STRING THEORY I

Lecture 6

$\boxed{3}$ $\cancel{\boxed{4}}$ Old covariant quantization

last lecture

- Hilbert space (without constraints)
- Normal ordering & Virasoro algebra
- Imposing the constraints and the phys
- Mass shell & level-matching conditions
- level 0 & 1 states dealing with ghosts

this lecture

- the phys, spurious states, null states, ghosts
- critical dimension

Summarize last lecture :

① Hilbert space (of states before imposing constraints)

vacuum annihilated by α^μ_n

+ve

Let

$$\mathcal{H}^{Fock} = \text{Span}\left\{ \prod_{i=1}^{K} \alpha^{\mu}_{-n_i} |0\rangle \right\} = \bigoplus_{N=1}^{\infty} \mathcal{H}^{Fock}[N]$$

Fock space of open string

creation op

eValue of number op N

Then

$$\mathcal{H}_{open} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}^{Fock}$$

ground state

$$\mathcal{H}_{closed} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}^{Fock} \otimes \widetilde{\mathcal{H}}^{Fock}$$

right

left $\tilde{\alpha}$

② Virasoro algebra

$$L_m = \frac{1}{2} \sum_{k=-\infty}^{\infty} \alpha_{m-k} \cdot \alpha_k \qquad m \neq 0$$

normal ordered

$$L_0 = \frac{1}{2} \alpha_0 \cdot \alpha_0 + N \qquad N = \sum_{k=1}^{\infty} \alpha_{-k} \cdot \alpha_k$$

number operator

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}$$

Virasoro algebra with central charge $c = D$

(similarly for $\tilde{L}$)

③ Physical states:

$$\mathcal{H}_{\text{phys}} = \Big[\underbrace{\bigcap_{m=1}^{\infty} \mathrm{Ker}(L_m)}_{\forall m \geq 1:\ L_m|\psi\rangle = 0} \Big] \cap \underbrace{\mathrm{Ker}(L_0 - a)}_{(L_0 - a)|\psi\rangle = 0}$$

physical states

$\forall m \geq 1: L_m|\psi\rangle = 0$ & $(L_0 - a)|\psi\rangle = 0$

normal ordering constant.

(4) level 0 & level 1 physical states

level zero : $|0;k\rangle$ ground state with $\alpha' k^2 = a$

giving states which are massive for $a<0$ $\quad k^2=-M^2$

massless for $a=0$

tachyons for $a>0$

level one : $|\xi;k\rangle = (\xi\cdot\alpha_{-1})|0;k\rangle$

general state $\quad \uparrow$ polarization $\xi^\mu \in \mathbb{R}^{1,D-1}$

These obey $\alpha' k^2 = a-1$, $\xi\cdot k = 0$ & norm $= \xi^2$

Need to require $a \leq 1$ to avoid ghosts

We studied the threshold case where $\underline{a=1}$ so $K^2=0$:

There is a state with zero norm
$$|K;K\rangle = (K\cdot\alpha_{-1})|0;K\rangle$$
which is a physical ($K^2=0$) "longitudinally polarized" state and, moreover, transverse to all physical states.
Hence, the longitudinal polarization decouples and we are left with $D-2$ physical polarizations

Note that
- the ground state for $a=1$ is a tachyon
- $|K;K\rangle$ is created by the action of L_{-1}
$$L_{-1}|0;K\rangle = e|K;K\rangle$$
ie $|K;K\rangle$ is "pure gauge"

Definition: a state is called spurious if it is orthogonal to all physical states and obeys

$$(L_0 - a)|\psi\rangle = 0$$

A spurious state which is also physical is orthogonal to itself, ie it has zero norm. These are called null states.

An example of a null state is the state $|K;k\rangle$ at level 1 & $a=1$ as we described.

$|\psi\rangle$ spurious $\Rightarrow$ $\langle\phi|\psi\rangle = 0$ $\forall\, |\phi\rangle \in \mathcal{H}_{phys}$

If $|\psi\rangle$ is also physical $\Rightarrow$ $\langle\psi|\psi\rangle = 0$

Recall that in gauge theories, a consequence of residual symmetries is that one expects to find states which are pure gauge states. These should be "quotiented out". That is we want to identify

$$|\psi\rangle_{phys} \sim |\psi\rangle_{phys} + |\chi\rangle_{null}$$

So

$$\mathcal{H} = \mathcal{H}_{phys} / \mathcal{H}_{null}$$

In the example at level 1 with $a=1$, null states are created by the action of L_{-1} on the ground state, and can be quotiented.

Consider the state $L_{-1}|0,K\rangle$

The state $L_{-1}|0;K\rangle$ is manifestly perpendicular to physical states.

$\langle\psi|L_{-1}|0;K\rangle = 0 \qquad |\psi\rangle \in \mathcal{H}_{phys}$

Now $(L_0 - a)L_{-1}|0;K\rangle = L_{-1}(L_0 - a + 1)|0;K\rangle$

Thus $L_{-1}|0;K\rangle$ is spurious when

$$L_0|0;K\rangle = (a-1)|0;K\rangle$$

$$L_0 L_{-1} = \underbrace{[L_0, L_{-1}]}_{L_1} + L_{-1}L_0 = L_{-1}(L_0 + 1)$$

The state $L_{-1}|0;k\rangle$ is also physical if $\underline{a=1}$

Check:

The state $L_{-1}|0;k\rangle$ is physical if $L_m L_{-1}|0;k\rangle = 0 \quad \forall m \geq 1$

$\boxed{L_1}$

$$L_1 L_{-1}|0;k\rangle = (2L_0 + L_{-1}L_1)|0;k\rangle$$

$$= 2L_0|0;k\rangle = 2(a-1)|0;k\rangle = 0 \quad \text{iff } \underline{a=1}$$

$\boxed{L_m}$ $m \geq 2$

$$L_m L_{-1}|0;k\rangle = \left((m+1)\underbrace{L_{m-1}}_{m-1\geq 1} + L_{-1}L_m\right)|0;k\rangle = 0$$

In fact: all spurious states are of the form

$$|\xi\rangle = \sum_{m=1}^{2} L_{-m} |\tilde{\chi}_m\rangle \quad \text{with} \quad L_0 |\tilde{\chi}_m\rangle = (a-m) |\tilde{\chi}_m\rangle$$

This series can be equivalently written as

$$|\xi\rangle = L_{-1} |\chi_1\rangle + L_{-2} |\chi_2\rangle \quad \text{with} \quad L_0 |\chi_m\rangle = (a-m) |\chi_m\rangle$$

This is because for $m \geq 3$: can replace L_{-m} by commutators

$$[L_{-p}, L_{-q}] = (-p+q) L_{-p-q}, \quad 1 < p, q < m$$

eg $\quad [L_{-1}, L_{-2}] = L_{-3}$

$$L_{-3} |\tilde{\chi}_3\rangle = [L_{-1}, L_{-2}] |\tilde{\chi}_3\rangle = L_{-1} (L_{-2} |\tilde{\chi}_3\rangle) - L_{-2} (L_{-1} |\tilde{\chi}_3\rangle)$$

$$= L_{-1} |\chi_1'\rangle + L_{-2} |\chi_2'\rangle$$

etc. (exercise)

<u>Check</u>: mostly an exercise

$$|\xi\rangle = L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle \qquad L_0|\chi_m\rangle = (a-m)|\chi_m\rangle$$

- clearly $\langle\varphi|\xi\rangle = 0 \quad \forall \ |\varphi\rangle \in \mathcal{H}_{phys}$
- $(L_0 - a)|\xi\rangle = 0$

so $|\xi\rangle$ is spurious. To see that all spurious states have this form see GSW p 83

Constructing null states

Consider $|\xi\rangle = L_{-1}|\chi\rangle$ s.t. $L_m|\chi\rangle = 0 \;\; \forall m>0$
$L_0|\chi\rangle = (a-1)|\chi\rangle$

Thus: check $L_m|\xi\rangle = 0 \;\; \forall m \geq 1$

$$L_{+1}|\xi\rangle = \left(\underbrace{[L_{+1}, L_{-1}]} + \cancel{L_{-1}L_{+1}}\right)|\chi\rangle$$
$$= 2L_0|\chi\rangle = 2(a-1)|\chi\rangle = 0 \quad \text{iff } \underline{a=1}$$

$L_m|\xi\rangle = 0 \;\; \forall m \geq 2$

eg $L_{+2}|\xi\rangle = ([L_{+2}, L_{-1}] + L_{-1}L_{+2})|\chi\rangle = 3L_{+1}|\chi\rangle = 0$

So for $\underline{a=1}$ we get an $\boxed{\text{infinite}}$ set of null states.
The example we had earlier $|K;K\rangle \sim L_{-1}|0;K\rangle$ is the simplest case in this family.

Consider now, the spurious state

$$|\xi\rangle = (L_{-2} + \gamma L_{-1}^2)|\chi\rangle \quad \text{with} \quad L_m|\chi\rangle = 0,\ m>0$$
$$L_0|\chi\rangle = (a-2)|\chi\rangle$$

Then $L_m|\xi\rangle = 0 \quad \forall\ m \geq 3$

$$L_1|\xi\rangle = \left([L_1, L_{-2}] + \cancel{L_{-2}L_{+1}} + \gamma([L_1, L_{-1}] + L_{-1}L_1)L_{-1}\right)|\chi\rangle$$
$$= \left(3L_{-1} + 2\gamma L_0 L_{-1} + \gamma L_{-1}([L_1, L_{-1}] + \cancel{L_{-1}L_1})\right)|\chi\rangle$$
$$= (3L_{-1} + 2\gamma(\underbrace{[L_0, L_{-1}]}_{+L_{-1}} + L_{-1}L_0) + 2\gamma L_{-1}L_0)|\chi\rangle$$
$$= ((3+2\gamma)L_{-1} + 4\gamma(a-2)L_{-1})|\chi\rangle$$
$$= (3 + 2\gamma(2a-3))L_{-1}|\chi\rangle$$

$L_1|\xi\rangle = 0$ iff $\gamma = \frac{3}{2(3-2a)}$ $\quad \left(\gamma = \frac{3}{2},\ a=1\right)$

$$L_{+2}|\xi\rangle = L_{+2}(L_{-2} + \gamma L_{-1}^2)\,|\chi\rangle$$

$$= \{\, [L_{+2}, L_{-2}] + \cancel{L_{-2}L_{+2}} + \gamma([L_{+2}, L_{-1}] + L_{-1}L_{+2})L_{-1} \}|\chi\rangle$$

$$= \{\, 4L_0 + \frac{D}{12}\cdot 6 + 3\gamma L_{+1}L_{-1} + \cancel{\gamma L_{-1}\cdot 3L_{+1}} \}|\chi\rangle$$

$$= \{\, 4L_0 + \frac{1}{2}D + 3\gamma\cdot 2L_0 \}\,|\chi\rangle$$

$$= \{\, 2(2+3\gamma)(a-2) + \frac{1}{2}D \}\,|\chi\rangle$$

$L_{+2}|\xi\rangle = 0$ iff $D = 4(2-a)(2+3\gamma)$

For $\underline{a=1}$: $\underline{D=26}$ critical bosonic string

← critical dim

there are $\boxed{\text{more}}$ null states of the form

$$(L_{-2} + \frac{3}{2}L_{-1}^2)\,|\chi\rangle, \quad L_m|\chi\rangle = 0 \;\; m>0, \;\; (L_0+1)|\chi\rangle = 0$$

let $a=1$:

This is the threshold value as for $a>1$ there are ghosts in the phys.

Consider the level-2 state $|\phi\rangle$ with momentum K

$$|\phi,K\rangle = [c_1\, \alpha_{-1}\cdot\alpha_{-1} + c_2\, \alpha_{-2}\cdot\alpha_0 + c_3\, (\alpha_{-1}\cdot\alpha_0)(\alpha_{-1}\cdot\alpha_0)]|0;K\rangle$$

onshell mass condition $-\alpha' K^2 = 1$ $\quad(\alpha' M^2 = N-a)$

$L_1|\phi,K\rangle = 0$ iff $c_1 + c_2 - 2c_3 = 0$ $\quad [L_m, \alpha_n^\mu] = -n\,\alpha_{m+n}^\mu$

$L_2|\phi,K\rangle = 0$ iff $Dc_1 - 4c_2 - 2c_3 = 0$

so $|\phi,K\rangle = [\alpha_1\cdot\alpha_1 + \frac{1}{5}(D-1)\,\alpha_{-2}\cdot\alpha_0 + \frac{1}{10}(D+4)(\alpha_{-1}\cdot\alpha_0)(\alpha_{-1}\cdot\alpha_0)]|0,K\rangle$

is a physical state for any D $\quad(c_1=1)$

Norm $\langle \phi; k | \phi; k' \rangle = \frac{2}{25}(D-1)(D-26)\,\delta(k-k')$

- no ghosts : $1 \leq D \leq 26$
- null state when $D=1$ or $D=26$

Therefore when $a=1$ & $D=26$ there are $\boxed{\text{more}}$ null states

1972 Brower ; Goddard, Thorn No ghost theorem

↳ For $a=1$ and $D=26$ the phys has no ghosts

Note : there are no ghosts for $a \leq 1$ & $1 \leq D \leq 25$
but these theories are inconsistent at the level of string loops (need to look at one loop interactions)

OCQ : $a = 1$ $D = 26$ needs 1-loop interactions to prove

(no ghosts, many null states)

LCQ : . $a = 1$ $D = 26$ follows by requiring Lorentz spacetime invariance

. manifestly ghost free

BRST quantization : $a = 1$, $D = 26$ required for quantum gauge (conformal) invariance.