STRING THEORY I

Lecture 8

4 Interactions

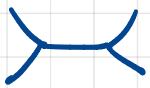
4.1 Generalities

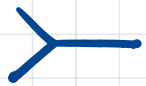
QFT:

- to understand interactions one adds non-linear terms to the action

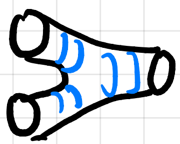
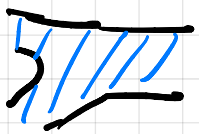
  ↳ · doesn't work for the string because anything you try to add breaks gauge invariance.

- scattering amplitudes → Feynman diagrams

eg  etc

interactions encoded at vertices say 

↳ in string theory this is replaced by, for instance

 or  : no such vertices!

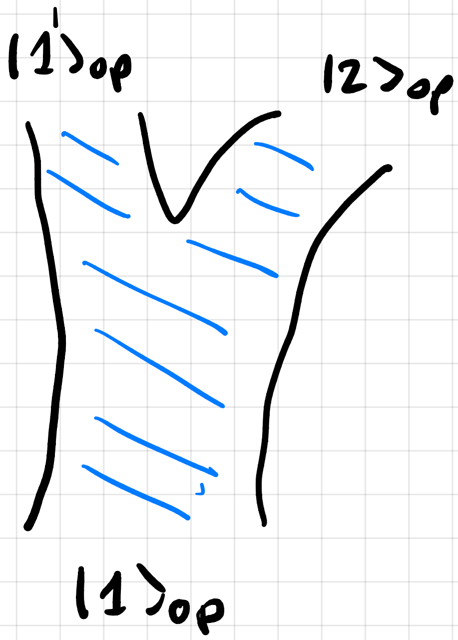
In string theory:

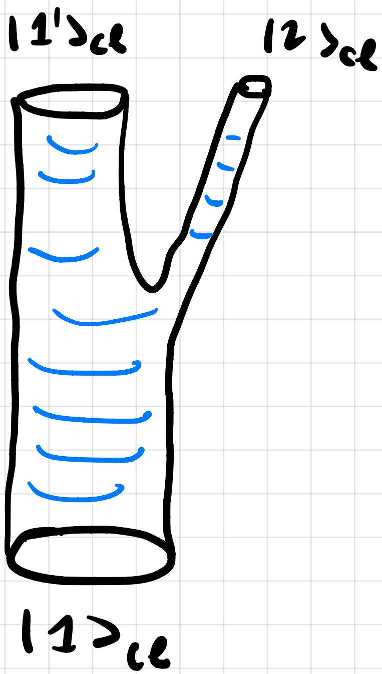
Want to compute for example the amplitude of a given configuration of quantized strings at an initial time to evolve into a new configuration at a later time

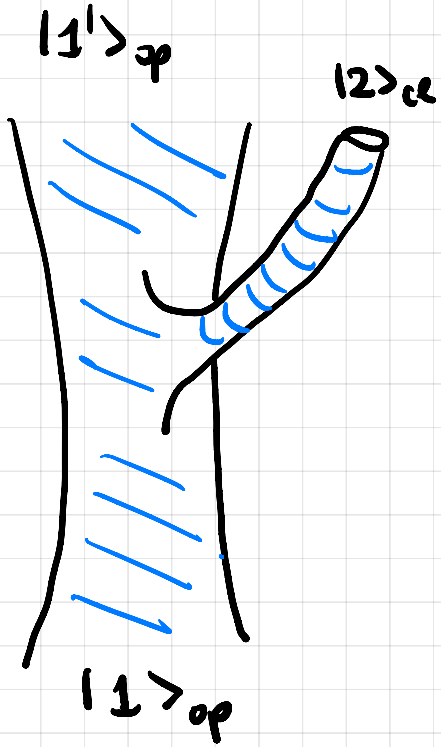
Problem: it is not known how to do this ....

we need to work with the first quantized formalism

We start by considering processes in which a string breaks into two (or two join to give a single string)



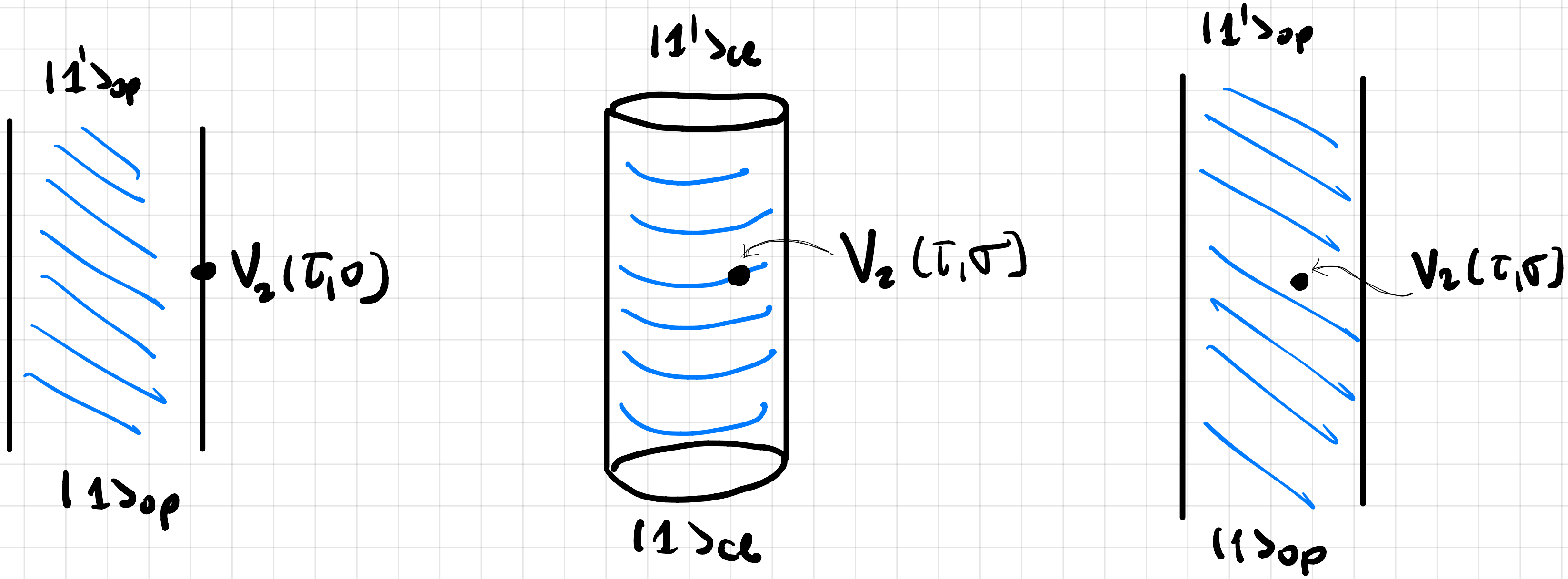




How ?

Suppose $|2\rangle$ is a physical state which is emitted/absorbed.

We describe emission or absorption of a given quantum state (say $|2\rangle$) from a string world sheet by the action of a local operator or Vertex operator



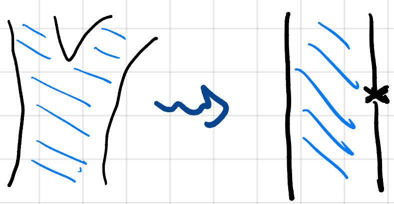
One explanation for this is that $|2\rangle$, as a physical state emitted or absorbed, is a quantum state with "mass squared and width

GS of order $\hbar$". Arguably in the classical limit these states behave like point particles.

Instead: (Chapter 1 GSW)

- Wick rotation of the world sheet (Lorentzian signature → Euclidean signature)
- conformal transformation

accept this!



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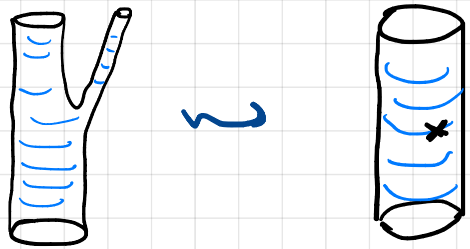
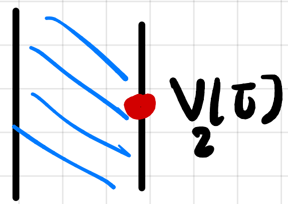


4.2 **Vertex operators : Introduction**

Require:

- time evolution on the worldsheet is a gauge transformation ⇒ position of the vertex operator should not be meaningful.

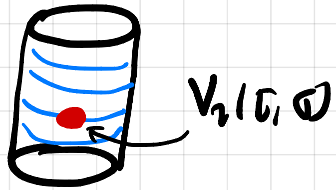
open string vertex operator



$$\int d\tau \underbrace{V_2(\tau)}_{\downarrow}$$

inserted on the boundary

closed string vertex operator



$$\int d\tau\, d\sigma \underbrace{V_2(\tau,\sigma)}_{\downarrow}$$

inserted on the interior

- absorption & emission string operators to map

$$\mathcal{H}_{phys} \longrightarrow \mathcal{H}_{phys}$$

and

$$\mathcal{H}_{null} \longrightarrow \mathcal{H}_{null}$$

Let's see how this works for the open strings first

Consider the action of the vertex operator on a state $|\phi\rangle$

$$\int d\tau \; V(\tau, 0) \; |\phi\rangle$$

physical state conditions:

$$(L_m - a\delta_{m,0}) \left[\int d\bar{\tau}\ V(\bar{\tau},0)\, |\phi_{phys}\rangle \right] \qquad m \geq 0$$

$$= \underbrace{\int d\bar{\tau}\ [L_m, V(\bar{\tau},0)]\, |\phi_{phys}\rangle}_{=0}$$

physical if this vanishes

if $\quad \underline{[L_m, V(\bar{\tau},0)] = \partial_{\bar{\tau}}(\text{local operator})}$

null states:

$$\int d\bar{\tau}\; V(\bar{\tau},0)\left[L_{-m}|\phi\rangle\right] \qquad m\geq 1$$

$$= \underbrace{\int d\bar{\tau}\left\{[V(\bar{\tau},0),L_{-m}]|\phi\rangle\right.}_{\text{need this to vanish up to a total derivative}} + \underbrace{\left.L_{-m}V(\bar{\tau},0)|\phi\rangle\right\}}_{\text{null state}}$$

$\langle\phi_{\text{phys}}|L_{-m}V(\bar{\tau},0)|\phi\rangle=0$

This is null if $\underline{[V(\bar{\tau},0),L_{-m}]\sim\partial_{\tau}(\text{local op})}$

Then $[L_m, V(\bar{\tau},0)]\sim\partial_{\bar{\tau}}(\cdots)$

Moreover, conformal transformations of the open string of the form $\tau \to \tilde{\tau}(\tau)$.

We want

$$\int V(\tau,0)\, d\tau \longrightarrow \int \tilde{V}(\tilde{\tau},0)\, d\tilde{\tau} = \int V(\tau,0)\, d\tau$$

$$= \int V(\tau,0) \frac{d\tau}{d\tilde{\tau}}\, d\tilde{\tau}$$

for $\int V(\tau,0)\, d\tau$ to be (invariant under $\tau \to \tilde{\tau}$

ie $\quad V(\tau,0) \longrightarrow \tilde{V}(\tilde{\tau},0) = V(\tau,0) \dfrac{d\tau}{d\tilde{\tau}}$

Definition: an operator $A(\tau)$ is a primary operator of weight h if under the transformation

$$\tau \to \tilde{\tau}(\tau)$$

it transforms as

$$A(\tau) \longrightarrow \tilde{A}(\tilde{\tau}) = A(\tau)\left(\frac{d\tau}{d\tilde{\tau}}\right)^{h}$$

For an operator with $h=1$

$$\int \tilde{A}(\tilde{\tau})\, d\tilde{\tau} = \int A(\tau)\frac{d\tau}{d\tilde{\tau}}\, d\tilde{\tau} = \int A(\tau)\, d\tau$$

ie the integrated operator is invariant.

For infinitesimal transformations

$$\tau \longrightarrow \tilde{\tau} = \tau + \epsilon(\tau)$$

we have

$$A(\tau) \longrightarrow \tilde{A}(\tilde{\tau}) = A(\tau)\left(1 + h\frac{d\epsilon}{d\tau}\right)$$

OTOH

$$\tilde{A}(\tilde{\tau}) = \tilde{A}(\tau+\epsilon) = \tilde{A}(\tau) + \epsilon\,\partial_\tau A(\tau) + \mathcal{O}(\epsilon^2)$$

Then we find the variation of A

$$\delta A(\tau) = \tilde{A}(\tau) - A(\tau) = -\epsilon\,\partial_\tau A - h(\partial_\tau \epsilon)A$$

$$= -\partial_\tau(\epsilon A) - (h-1)\,\partial_\tau \epsilon\, A$$

[which is a total derivative when $h=1$]

The Virasoro operators generate the transformations

$$\tau \longrightarrow \tilde{\tau} = \tau + \epsilon(\tau) \qquad \text{with} \qquad \epsilon = i e^{im\tau}$$

Then

$$\delta \phi(\tau) = e^{im\tau}\left(-i\partial_\tau \phi + h m \phi\right)$$

So the action of the Virasoro operators is

$$\boxed{[L_m, \phi(\tau)] = e^{im\tau}(-i\partial_\tau + mh)\,\phi(\tau)}$$

Equivalently, this is the condition for ϕ to have conformal weight h.

$h=1$ $\qquad [L_m, \phi(\tau)] = \partial_\tau(-i e^{im\tau}\phi(\tau))$ ✓

We need to identify primaries of weight 1 which correspond to the physical states in the string Hilbert space. We use this to compute string amplitudes.

Closed strings:

A primary operator of dimension $(h,\tilde{h})$ is an operator transforming under conformal transformations as

$$\phi(\sigma_+,\sigma_-) \longrightarrow \tilde{\phi}(\tilde{\sigma}_+,\tilde{\sigma}_-) = \left(\frac{d\sigma_+}{d\tilde{\sigma}_+}\right)^{\tilde{h}} \left(\frac{d\sigma_-}{d\tilde{\sigma}_+}\right)^{h} \phi(\sigma_+,\sigma_-)$$

The corresponding infinitesimal transformations are

$$\delta\phi(\sigma,\sigma) = -\partial_+(\tilde{\epsilon}\phi) - (\tilde{h}-1)(\partial_+\tilde{\epsilon})\phi - \partial_-(\epsilon\phi) - (h-1)(\partial_-\epsilon)\phi$$

(This is a total derivative if $h = \tilde{h} = 1$)

For $\epsilon = \frac{i}{2} e^{2im\sigma_+}$ this gives the action of L_m:

$$[L_m, \phi(\sigma_\pm)] = \frac{1}{2} e^{2im\sigma_+}\left(-i\partial_+ + 2mh\right)\phi(\sigma_\pm)$$

$$[\tilde{L}_m, \phi(\sigma_\pm)] = \frac{1}{2} e^{2im\sigma_-}\left(-i\partial_- + 2m\tilde{h}\right)\phi(\sigma_\pm)$$

$\epsilon = \frac{i}{2} e^{2im\sigma_-}$, $\tilde{L}_m$

$\boxed{4.3}$ Vertex operators for the open string

Consider the boundary scalar operator $X^\mu(\tau, 0)$:

$(l=1$
$(\alpha'=1/2)$

$$X^\mu(\tau,0) = \underbrace{x^\mu + \tau p^\mu}_{x(\tau)\text{ center of mass}} + i \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau}$$

We can check conformal transformations

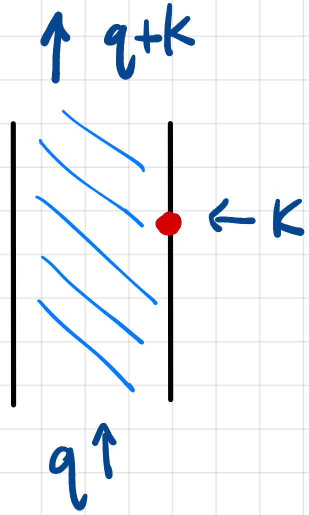
$$L_m = \frac{1}{\alpha} \sum_n \alpha_{m-n} \cdot \alpha_n$$

By direct computation

$$[L_m, X^\mu(\tau,0)] = -i \sum_n \alpha_n^\mu e^{-i(n-m)\tau} = -i\left(\frac{d}{d\tau} X^\mu(\tau,0)\right) e^{im\tau}$$

so $h=0$ ✓

Building a vertex operator associated to a tachyon of momentum k: (level 0, ground state)



so

$$V_k(\tau)\,|0;q\rangle \sim |\phi;q+k\rangle$$

Thus we should demand

$$[P^\mu, V_k(\tau)] = k^\mu V_k(\tau)$$

action of momentum operator on V

This means $V_k(\tau)$ must depend on $e^{ik\cdot x(\tau)}$

A naive guess for $V_k(\tau)$ is

$$V_k(\tau) = e^{ik\cdot X(\tau)}$$

As it stands, it is not well defined: it still needs normal ordering.

Consider the normal ordered expression

$$:e^{ik\cdot X(\tau)}: = e^{k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-m}}{m}e^{im\tau}}\, e^{ik\cdot x(\tau)}\, e^{-k\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}}$$

Remark:

$k_\mu k_\nu [\alpha^\mu_m, \alpha^\nu_n]$

$$e^{ik\cdot\alpha_m} e^{ik\cdot\alpha_n} = e^{ik\cdot(\alpha_m+\alpha_n)}\, e^{-\frac{1}{2}\boxed{k\cdot k\, m\, \delta_{m+n,0}}}$$

so reordering is free if $k^2=0$

Next, we need to compute the conformal dimension

$$[L_m, :e^{iK\cdot X(\tau)}:] = e^{im\tau}\left(-i\frac{d}{d\tau} + \frac{1}{2}m(K\cdot K)\right):e^{iK\cdot X(\tau)}:$$

PS3

from reordering, gives $h = \frac{1}{2}(K\cdot K)e^2$

This is a good vertex operator for $e^2K^2 = 2$

same as tachyon mass-shell condition ($\alpha' K^2 = 1$)

$\alpha' = e^2/2$

Building vertex operators for level one states

For these states $k^2=0$

so we are looking for the photon emission/absorption vertex operator

$$V_S(\tau) = \left(\cdots \right) :e^{ik\cdot X(\tau)}:$$

- some appropriate local operator
- no zero mode dependence
  ↳ consider $\partial_\tau X^\mu$

as $k^2=0$: • $h=0$

• no ordering issue

$$[L_m, \partial_\tau X^\mu(\tau)] = \frac{\partial}{\partial \tau}[L_m, X^\mu(\tau)]$$

$$= \frac{\partial}{\partial \tau}\left(-i\left(\partial_\tau X^\mu(\tau)\right)e^{im\tau}\right)$$

$$= e^{im\tau}\left(-i\,\partial_\tau + m\right)\partial_\tau X^\mu(\tau)$$

Then $\partial_\tau X^\mu(\tau)$ is primary of weight $h=1$

We could try

polarization

$$V_\varsigma(\tau) = (\varsigma \cdot \partial_\tau X(\tau)) : e^{ik\cdot X(\tau)} :$$

and deal with normal ordering issues.

(Note that $h(AB)$ is not necessarily $h_A + h_B$)

In $\zeta \cdot \partial_\tau X(\tau)$: each oscillator operator is contracted with ζ

In $e^{iK\cdot X(\tau)}$: each oscillator operator is contracted with K

Moreover:

$$[\alpha_m \cdot \zeta, \alpha_n \cdot K] = m\, \delta_{m+n,0}\, (\zeta \cdot K)$$

$\therefore$ when $\zeta \cdot K = 0$ (physical polarization) the operator product

$$(\zeta \cdot \partial_\tau X(\tau)) :e^{iK\cdot X(\tau)}: \quad \text{is well defined}$$

(no corrections from ordering ...)

Then we have:

$$V_\zeta(\tau) = (\zeta \cdot \dot{X}(\tau)) :e^{iK\cdot X(\tau)}: \text{ with } K^2=0,\ \zeta\cdot K = 0 \quad (h=1)$$

Note that

$$(K\cdot \dot{X}) :e^{iK\cdot X}: = -i\,\partial_\tau (e^{iK\cdot X})$$

which vanishes after integrating over τ.

This means that the longitudinal mode decouples

- It is not a coincidence that the rules for constructing vertex operators is closely parallel to the construction of physical states.

Here we have for level 1

$$\underset{h=1}{\text{vertex operators with}} \;\overset{1-1}{\longleftrightarrow}\; \text{physical states}$$

Building vertex operators for level two states

Harder. At level two we guess

$$\gamma_{\mu\nu}\, \alpha_{-1}^{\mu}\, \alpha_{-1}^{\nu}\, |0;K\rangle \longleftrightarrow \gamma_{\mu\nu}\, \dot{X}^{\mu}\, \dot{X}^{\nu} \underbrace{:e^{iK\cdot X}:}_{h=-1}$$

$$K^2 = -2$$

normal ordering corrections:

$\gamma_{\mu\nu}\, \dot{X}^{\mu}\, \dot{X}^{\nu}$ oscillator reorderings are proportional to $\gamma^{\mu}{}_{\mu}$ so require traceless γ

$\gamma_{\mu\nu}\, \dot{X}^{\mu}\, \dot{X}^{\nu}$ & $:e^{iK\cdot X}:$ oscillator reorderings proportional to $K\cdot\gamma$ so require $K\cdot\gamma = 0$ ie transverse polarization

$$\gamma_{\mu\nu}\, \alpha_{-1}^{\mu}\, \alpha_{-1}^{\nu}\, |0;K\rangle \longleftrightarrow \gamma_{\mu\nu}\, \dot{X}^{\mu}\, \dot{X}^{\nu} \underbrace{:e^{iK\cdot X}:}_{h=-1}$$

$$K^2 = -2$$

with $K\cdot\gamma = 0$ & γ traceless

These are the conditions for a massive spin two state, that is a symmetric traceless 2-tensor of $SO(D-1) = SO(25)$

It is the only possibility in $D=26$ (see GSW)

$$\ddot{X}^{\mu} :e^{iK\cdot X}: \quad ??$$

__Next:__ Vertex operator $\leftrightarrow$ physical states