen from

$$P^\mu = p^\mu$$

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \frac{1}{n} \left( \alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_{-n} \alpha^\mu_n \right)$$

open string

$$M^{\mu\nu} = \int d\sigma \left( X^\mu P^\nu - P^\nu X^\mu \right)$$

Moreover: $[P^\mu, L_m] = [M^{\mu\nu}, L_m] = 0 \quad \forall m$

$[L_m, \alpha^\mu_n] = -n\, \alpha^\mu_{m+n}$

$P^\mu$ & $M^{\mu\nu}$ preserve the physical state conditions. This means that the phys. decomposes into representations of $ISO(1, D-1)$ (covariant quantization).

# Massshell & level-matching conditions: the $L_0$ (& $\tilde{L}_0$) condition

## Closed string

mass shell condition

$$(L_0+\tilde{L}_0-2a)|\varphi,k\rangle=0 \iff \overbrace{\left(\frac{\ell^2p^2}{4}+N+\tilde{N}-2a\right)}|\varphi,k\rangle=0$$

so
$$\frac{\ell^2M^2}{4}=\underline{N+\tilde{N}-2a} \Rightarrow \underline{\alpha'M^2=2(N+\tilde{N}-2a)}$$

$\alpha'=\ell/2$

level matching condition

$$(L_0-\tilde{L}_0)|\varphi,k\rangle=0 \iff (N-\tilde{N})|\varphi,k\rangle=0 \text{ so } \underline{N=\tilde{N}}$$

## Open string

mass shell condition

$$(L_0-a)|\varphi,k\rangle=0 \iff \left(\frac{\ell^2p^2}{2}+N-a\right)|\varphi,k\rangle=0 \text{ so } \underline{\alpha'M^2=N-a}$$

## Level 0    ground $|0, k\rangle$

As an exercise, you can show that

$$[N, L_m] = -m L_m$$

so $L_m$ shifts N-level by $-m$    (similar for $\tilde{N}$ & $\tilde{L}$)

$\Rightarrow$ at level zero we only need to check the $L_0$ conditions.

The $L_0$-conditions at level $N=0$ are

open strings    $\alpha' M^2 = -a$

closed strings    $\alpha' M^2 = -4a$

$\therefore$    $a < 0$    massive ground state

$a = 0$    massless ground state

$a > 0$    tachyonic ground state (!)

# Level one states & dealing with ghosts

Recall: earlier we encountered a problem with negative norm states (ghosts) at level one of the open string (eg: $\alpha^0_{-1}|0;K\rangle$). We want to see if this issue remains after applying the constraints

Consider a general level one open string state

$$\zeta \cdot \alpha_{-1}|0;K\rangle = |\zeta;K\rangle \qquad \zeta \in \mathbb{R}^{1,D-1}$$

polarization vector

and impose the physical state conditions for $L_0$ & $L_{+1}$.

The $L_n$ conditions for $n \geq 2$ are satisfied automatically once the $L_0$ & $L_{+1}$ are imposed.

- mass-shell: $-\alpha' k^2 = \alpha' M^2 = 1 - a$
- $L_{+1}$ condition: $L_{+1}\,(\zeta \cdot \alpha_{-1})|0;k\rangle = \eta_{\mu\nu}\zeta^\mu [L_{+1}, \alpha^\nu_{-1}]|0;k\rangle$

$$= \eta_{\mu\nu}\zeta^\mu \alpha_0^\nu |0;k\rangle = \ell(\zeta \cdot k)|0;k\rangle$$

$$L_{+1}|\zeta;k\rangle = 0 \iff \zeta \cdot k = 0$$

OTOH

The norm of a general level 1 state $|\zeta, k\rangle$ is

$$\langle \zeta; k|\zeta'; k'\rangle = \langle 0; k|\,(\zeta \cdot \alpha_{+1})\,(\zeta' \cdot \alpha_{-1})|0;k'\rangle$$

$$= (\zeta \cdot \zeta')\langle 0; k|0;k'\rangle = (\zeta \cdot \zeta')\,\delta(k-k')$$

For $\zeta = \zeta'$: require $\zeta^2 \geq 0$ to avoid ghosts

In summary we require :

$\xi^2 \geq 0$ polarization is light like or space like

$\xi \cdot K = 0$ transverse polarization

$\alpha' K^2 = a - 1$ K is spacelike if $a > 1$
light like if $a = 1$
timelike if $a < 1$

If $a < 1$ K is timelike, $\xi \cdot K = 0 \Rightarrow \xi$ spacelike

$a = 1$ K is null , $\xi$ spacelike or null

$a > 1$ K is spacelike, $\xi \cdot K = 0 \Rightarrow \xi$ timelike

$\Rightarrow$ ghosts !

Thus we reject $a > 1$ ie require $a \leq 1$

However the ground state is a tachyon for $0 < a \leq 1$

Critical theory: $a=1$ : $K^2=0$ $(\zeta^2 \geq 0)$

$\zeta \cdot \alpha_{-1}|0,K\rangle = |\zeta, K\rangle$ is a massless state

Consider now the state $|K;K\rangle = K\cdot\alpha_{-1}|0,K\rangle$
Clearly, this state has zero norm (so it is physical)
and has "longitudinal" polarization

Moreover, this state is orthogonal to <u>all</u> physical states

$$\langle K;K|\zeta;K'\rangle = (K\cdot\zeta)\,\delta(K-K') = 0$$

So the longitudinal polarization decouples
leaving only $D-2$ physical polarizations (like a photon!)

However: for $a=1$ ground state is a tachyon

Note that for the longitudinal degree of freedom the decoupling is due to fact that $|K;K\rangle$ is a "pure gauge" state in the following sense. Consider

$$L_{-1}|0;K\rangle = eK\cdot\alpha_{-1}|0;K\rangle = e|K;K\rangle$$

ie $|K;K\rangle$ created by the action of $L_{-1}$.

Recalling that $L_{-1}$ is a generator of a conformal transformation, we say that $|K;K\rangle$ is a pure gauge" state.

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# Old covariant quantization

- Hilbert space (without constraints)
- Normal ordering
- Virasoro algebra
- Imposing the constraints and the phys
- Mass shell & level-matching conditions
- level 0 & 1 ; dealing with ghosts

Next: more on physical states