STRING THEORY I

Lecture 7

3

Old covariant quantization

so far

- Hilbert space (without constraints)
- Normal ordering & Virasoro algebra
- Imposing the constraints and the phys
- Mass shell & level-matching conditions
- level 0 & 1 states dealing with ghosts
- the phys, spurious states, null states, ghosts
- critical dimension

Today: comments on the no ghost theorem for subcritical strings, closed string spectrum

<u>Open strings</u>: we have

- $a > 1$ $D > 26$: there are ghosts in $\mathcal{H}_{phys}$

- $a = 1$ $D = 26$ critical string
  there are two infinite families of null states

  $\mathcal{H}_{phys}$: we have only imposed half the constraints
  $(L_m |\phi\rangle = 0 \quad m \geq 1 \quad (L_0 - a)|\phi\rangle = 0)$
  The remaining charges L_{-m} give redundancies associated to residual gauge symmetries and we consider
  $$|\psi\rangle \sim |\psi\rangle + |\varphi\rangle_{null}$$

- $a \leq 1$ $D \leq 25$ (subcritical strings)
  no inconsistencies at tree-level [no ghosts]

sub Critical string states: no ghosts for $1 \leq D \leq 25$, $a \leq 1$

Consider states with

$$k = (k^0, k^1, \dots, k^{25}, \xi) \qquad \text{for fixed } \xi \in \mathbb{R}$$

$$N_{25} = \sum_{k=1}^{\infty} \alpha^{25}_{-k} \alpha^{25}_{k} \equiv 0 \qquad \text{so no } X^{25} \text{ oscillators activated}$$

These states obey

$a=1$

$$\alpha' k^2 = \alpha'(k_i k^i + \xi^2) = 1 - N = 1 - N_{0\text{-}24}$$

$i = 0, \dots, 25$ number operator with 25 operator omitted

$$\Rightarrow \quad \alpha' k_i k^i = (1 - \alpha' \xi^2) - N_{0,24}$$

which can be identified to $D=25$ mass shell condition for $a = 1 - \alpha' \xi^2$ < 1

$\Rightarrow$ no ghost theorem for the critical string implies the no ghost theorem for the subcritical string.

Physical states of the closed string

$N = 0, 1$

$$\mathcal{H}_{closed} = L^2(\mathbb{R}^{1,D-1}) \otimes \mathcal{H}_{Fock} \otimes \tilde{\mathcal{H}}_{Fock}$$

$$\prod_{j=1}^{\ell} \alpha_{-n_i}^{m_i} \quad \prod_{i=1}^{\ell} \tilde{\alpha}_{-m_i}^{\nu_i} \quad |0; k\rangle$$

Physical state conditions

$$(L_0 + \tilde{L}_0 - 2a)|\phi\rangle = 0 \iff \alpha' k^2 = 4a - 2(N + \tilde{N})$$

$a = 1$

$$= 2(2 - N - \tilde{N})$$

$$(L_0 - \tilde{L}_0)|\phi\rangle = 0 \iff N = \tilde{N}$$

$$L_m|\phi\rangle = 0 \qquad \tilde{L}_m|\phi\rangle = 0 \qquad m \geq 1$$

level - 0

$N = \tilde{N} = 0$

ground

$|0; k\rangle$ $\quad \alpha' k^2 = 4a \overset{a=1}{=} 4$

level - 1

$N = \tilde{N} = 1$

$$|\Omega; k\rangle = \Omega_{\mu\nu} \alpha^{\mu}_{-1} \tilde{\alpha}^{\nu}_{-1} |0; k\rangle, \quad \alpha' k^2 = 4a - 4 \overset{a=1}{=} 0$$

spacetime two tensor $\qquad$ massless

Decompose $\Omega_{\mu\nu}$ into Lorentz irreps

$$\Omega_{\mu\nu} = \underbrace{\gamma_{(\mu\nu)}}_{\text{traceless symmetric}} + \underbrace{\varphi \eta_{\mu\nu}}_{\text{trace part}} + \underbrace{b_{[\mu\nu]}}_{\text{antisymmetric}}$$

$$D^2 = \frac{1}{2}D(D+1) - 1 + 1 + \frac{1}{2}D(D-1)$$

massive states in reps $SO(D-1)$; massless states $SO(D-2)$ (little group

Impose constraints: L_{+1} & $\tilde{L}_{+1}$

$(\Rightarrow L_m = 0$ & $\tilde{L}_m = 0 \quad \forall m \geq 2)$

$$L_1 |\Omega; K\rangle = \Omega_{\mu\nu} L_1 \alpha^{\mu}_{-1} \tilde{\alpha}^{\nu}_{-1} |0; K\rangle$$

$$= \Omega_{\mu\nu} (\alpha^{\mu}_{0} \tilde{\alpha}^{\nu}_{-1}) |0; K\rangle$$

$$= \frac{\ell}{2} K^{\mu} \Omega_{\mu\nu} \tilde{\alpha}^{\nu}_{-1} |0; K\rangle$$

$$\tilde{L}_1 |\Omega; K\rangle = \frac{\ell}{2} K^{\nu} \Omega_{\mu\nu} \alpha^{\mu}_{-1} |0; K\rangle$$

$\boxed{\gamma}$

$$\frac{\ell}{2} K^\mu \gamma_{\mu\nu} \tilde{\alpha}^\nu_{-1} |0; K\rangle = 0 \quad \text{if} \quad K^\mu \gamma_{\mu\nu} = 0$$

$$\frac{\ell}{2} K^\nu \gamma_{\mu\nu} \alpha^\mu_{-1} |0; K\rangle = 0 \quad \text{if} \quad K^\nu \gamma_{\mu\nu} = 0$$

$\gamma_{\mu\nu}$ polarization is transverse traceless

$\boxed{b}$

$$\frac{\ell}{2} K^\mu b_{\mu\nu} \tilde{\alpha}^\nu_{-1} |0; K\rangle = 0 \quad \text{if} \quad K^\mu b_{\mu\nu} = 0$$

$$\frac{\ell}{2} K^\nu b_{\mu\nu} \alpha^\mu_{-1} |0; K\rangle = 0 \quad \text{if} \quad K^\nu b_{\mu\nu} = 0$$

$b_{\mu\nu}$ polarization is transverse antisymmetric

$\boxed{\varphi}$

$$\frac{\ell}{2} \varphi K \cdot \tilde{\alpha}_{-1} |0; K\rangle = 0 \quad \text{if} \quad K = 0$$

$$\frac{\ell}{2} \varphi K \cdot \alpha_{-1} |0; K\rangle = 0 \quad \text{if} \quad K = 0$$

this is not a physical state!

↳ see later

Redundancies: consider $|\chi\rangle = (\xi\cdot\tilde{\alpha}_{-1})|0;K\rangle$

$$L_{-1}\left((\xi\cdot\tilde{\alpha}_{-1})|0;K\rangle\right) = (\xi\cdot\tilde{\alpha}_{-1})\,L_{-1}|0;K\rangle$$

$$= (\xi\cdot\tilde{\alpha}_{-1})\,\frac{1}{2}\sum_{n\in\mathbb{Z}}\alpha_{-1-n}\cdot\alpha_n|0;K\rangle$$

$$= (\xi\cdot\tilde{\alpha}_{-1})\,\frac{1}{2}\left(\alpha_0\cdot\alpha_{-1} + \alpha_{-1}\cdot\alpha_0\right)|0;K\rangle$$

$$= \frac{\ell}{2}(\xi\cdot\tilde{\alpha}_{-1})(K\cdot\alpha_{-1})|0;K\rangle$$

$$= \frac{\ell}{2}\underbrace{K_\mu\,\xi_\nu}\,\alpha_{-1}^\mu\,\tilde{\alpha}_{-1}^\nu|0;K\rangle$$

so states with $\Omega_{\mu\nu} = K_\mu\xi_\nu$ are spurious.

They are also physical if $K^\mu(K_\mu\xi_\nu) = 0$ $\quad K^\nu K_\mu \xi_\nu$

ie if $K^2 = 0$ & $K\cdot\xi = 0$

Then we have the invariances

$$\gamma_{\mu\nu} \longrightarrow \gamma_{\mu\nu} + \xi_\mu K_\nu + \xi_\nu K_\mu \qquad \xi \cdot K = 0$$

$$b_{\mu\nu} \longrightarrow b_{\mu\nu} + \xi_\mu K_\nu - \xi_\nu K_\mu \qquad (\xi_\mu \sim \xi_\mu + K_\mu)$$

Interpretation

massless transversely polarized spin 2 particle

- traceless symmetric state $\rightsquigarrow$ graviton

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \gamma_{\mu\nu}(x)$$

$$\gamma_{\mu\nu}(x) \sim \gamma_{\mu\nu}(x) + \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x)$$

In momentum space $\gamma_{\mu\nu}(K) \sim \gamma_{\mu\nu}(K) + K_\mu \xi_\nu + K_\nu \xi_\mu$

with $K \cdot \xi = 0$

$$\frac{1}{2} D(D+1) - 1 - \underbrace{(\qquad\qquad)}_{K^\mu \gamma_{\mu\nu} = 0} - \underset{\xi \cdot K = 0}{\ } = \frac{1}{2}(D-1)(D-2) - 1$$

• anti-symmetric state $\rightsquigarrow$ 2 form gauge field (Kalb–Ramond field)

$$b^{(2)} = b_{\mu\nu}(x)\, dx^\mu \wedge dx^\nu \sim b^{(2)} + da^{(1)} \qquad a^{(1)} \sim a^{(1)} + d\lambda$$

in momentum space $\quad b_{\mu\nu}(k) \sim b_{\mu\nu}(k) + k_\mu \xi_\nu - k_\nu \xi_\mu$

with $\xi_\mu \sim \xi_\mu + k_\mu \phi$

$b \sim$ 2 form with "field strength"

$$\left.\begin{array}{l} H = db = \frac{1}{2} \partial_\mu b_{\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \\ H \text{ invariant under } \quad b \to b + da \end{array}\right\}$$

$$\frac{1}{2} D(D-1) - (\qquad) - (\qquad) = \frac{1}{2}(D-2)(D-3) \quad \checkmark$$

$SO(D-2)$

How about the scalar? We have $\Omega_{\mu\nu} = \frac{1}{2}\varphi\eta_{\mu\nu}$

We have the state $\frac{1}{2}\varphi\, \alpha_{-1}\cdot\tilde{\alpha}_{-1}|0;k\rangle$

However

$$L_1\left(\alpha_{-1}\cdot\tilde{\alpha}_{-1}|0;k\rangle\right) = \alpha_0\cdot\tilde{\alpha}_{-1}|0;k\rangle = \frac{\ell}{2}k\cdot\tilde{\alpha}_{-1}|0;k\rangle \neq 0$$

(and similarly for $\tilde{L}_1$)

Then this is not a physical state.

However we can construct a physical level one state

Define the state

$$|\varphi_{\xi,\tilde{\xi}};k\rangle = \varphi\left[(\xi\cdot\alpha_{-1})\left(\frac{\ell k}{2}\cdot\tilde{\alpha}_{-1}\right)+\left(\frac{\ell k}{2}\cdot\alpha_{-1}\right)(\tilde{\xi}\cdot\tilde{\alpha}_{-1}) + \alpha_{-1}\cdot\tilde{\alpha}_{-1}\right]|0;k\rangle$$

Virasoro constraints: $\quad K^2 = 0$

$$L_1 |\varphi_{\zeta,\tilde\zeta}; K\rangle = \varphi\Big[\zeta\cdot(\alpha_0 + \cancel{\alpha_{-1}L_1})\left(\tfrac{\ell K}{2}\cdot\tilde\alpha_{-1}\right)$$

$$+ \tfrac{\ell K}{2}(\alpha_0 + \cancel{\alpha_{-1}L_1})(\tilde\zeta\cdot\tilde\alpha_{-1}) + (\alpha_0 + \cancel{\alpha_{-1}L_1})\cdot\tilde\alpha_{-1}\Big]|0,K\rangle$$

$$= \varphi\frac{\ell}{2}\Big[(\zeta\cdot K)\left(\ell\tfrac{K}{2}\cdot\tilde\alpha_{-1}\right) + \frac{\ell}{2}\underbrace{\cancel{K^2}}_{0}(\tilde\zeta\cdot\tilde\alpha_{-1}) + K\cdot\tilde\alpha_{-1}\Big]|0,K\rangle$$

$$= \varphi\frac{\ell}{2}\left(\frac{\ell}{2}(\zeta\cdot K) + 1\right)(K\cdot\tilde\alpha_{-1})|0,K\rangle = 0$$

iff $\quad \zeta\cdot K = -\dfrac{2}{\ell}$

$$\tilde L_1 |\varphi_{\zeta,\tilde\zeta}; K\rangle = \varphi\frac{\ell}{2}\left(\frac{\ell}{2}(\tilde\zeta\cdot K) + 1\right)(K\cdot\alpha_{-1})|0,K\rangle = 0$$

iff $\quad \tilde\zeta\cdot K = -\dfrac{2}{\ell}$

So $|\varphi_{\zeta,\tilde\zeta}; K\rangle$ is a physical state when $\zeta\cdot K = -2/\ell$, $\tilde\zeta\cdot K = -\dfrac{2}{\ell}$

This state corresponds to a scalar despite the fact that it seems to depend on s & $\tilde{s}$!

To see this consider how the state depends on the choice of s & $\tilde{s}$:

$$|\varphi_{s,\tilde{s}}\,;K\rangle - |\varphi_{s',\tilde{s}}\,;K\rangle$$

and similarly for $\tilde{s}$, $\tilde{s}'$

$$= \varphi\,((s-s')\cdot\alpha_{-1})\left(\frac{\ell K}{2}\cdot\tilde{\alpha}_{-1}\right)|0;K\rangle$$

$$= \varphi\,\tilde{L}_{-1}((s-s')\cdot\alpha_{-1})\,|0;K\rangle$$

In fact, setting $\lambda = s-s'$

$$\tilde{L}_{-1}(\lambda\cdot\alpha_{-1})|0;K\rangle = (\lambda\cdot\alpha_{-1})(\tilde{\alpha}_0\cdot\tilde{\alpha}_{-1})|0;K\rangle$$

$$= (\lambda\cdot\alpha_{-1})\left(\frac{\ell K}{2}\cdot\tilde{\alpha}_{-1}\right)|0;K\rangle$$

and similarly for $\tilde{s}$ & $\tilde{s}'$

Then

$$\left.\begin{array}{l} |\varphi_{s,\tilde{s}}\,; K\rangle - |\varphi_{s',\tilde{s}}\,; K\rangle \\ |\varphi_{s,\tilde{s}}\,; K\rangle - |\varphi_{s,\tilde{s}'}\,; K\rangle \end{array}\right\} \text{ are spurious}$$

Moreover, they are null

$$(s-s')\cdot K = \underset{-1/2}{s\cdot K} - \underset{-1/2}{s'\cdot K} = 0 \;; \qquad (\tilde{s}-\tilde{s}')\cdot K = 0$$

Hence

$$|\varphi_{s,\tilde{s}}\,; K\rangle \sim |\varphi_{s',\tilde{s}}\,; K\rangle + \varphi\, \tilde{L}_{-1}\left((s-s')\cdot\alpha_{-1}\right)|0; K\rangle$$

$$|\varphi_{s,\tilde{s}}\,; K\rangle \sim |\varphi_{s,\tilde{s}'}\,; K\rangle + \varphi\, L_{-1}\left((\tilde{s}-\tilde{s}')\cdot\tilde{\alpha}_{-1}\right)|0; K\rangle$$

These are then independent of the choice of s (or $\tilde{s}$) up to gauge equivalence.

This scalar φ is called the dilaton:
it plays a very important role in the context of string interactions.

Appendix on the light-cone quantization (see GSW)

The fastest way to construct the phys/null of the quantized string is to fix the remaining gauge freedom by choosing the light-cone gauge.

One defines $\quad X^{\pm}(\tau,\sigma) = \frac{1}{2}(X^0 \pm X^{D-1})$

and uses the remaining gauge freedom to set

$$X^+ = x^+ + p^+\tau \qquad \text{no oscillators!}$$

Then one uses the Virasoro constraints for X^-:

The result is that the space of physical states is constructed from

- $D-2$ spacelike transverse oscillators

$$\alpha_n^i \qquad i = 1, \dots, D-2$$

(and $\tilde{\alpha}_n^i$ for the closed string)

There are no negative norm states

- $M^2 = \frac{1}{\alpha'}(N^\perp - 1)$ (and $M^2 = \frac{1}{\alpha'}(N^\perp + \bar{N}^\perp - 2)$ for the closed string)

$N^\perp = \sum_{n=1}^{\infty} \delta_{ij} \alpha_{-n}^i \alpha_n^j$, transverse number operator

Counting states is very fast:

open string at level 1 $\quad \alpha_{-1}^i |0;k\rangle \quad i=1,\dots,D-2$

D-2 states transforming as vector of SO(D-2)
which only makes sense if this is massless
(for which the little group is SO(D-2))

after some computations } This implies $a=1$ for Lorentz invariance
Also one needs $D=26$ (see GSW)

open string level 2 (see problem sheet)

The states $\alpha^i_{-1}\,\alpha^j_{-1}\,|0;k\rangle, \qquad \alpha^i_{-2}\,|0;k\rangle$

give $\frac{1}{2}(D-1)(D-2) + D-2 = \frac{1}{2}(D+1)(D-2)$ states

$$= \underbrace{\frac{1}{2}D(D-1)}_{\text{symmetric 2-tensor of } SO(D-1)} \underbrace{-\,1}_{\text{trace}}$$

this is interpreted as a <u>massive</u> spin-2

(which under the full $SO(1,D-1)$ corresponds to a transverse, traceless symmetric 2 Tensor)

Final remark:

We have gone over the quantization of the relativistic string

This is a first quantization meaning that the states are one $\boxed{\text{particle states.}}$

Second quantization has to do with string field theory (no well understood !)

Next : vertex operators, interactions.