STRING THEORY I

Lecture 10

4 Interactions

4.1 Generalities

4.2 Vertex operators: introduction

4.3 Vertex operators: open string

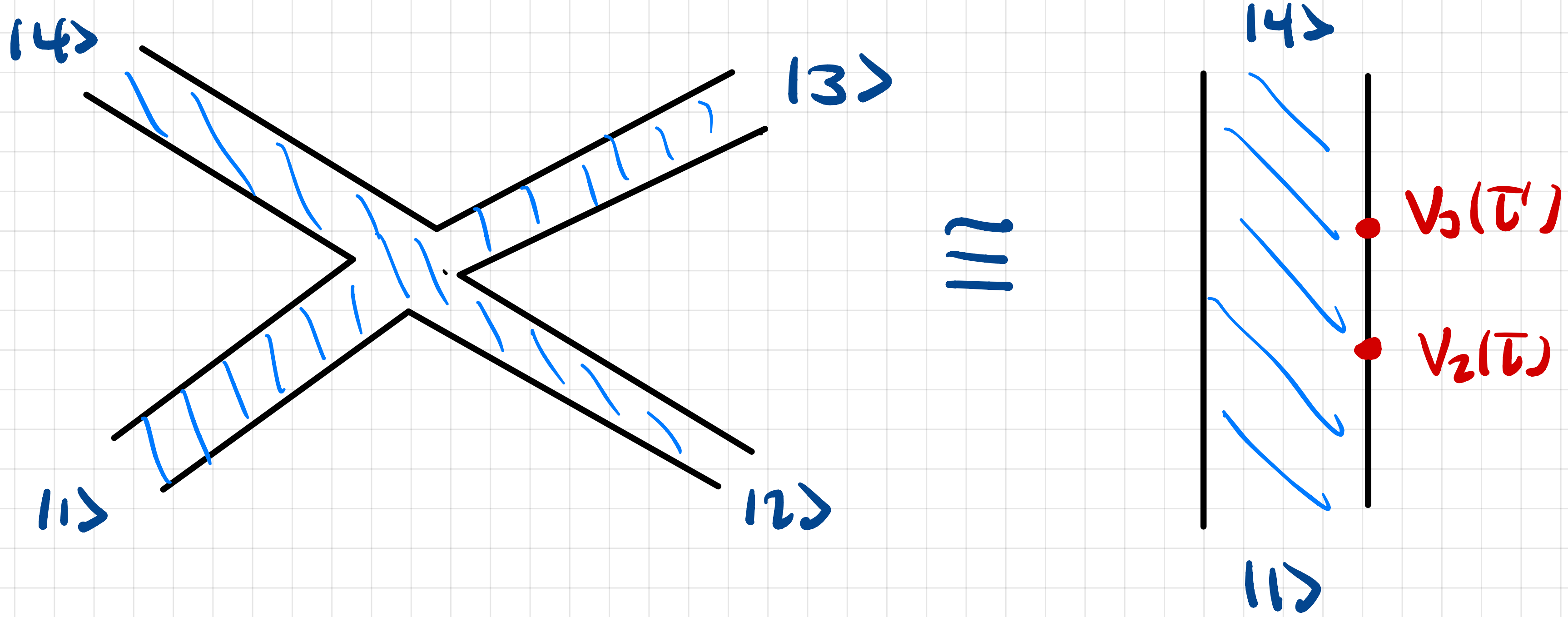
4.4 The state vertex correspondence

4.5 3-point interactions

4.6 4-point tachyon amplitude (open string)

4.7 Comments on the general picture

4.6 4-point tachyon amplitude (open string)



$$A_4(k_1,k_2,k_3,k_4) = g_o^2 \int_{\tau'>\tau} d\tau'\, d\tau \, \langle 0;-k_4 | V_3(\tau') V_2(\tau) | 0;k_1 \rangle \Big/ \mathrm{Vol(conf)}$$

$$= g_o^2 \int_{\tau'>0} d\tau'\, d\tau \, \langle 0;-k_4 | e^{i\tau' L_0} V_3(0)\, e^{-i\tau' L_0} \; V_2(\tau) \, | 0;k_1 \rangle \Big/ \mathrm{Vol(conf)}$$

Use the residual gauge freedom to fix $\tau' = 0$

$$A_4(k_1, k_2, k_3, k_4) = g_0^2 \int_{-\infty}^{0} d\tau \, \langle 0; -k_4 | V_3(0) V_2(\tau) | 0; k_1 \rangle$$

$$= g_0^2 \int_{-\infty}^{0} d\tau \, \langle 0, -k_4 | V_3(0) e^{i\tau L_0} V_2(0) \underbrace{e^{-i\tau L_0} | 0; k_1 \rangle}_{e^{-i\tau} | 0; k_1 \rangle}$$

$$= g_0^2 \int_{-\infty}^{0} d\tau \, \langle 0; -k_4 | V_3(0) e^{i\tau(L_0 - 1)} V_2(0) | 0; k_1 \rangle$$

$$= g_0^2 \langle 0; -k_4 | V_3(0) \underbrace{\left(\int_{-\infty}^{0} d\tau \, e^{i\tau(L_0-1)} \right)}_{\text{propagator}} V_2(0) | 0; k_1 \rangle$$

What do we do with $\int_{-\infty}^{0} d\tau\, e^{i\tau(L_0-1)}$?

$$\int_{-\infty}^{0} d\tau\, e^{i\tau(L_0-1)} \stackrel{?}{=} \frac{-i + \text{oscillatory}}{L_0 - 1} \qquad ??$$

Analogy in QFT in Schwinger proper-time formalism:

$$\int_{-\infty}^{0} d\tau\, e^{i\tau(p^2+m^2-i\epsilon)} = -\frac{i}{p^2+m^2-i\epsilon}$$

($-i\epsilon$: to avoid poles due to on shell particles)

rotate the contour of integration $t = i\tau$

as long as $p^2+m^2>0$ i.e. below threshold for production

$$\int_{-\infty}^{0} dt\, e^{t(p^2+m^2)} = \frac{1}{p^2+m^2}$$

$$\mathcal{A}_4(k_1,k_2,k_3,k_4) = g_o^2 \langle 0; -k_4 | V_3(0) \left(\int_{-\infty}^{0} d\tau \, e^{i\tau(L_0-1)} \right) V_2(0) | 0; k_1 \rangle$$

$$= g_o^2 \langle 0; -k_4 | V_3(0) \left(\int_{-\infty}^{0} d\tau \, e^{i\tau(L_0-1-i\epsilon)} \right) V_2(0) | 0; k_1 \rangle$$

↓ rotate contour $\tau = -it$

$$= g_o^2 \langle 0; -k_4 | V_3(0) \left(\int_{-\infty}^{0} dt \, e^{t(L_0-1)} \right) V_2(0) | 0; k_1 \rangle$$

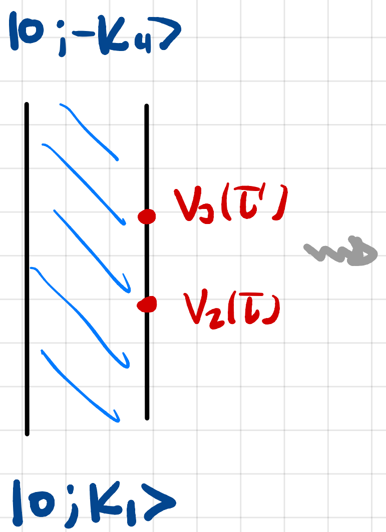
$$= g_o^2 \int_{-\infty}^{0} dt \, \langle 0; -k_4 | V_3(0) \, e^{tL_0} \, V_2(0) \, e^{-tL_0} | 0; k_1 \rangle$$

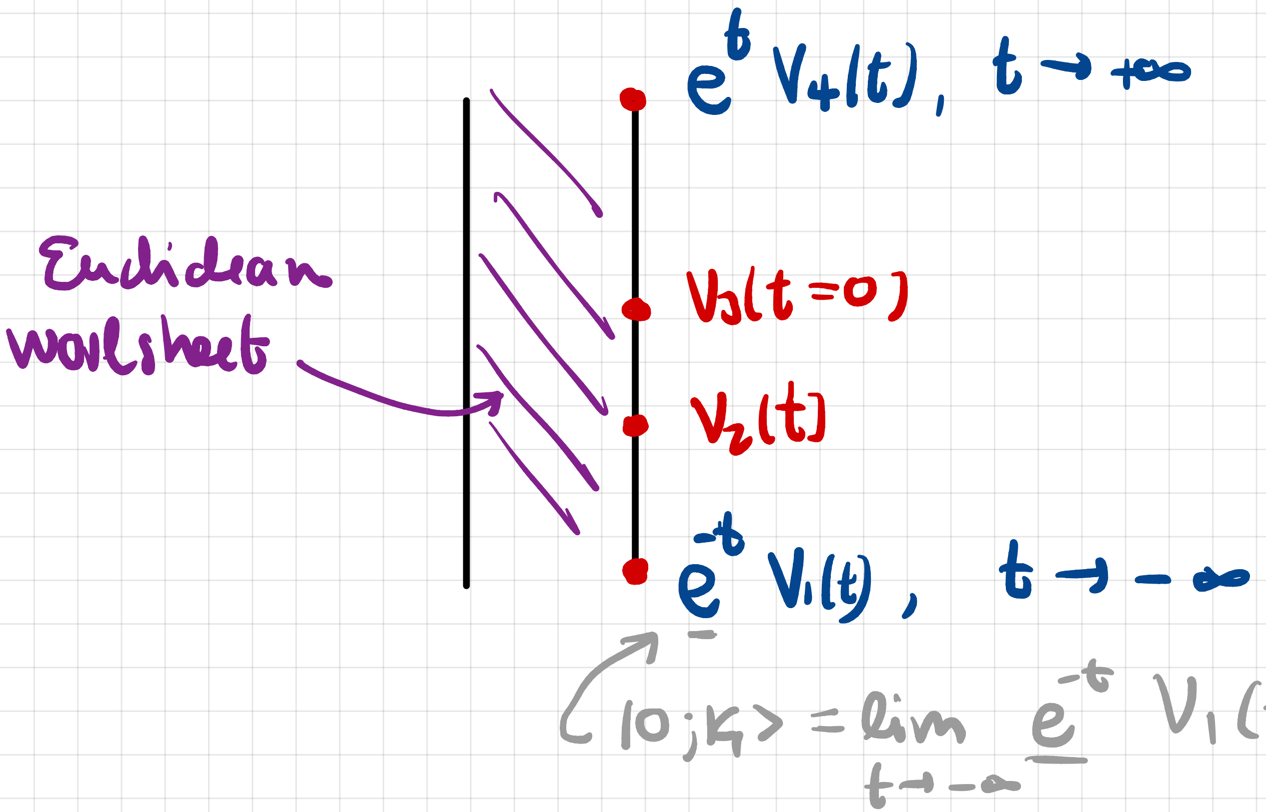
$$= g_o^2 \int_{-\infty}^{0} dt \, \langle 0; -k_4 | V_3(0) \, V_2(it) \, | 0; k_1 \rangle$$

all operators are now in <u>Euclidean</u> worldsheet time.

The amplitude now has an interpretation in Euclidean worldsheet.

$$A_4(k_1, k_2, k_3, k_4) = g_o^2 \int_{-\infty}^{0} dt \, \langle 0; -k_4 | V_3(0) \, V_2(it) | 0; k_1 \rangle$$





Consider now a Euclidean conformal map

$$z = e^{t+i\sigma} \quad , \quad \bar{z} = e^{t-i\sigma}$$

so $$dz\,d\bar{z} = e^{2t}(dt^2 + d\sigma^2) \longrightarrow dt^2 + d\sigma^2$$

Weyl transformation

$|z|^2 = e^{2t}$

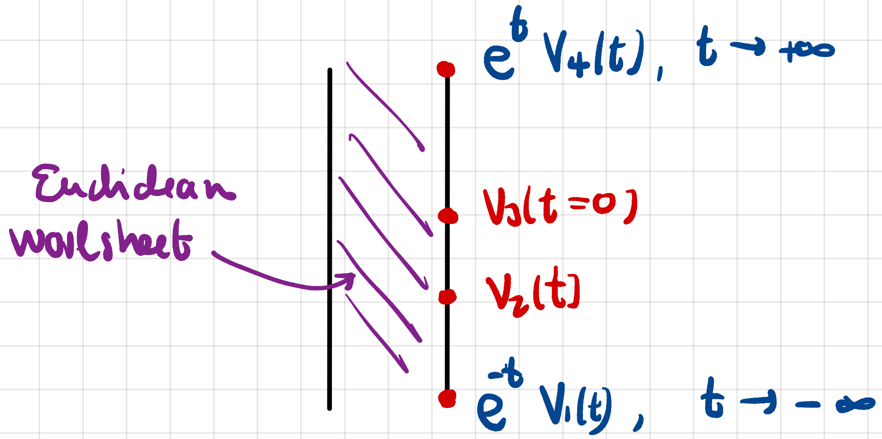
ie $$\frac{dz\,d\bar{z}}{|z|^2} = dt^2 + d\sigma^2$$

This is the metric on the upper half-plane (UHP)

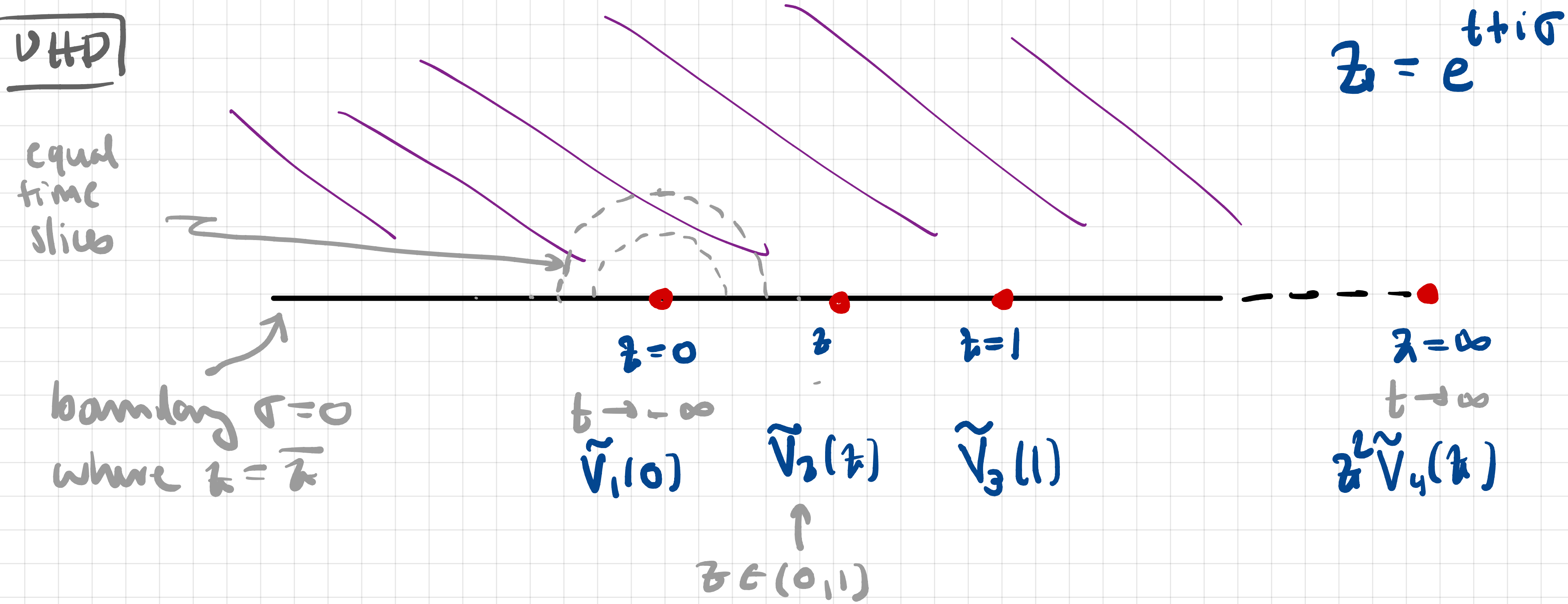
$\mathrm{Im}\, z = e^{t} \sin\sigma \qquad 0 \leq \sigma < \pi$

$\mathrm{Im}\, z \geq 0$

The strip



is mapped to the UHP with four marked points



Modification to the transformation rules of Vertex operators:

Recall that a vertex operator $V(\tau)$ transforms as

$$V(\tau) \longrightarrow \tilde{V}(\tilde{\tau}) = \left(\frac{d\tau}{d\tilde{\tau}}\right)^{1} V(\tau)$$

Then

$z = e^{t+i\sigma}$

$$\tilde{V}(\underbrace{z = \bar{z}}_{\sigma = 0}) = \frac{dt}{dz} V(t) = z^{-1} V(t)$$

Thus for the operators V_i we have:

- $V_2(t)$: $$V_2(t)\,dt = z\,\tilde{V}_2(z)\,\frac{1}{z}\,dz = \tilde{V}_2(z)\,dz$$

- $V_3(t=0)$: $$V_3(t=0) = \tilde{V}_3(1)$$

- incoming state $$|0;k_1\rangle = \lim_{t\to-\infty} e^{-t}\,V_1(it)\,|0;0\rangle$$
$$\lim_{t\to-\infty} e^{-t}\,V_1(t) = \lim_{z\to 0} \tilde{V}_1(z)$$

- outgoing state $$\langle 0;-k_4| = \lim_{t\to\infty} e^{t}\,V_4(t)$$
$$\lim_{t\to\infty} e^{t}\,V_4(t) = \lim_{z\to\infty} z^2\,\tilde{V}_4(z)$$

Let's try to discuss conformal transformations and the gauge fixing procedure in more generality:

The group of conformal transformations of the upper-half plane is $PSL(2,\mathbb{R})$ — L_{-1}, L_0, L_1 Möbius group

$$z \longmapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{R}, \quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1$$

This is a three dimensional group of residual gauge symmetries.

One can find a transformation which maps any three points z_1, z_3, z_4

$$z_1 \longrightarrow 0 \qquad z_3 \longrightarrow 1 \qquad z_4 \longrightarrow \infty$$

Indeed, for any three points z_1, z_3, z_4

set $$z_{ij} = z_i - z_j \quad (i \neq j)$$

Then $$z \longmapsto \frac{z_{34}}{z_{13}} \, \frac{z_1 - z}{z - z_4}$$

maps $z_1 \to 0$, $z_3 \to 1$ and $z_4 \to \infty$.

One can use this to gauge fix the three point amplitude.

Of particular interest for us is the fact that the group of conformal transformations $PSL(2,\mathbb{R})$ of the UHP preserves the cyclic ordering of any four points on the boundary.

Consider a fourth point z_2, with $z_1 < z_2 < z_3 < z_4$. Then

$$z_2 \longmapsto \frac{z_{12}\, z_{34}}{z_{13}\, z_{24}} \in (0,1)$$

conformal cross ratio

Fixing three points z_1, z_3 & z_4 at 0, 1 and ∞, then, the fourth point $0 < z_2 < 1$

This is what we are integrating over in the four point amplitude!

$$\int_{-\infty}^{0} \longrightarrow \int_{0}^{1}$$

We say that $(0,1)$ is the moduli space of conformal structures on the UHP with four marked points.

Fadeev-Popov gauge fixing:

We could have written the four point amplitude as

overcounts configurations

$$g_o^2 \int dz_1 dz_2 dz_3 dz_4 \langle \tilde{V}_4(z_4) \tilde{V}_3(z_3) \tilde{V}_2(z_2) \tilde{V}_1(z_1) \rangle_{UHP} \Big/ \mathrm{Vol}(SL(2;\mathbb{R}))$$

and then use the Fadeev-Popov procedure to fix the gauge.

Imposing: $z_1 = z_1^0$ $\quad z_3 = z_3^0$ $\quad z_4 = z_4^0$

$$g_o^2 \int dz_1 dz_2 dz_3 dz_4 \, \delta(z_1 - z_1^0) \delta(z_3 - z_3^0) \delta(z_4 - z_4^0) \left| \mathrm{Det} \frac{\partial(z_1, z_3, z_4)}{\partial(\lambda_{-1}, \lambda_0, \lambda_1)} \right|$$

$$\times \langle \tilde{V}_4(z_4) \tilde{V}_3(z_3) \tilde{V}_2(z_2) \tilde{V}_1(z_1) \rangle_{UHP}$$

Fadeev-Popov determinant

$$\left| \mathrm{Det} \frac{\partial(z_1, z_3, z_4)}{\partial(\lambda_{-1}, \lambda_0, \lambda_1)} \right| = \text{Jacobian of the transformation from } z_1, z_2, z_3 \text{ to } \lambda_1, \lambda_0, \lambda_{-1}$$

$$= (z_4 - z_3)(z_3 - z_1)(z_4 - z_1)$$

where $\delta z = \lambda_{-1} + \lambda_0 z + \lambda_1 z^2$

$\uparrow L_{-1}$ $\uparrow L_0$ $\uparrow L_{+1}$

(Infinitesimal Mobius transformation : $\lambda_{-1} = \delta b$

& expand around $a = d = 1$, $c = b = 0$ $\quad \lambda_0 = 2\delta a$, $\lambda_1 = -\delta c$

$$\mathrm{Jac} = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_3 & z_4 \\ z_1^2 & z_3^2 & z_4^2 \end{vmatrix} \qquad \partial z^i / \partial \lambda$$

$$= z_{43} \, z_{31} \, z_{41}$$

$d\mu$ = measure on UHP

$$g_o^2 \int d z_1 \, d z_2 \, d z_3 \, d z_4 \, \delta(z_1 - z_1^0) \, \delta(z_3 - z_3^0) \, \delta(z_4 - z_4^0) \, |(z_4 - z_3)(z_3 - z_1)(z_4 - z_1)|$$

$$\times \langle \tilde{V}_4(z_4) \tilde{V}_3(z_3) \tilde{V}_2(z_2) \tilde{V}_1(z_1) \rangle_{UHP}$$

choose $z_1^0 = 0$, $z_3^0 = 1$, $z_4^0 = \Lambda \to \infty$

$$= \lim_{\Lambda \to \infty} g_o^2 \int d z_2 \, (\Lambda - 1) \Lambda \, \langle \tilde{V}_4(\Lambda) \tilde{V}_3(1) \tilde{V}_2(z_2) \tilde{V}_1(0) \rangle_{UHP}$$

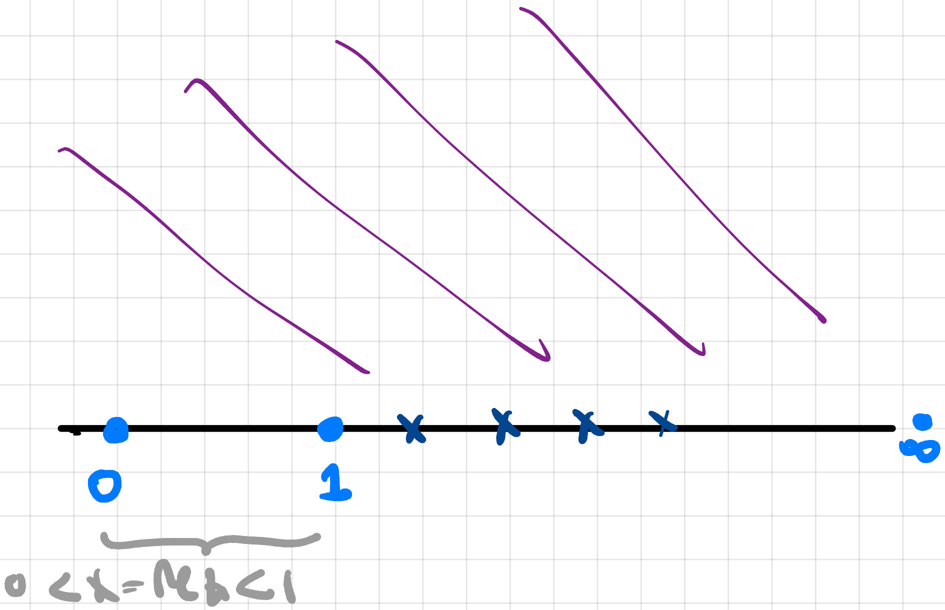
$$= g_o^2 \int d z \, \langle 4 | \tilde{V}_3(1) \tilde{V}_2(z) | 1 \rangle$$

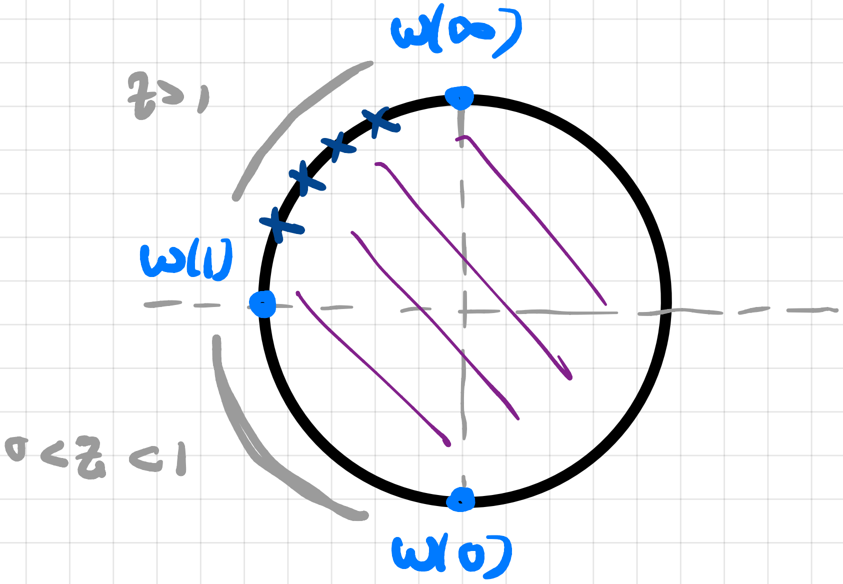
PS 3 This is the Veneziano amplitude.

cyclic symmetry: an elegant way to make the cyclic symmetry of amplitudes manifest is by introducing the map

$$z \longmapsto w = \frac{iz-1}{z-i}$$

which maps UHP $\longrightarrow$ unit disk





Next **4.7** General comments on amplitudes