STRING THEORY I

Lecture 12

5 Strings in background fields

5.1 Background field expansion and the Weyl anomaly

We have identified various massless fields in the bosonic string spectrum.

We identified a graviton in the string spectrum so we suppose target spacetime should be allowed a non trivial metric or indeed non trivial topology.

We should be able to describe the dynamics of string excitations propagating in non-trivial backgrounds for these fields.

We know what the action for such a configuration is

$$S_p[\gamma, X] = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma}\, \gamma^{ab}\, \partial_a X^\mu \partial_b X^\nu \underbrace{G_{\mu\nu}(X)}_{\text{target space metric}}$$

Classically this is Weyl invariant so take

$$\gamma_{ab} = e^{2\phi(\sigma)} \eta_{ab}$$

$\boxed{\text{NON-LINEAR } \sigma\text{-MODEL}}$

NLσM

$$S_p[\gamma, X] = -\frac{1}{4\pi\alpha'} \int d^2\sigma\, \partial_a X^\mu \partial^a X^\nu G_{\mu\nu}(X)$$

describes an interacting 2dim QFT with couplings encoded in the target space metric $G_{\mu\nu}(X)$

complicated! commonly $G_{\mu\nu} = \eta_{\mu\nu} \Rightarrow$ free field theory

Consider then the case where

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \underbrace{\chi_{\mu\nu} e^{ik\cdot X}}_{\text{small perturbation of flat space}}$$

We can then compute amplitudes in this background treating the fluctuation $\chi_{\mu\nu}$ as a perturbative "parameter":

$$\langle A_1, \ldots, A_n \rangle = \langle A_1, \ldots A_n \rangle_0 - \frac{1}{2\pi} \langle A_1 \cdots A_n \underbrace{\int d^2z\, \chi_{\mu\nu} e^{ik\cdot X} \partial X^\mu \bar{\partial} X^\nu}_{\text{graviton vertex operator}} \rangle_0 + \mathcal{O}(\chi^2)$$

This is a physical vertex operator as long as $\chi_{\mu\nu} e^{ik\cdot X}$ satisfies the appropriate conditions

We would like to require that the 2 dim QFT on the world sheet (ie NLσ-M) to be Weyl invariant at the quantum level.

This implies, in particular, that the theory is conformally invariant.

This requirement places restrictions on target space fields.

However the NLσ-M is not so easy to analyze.

To analyse the quantum NLσM we use the covariant background field expansion, which is a perturbation theory in which one separates the 2 dim fields

$$X^\mu(\xi) = X_0^\mu(\xi) + x^\mu(\xi)$$

background part or "expectation value" satisfying EOM

quantum part (dynamical)

Next, one expands the NLσM action around X_0^μ to get an expansion in powers of the quantum field x^μ

$\Rightarrow$ σ-model perturbation theory (ie perturbative QFT in 2 dim)

We first need to be clear about the meaning of the expansion: we need to expand in terms of a dimensionless parameter.

The quantum perturbation theory is an expansion in powers of α'!

Note that rescaling the metric $G_{\mu\nu} \to L^2 G_{\mu\nu}$ in the Polyakov action is the same as $\alpha' \longrightarrow L^{-2}\alpha'$

Then a small α' expansion corresponds to a large distance in space-time so the dimensionless expansion parameter is $\frac{1}{r}\sqrt{\alpha'}$ with

$r \sim$ characteristic radius of curvature of target space

In spacetime we obtain an EFT-like large radius expansion with cutoff $M_s \sim (\alpha')^{-1/2}$

It is a σ-model perturbation theory in the usual sense of a perturbative QFT framework, and from this one can read off the Feynman rules for diagrams.

However, generally the couplings in QFT get renormalized. The σ-model action can be regularized by dimensional regularization but this violates scale invariance

The lack of scale invariance in a QFT is described in terms of the β-function which arises from UV divergences in Feynman diagrams.

We are interested in computing the (one loop) β-function to obtain conditions on the fields necessary to preserve Weyl invariance at the quantum level

Returning to the NLσM action

$$S_p[\gamma, X] = -\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{\gamma}\, \gamma^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)$$

let $$X^\mu(\xi) = X_0^\mu(\xi) + x^\mu(\xi)$$ and

expand around X_0^μ.

To expand $G_{\mu\nu}(X)$ we use Riemann normal coordinates

$$G_{\mu\nu}(X) = G_{\mu\nu}(X_0) - \frac{1}{3} \underbrace{R_{\mu\rho\nu\sigma}(X_0)}_{\substack{\text{Riemann tensor} \\ \text{of } M \text{ at } X_0}} x^\rho x^\sigma + \mathcal{O}(x^3)$$

Normal coordinates: $\Gamma^\mu{}_{\nu\rho}\big|_x = 0$

$$R^\mu{}_{\nu\sigma\rho}\big|_x = \partial_\sigma \Gamma^\mu{}_{\nu\rho}\big|_x - \partial_\rho \Gamma^\mu{}_{\nu\sigma}\big|_x$$

simplifies computations!

Thus the σ-model expansion is given by

$$\underbrace{S_\sigma[X]}_{\boxed{\text{interacting QFT}}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma \Big(\underbrace{G_{\mu\nu}(X_0)\,\partial_a x^\mu \partial^b x^\nu}_{\text{kinetic term}} - \frac{1}{3}\underbrace{R_{\mu\rho\nu\sigma}(X_0)\,\partial_a x^\mu \partial^a x^\nu x^\rho x^\sigma}_{\text{quartic interaction term}} + \mathcal{O}(x^5)\Big\}$$

...

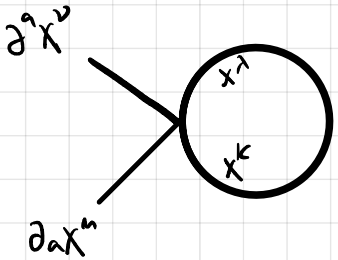
We are now ready to read off Feynman rules for diagrams and do perturbation theory. Moreover we can compute one loop divergences that contribute to renormalization of couplings.

one loop logarithmic divergences in $2+\epsilon$ dims

counterterms

terms that need to be substracted from the action to obtain a finite theory

Kinetic term

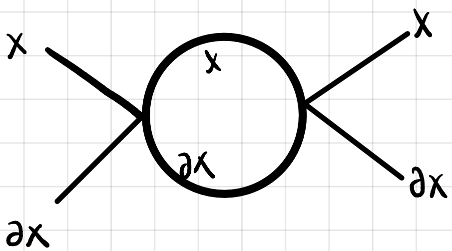


$$R_{\mu\lambda\nu\kappa}\int \frac{d^{2+\epsilon}p}{(2\pi)^{2+\epsilon}}\,\frac{\eta^{\lambda\kappa}}{p^2}$$

$$-\frac{1}{2\epsilon}R_{\mu\nu}(X_0)\,\partial X^\mu \partial X^\nu$$

Ricci tensor

Interaction vertex



$$\frac{1}{9}R_{\mu\lambda\nu\sigma}R_{\rho}{}^{\lambda}{}_{\kappa}{}^{\sigma}\int \frac{d^{2+\epsilon}}{(2\pi)^{2+\epsilon}}\,\frac{p^2}{p^4}$$

$$-\frac{1}{18\epsilon}(R^2)_{\mu\lambda\nu\kappa}\,\partial X^\mu \partial X^\nu X^\lambda X^\kappa$$

These divergences (and those from higher loops!) can be absorbed by (not entirely easy computation)

- a wave function renormalization of the fields x

$$x^\mu \longrightarrow x^\mu + \frac{1}{6\epsilon} R^\mu{}_\nu x^\nu + \mathcal{O}(x^2)$$

together with

- a renormalization of the coupling

$$G_{\mu\nu} \longrightarrow G_{\mu\nu} - \frac{1}{2\epsilon} R_{\mu\nu}$$

This gives the one-loop β-functional

$$\beta_{\mu\nu}(x) = -\frac{1}{2\pi} R_{\mu\nu}$$

← obtained from $\frac{1}{\epsilon}$ poles

The condition for conformal invariance to leading order in α' is

$$\beta_{\mu\nu} = 0 \quad \Rightarrow \quad R_{\mu\nu} = 0$$

target space must be Ricci-flat !

that is, the string moves in a background spacetime which satisfies vacuum Einsteins eqs

We have recovered spacetime dynamics from a world sheet consistency condition

Higher orders in α': one gets stringy corrections to Einsteins eqs

$$R_{\mu\nu} + \underbrace{\frac{\alpha'}{2} R_{\mu\kappa\lambda\sigma} R_{\nu}{}^{\kappa\lambda\sigma}}_{} = 0 \qquad \text{to } \mathcal{O}(\alpha'^2)$$

string theory predicts specific small corrections to Einstein's in D=26 at large radius.

Remark

Thus far we have been discussing a perturbative two dimensional QFT on the worldsheet.
Notice however that we can see an exact version developing:

abstract

$$\mathbb{R}^{1,25}_{\eta} \longrightarrow \mathbb{R}^{1,25-n}_{\eta} \times M^{n}_{G} \longrightarrow \mathbb{R}^{1,25-n}_{\eta} \times (\mathrm{CFT}^{2d})$$

special case of

Ricci flat to leading order

non-perturbative in α'

string theory goes beyond target space (semiclassical geom)

free bosonic string

$G = \eta$

machinery carries over to abstract CFT
(no ghost theorem, vertex operator formalism, etc
requires deeper study of CFTs

15.2 Including other massless modes

Apart from the graviton, we identified other massless fields in the closed string bosonic spectrum:

- The Ramond-Kalb antisymmetric field

$$B_{\mu\nu}\, dx^\mu\, dx^\nu$$

One can add to the Polyakov action the term

$$S^{(B)}[X] = -\frac{1}{4\pi\alpha'} \int d^2\sigma\ \epsilon^{\alpha\beta}\, B_{\mu\nu}(X)\, \partial_\alpha X^\mu \partial_\beta X^\nu$$

which is reparametrization and Weyl invariant

Moreover under spacetime gauge transformations

$$B \longrightarrow B + d\Lambda\ , \quad \Lambda \text{ a 1-form}$$

the action $S^{(B)}$ changes by a total derivative (exercise)

- The dilaton background we add

$$S^{[\Phi]}[X;\gamma] = \frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{\gamma}\,\Phi(X)\,R^{(2)}(\gamma)\,\alpha'$$

[We had previously ignored this term for Φ = constant as in this case the integrand is a total derivative.]

This term however is <u>not</u> Weyl invariant:

$$\gamma \to e^{2\omega(\sigma)}\gamma \implies R^{(2)} \to e^{-2\omega}\left(R^{(2)} - 2\nabla^2\omega\right)$$

the integrand so not a total derivative if $\Phi \neq$ constant

One can show however that a classical Weyl variation of S^{Φ} can be cancelled by an $\mathcal{O}(\alpha')$ variation of $S^{(G)} + S^{(B)}$.

An involved computation of the β-functional extending the one-loop computation for $S^{[G]}$ gives for the full σ-model action $S^{[G]} + S^{[\sigma]} + S^{[\Phi]}$:

$$\beta^G_{\mu\nu} = \alpha' \Big(\underbrace{R_{\mu\nu} - \tfrac{1}{4} H_{\mu\lambda\sigma} H_\nu{}^{\lambda\sigma}}_{\text{1-loop } G+B} + \underbrace{2 D_\mu D_\nu \Phi}_{\text{classical } \Phi} \Big)$$

$H = dB$

$H_{\mu\nu\rho} = 3\,\partial_{[\mu} B_{\nu\rho]}$

$$\beta^B_{\mu\nu} = \alpha' \Big(\underbrace{-\tfrac{1}{2} D^\lambda H_{\lambda\mu\nu} + D^\lambda \Phi\, H_{\lambda\mu\nu}}_{\text{1-loop } G+B} \Big)$$

D_μ = covariant derivative on space time

$$\beta^\Phi = \underbrace{\tfrac{1}{6}(D-26)}_{\text{1-loop } G+B} + \alpha' \Big(\underbrace{(D_\mu \Phi)(D^\mu \Phi)}_{\text{1-loop } \Phi} - \underbrace{\tfrac{1}{2} D^2 \Phi - \tfrac{1}{24} H_{\mu\nu\rho} H^{\mu\nu\rho}}_{\text{two loop } G+B} \Big)$$

references: Friedan's thesis; Callan & Thorlacius "sigma models & string theory"; Tseytlin "Conformal anomaly in a 2dim σ-model"

Add a comment:

non vanishing of $\beta \longleftrightarrow$ Weyl anomaly

classical: $T_{+-} = 0 \longleftrightarrow$ Weyl symmetry

preserving Weyl symmetry at the q-level involves computing

$$T_{+-} \sim \frac{\delta S}{\delta \gamma} \sim \underbrace{\text{Beta function}}\ = 0$$

5.3 Space-time effective action

We want to interpret the vanishing of the β-function as spacetime equations of motion.

Indeed, one can show that they arise as the Euler-Lagrange equations for the effective action

$$S^S_{26} = \frac{1}{2\kappa_0^2}\int d^{26}x \sqrt{-G}\, e^{-2\Phi}\left(R - \frac{1}{12}|H|^2 + 4|D\Phi|^2\right)$$

"string frame" action (G, B, Φ in S^S_{26} are the fields that appear in the σ-model action)

For space-time computations one often uses the "Einstein frame":

let $\tilde{\Phi} = \Phi - \Phi_0$, $\tilde{G} = e^{\frac{1}{6}\tilde{\Phi}} G$

$$S^{(E)}_{16} = \frac{1}{2\kappa^2} \int d^{16}x \sqrt{-\tilde{G}} \left(\tilde{R} - \frac{1}{12} e^{-\frac{1}{3}\tilde{\Phi}} |H|^2 - \frac{1}{6} |D\tilde{\Phi}|^2 \right)$$

$\kappa = \kappa_0 e^{\Phi_0}$

indices raised and lowered with $\tilde{G}$

Einstein-Hilbert term takes the canonical form with gravitational coupling $\kappa = (4\pi G_N)^{1/2}$

The spacetime action should capture the classical limit when $E \ll M_s$. The stringy corrections to this can be seen from the corrected β function

$$\beta = \beta^{(0)} + \alpha' \beta^{(1)} + (\alpha')^2 \beta^{(2)} + \cdots$$

$\uparrow$ harder ..

(For example $\quad \beta^G_{\mu\nu} \sim \alpha' R_{\mu\nu} + \frac{(\alpha')^2}{2} R_{\mu\nu\lambda\sigma} R_\nu{}^{\lambda\rho\sigma} + \cdots$)

The corrected β-functional gets interpreted as Euler-Lagrange equations for a corrected action

$$\underbrace{S_{26}}_{\text{EFT expansion}} = S_{26}^{(0)} + \underbrace{\alpha'}_{\frac{1}{M_s^2}} \underbrace{S_{26}^{(1)}}_{\text{4-derivative terms}} + \underbrace{(\alpha')^2}_{\frac{1}{M_s^4}} \underbrace{S_{26}^{(2)}}_{\text{6-derivative terms}} + \cdots$$

EFT expansion, cutoff scale M_s,
effective action obtained after integrating out massive modes

Next: strings in background fields continued.