STRING THEORY I

Lecture 14

$\boxed{6}$ Compactifications

Consider S^1-compactifications of the bosonic string

$$\mathbb{R}^{1,25} \longrightarrow \mathbb{R}^{1,24} \times S^1_R$$

circle of radius R

We will discuss this from our two perspectives

① From the spacetime EFT:

We do a Kaluza-Klein reduction to see how one can obtain an effective theory on $\mathbb{R}^{1,24}$

② From the world sheet perspective

$$X^\mu \longrightarrow X^i \qquad i = 0, \dots, 24$$

$$X^{25} \sim X^{25} + 2\pi R$$

so target space will "look" the same as for flat $\mathbb{R}^{1,25}$ but with non-trivial topology

What are the consequences ?

Spacetime EFT approach: Kaluza-Klein mechanism

Kaluza-Klein ansatz for the fields to obtain a effective action in (1+24) dimensions.

Fields X^μ $<$ X^i $\quad i=0,\dots,24$
X^{25} coordinate on S^1_R

$G_{ij}(X^i, X^{25})$

metric $\quad G_{\mu\nu}\,dX^\mu dX^\nu = G_{ij}\,dX^i dX^j + e^{2\sigma}(dX^{25}+A_i dX^i)^2$

$\left(\text{so } G_{25,25} = e^{2\sigma}, \; G_{25,i} = e^{2\sigma}A_i\right)$

KR field $\quad B_{\mu\nu}\,dX^\mu \wedge dX^\nu = B_{ij}\,dX^i \wedge dX^j + \tilde{A}_i\,dX^i \wedge dX^{25}$

$\left(\text{so } B_{i25} = \tilde{A}_i\right)$

dilaton $\quad \Phi_{(25)} = \Phi - \frac{1}{2}\sigma$

One then rewrites the effective action $S_{(26)}$ in terms of $G_{ij}, e^{2\sigma}, A_i$

$B_{ij}, \tilde{A}_i$

and $\Phi_{(25)}$

This is a long computation, but that is ok. All these fields depend on X^i but also on X^{25}. Due to the identification $X^{25}(\tau,\sigma) \sim X^{25}(\tau,\sigma) + 2\pi R$, we expand them in Fourier modes with respect to X^{25} ie

$$F(X^i, X^{25}) = \sum_{n\in\mathbb{Z}} e^{in\frac{1}{R}X^{25}} F_n(X^i)$$

Finally we integrate over X^{25} to obtain a theory in 25-dimensions (····).

We will not do all this (but see below for the dilaton)

$\rightsquigarrow$ long computation indeed.

Note however the two make typically give the massless sector of the theory

These are:

metric $G_{ij}(X^i)$

KR field $B_{ij}(X^i)$

2x 1-form gauge fields A and $\tilde{A}$

what is the gauge symmetry?

A (graviphoton) & $\hat{A}$ (KR-photon) correspond to $u(1)$ gauge fields and the gauge symmetries descend from 26-dim diffeomorphism.

2 scalars σ, $\Phi_{(2D)}$

Let's look at the dilaton more carefully.

$$\Phi(X^\mu) = \Phi(X^i, X^{25})$$

We can expand this field (and any other fields) in Fourier modes with respect to X^{25}:

$$\Phi(X^\mu) = \sum_{n\in\mathbb{Z}} e^{in\frac{1}{R}X^{25}} \phi_n(X^i)$$

$$\underbrace{\phi_n = \phi^*_{-n}}_{\text{because } \Phi \text{ is real-valued.}}$$

Dilaton terms in the action

$$\partial_\mu \phi \, \partial^\mu \Phi = \partial_i \Phi \, \partial^i \Phi + (\partial_{25}\Phi)^2$$

$$= \sum_{n,m} e^{i(n+m)\frac{1}{R}X^{25}} \underbrace{\left\{ \partial_i \phi_n \, \partial^i \phi_m - \frac{nm}{R^2} \phi_n \phi_m \right\}}_{\text{indep of } X^{25}}$$

Ignoring the coupling to gravity

$$\int d^{26}x \; \partial_m \phi \, \partial^m \phi$$

$$= \int d^{25}x \; 2\pi R \sum_{n=-\infty}^{\infty} \Big\{ \partial_i \phi_n \partial^i \phi_{-n} + \frac{n^2}{R^2} \underbrace{\phi_n \phi_{-n}}_{|\phi_n|^2} \Big\}$$

$\Rightarrow$ the massless dilaton $\Phi(X^\mu)$ of the 26-dimensional

EFT gives rise to a discrete infinite tower of scalar fields ϕ_n with mass $M_n^2 = \dfrac{n^2}{R^2}$

(Kaluza-Klein modes)

For small R all are heavy modes except the massless mode $(n=0)$

Now, note that, as $\delta G_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ under a space-time diffeomorphism $\delta X^\mu = \epsilon^\nu(X)$, we have that under $\delta X^{25} = \epsilon(X^i)$

$A_i \sim G_{25,i}$ transforms as $\delta A_i = \partial_i \epsilon$

so the gauge symmetry descends from the 26-dimensional diffeomorphism invariance.

A is a $U(1)$ gauge field

The massive KK modes ϕ_n $(n \neq 0)$ are **charged** under this gauge field:

$$\Phi(X^\mu) \to \sum_{n \in \mathbb{Z}} e^{in\frac{1}{R}(X^{25}+\epsilon)} \phi_n(X^i)$$

hence $\phi_n \longrightarrow e^{in\epsilon/R} \phi_n$ KK-momentum is charge for the graviphoton

One can show that there are no excitations charged under the $U(1)$ symmetry associated to the Ramond-Kalb-photon

Note that we have introduced a new scale

$$M_{KK} \sim \frac{1}{R}$$

We should not trust the EFT analysis for $\boxed{M_{KK} \sim M_s}$: however in this case one can perform an exact analysis of worldsheet CFT.

For the other fields, we do get a massless sector

$G_{\mu\nu}(x) \longrightarrow G_{\mu\nu}(x^i) : \{ G_{ij}(x^i),\ G_{i,25}(x^i),\ G_{25,25}(x^i) \}$

25 dim graviton | graviphoton $e^{2\sigma} A$ | radion $e^{2\sigma}$

$B_{\mu\nu}(x) \longrightarrow B_{\mu\nu}(x^i) : \{ B_{ij}(x^i),\ B_{i,25}(x^i) \}$

25 dim KR field | KR-photon $\tilde{A}$

$\Phi(x) \longrightarrow \Phi(x^i)$ 25 dim dilaton

World-sheet perspective

The target space for the two dimensional NLσM is $\mathbb{R}^{1,25}$ however X^{25} is a field on S^1_R which is periodic ie $X^{25} \sim X^{25} + 2\pi R$

This non trivial topology has consequences

① a space-time translation by $2\pi R$: $e^{2\pi i R \hat{P}_{25}}$ should act as identity:

$$e^{2\pi i R \hat{P}_{25}} |\cdots, k_{25}\rangle = e^{2\pi i R k_{25}} |\cdots, k_{25}\rangle = |\cdots, k_{25}\rangle$$

iff $\boxed{k_{25} = \frac{m}{R}}$ $m \in \mathbb{Z}$

just as in the EFT analysis

② $X^{25}(\tau, \sigma+\pi) = X^{25}(\tau,\sigma) + 2\pi R w \qquad w \in \mathbb{Z}$

(that is X^{25} only needs to be periodic $\sigma \to \sigma+\pi$ up to $2\pi R$ shifts)

w is called the winding number

(counts how many times the string wraps around S^1_R)

winding is a stringy effect: there is nothing like this in the EFT

Now we need to study the spectrum of the string on the spacetime $\mathbb{R}^{1,24} \times S^1_R$

Mode expansion of X^{25} (which respects $X^{25}(\tau,\sigma+\pi) = X^{25}(\tau,\sigma) + 2\pi R w$)

$$X^{25}(\tau,\sigma) = x^{25} + \tau\, p^{25} + 2wR\sigma + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^{25} e^{-2in\sigma^-} + \tilde{\alpha}_n^{25} e^{-2in\sigma^+}\right)$$

$$= X_R^{25}(\sigma^-) + X_L^{25}(\sigma^+)$$

with $$X_L^{25}(\sigma^-) = x_R^{25} + \frac{1}{2}p_L^{25}\,\sigma^- + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25} e^{-2in\sigma^-}$$

$$X_L^{25}(\sigma^+) = x_L^{25} + \frac{1}{2}p_R^{25}\,\sigma^+ + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{25} e^{-2in\sigma^+}$$

and $$p_L^{25} = p^{25} + 2Rw \qquad p_R^{25} = p^{25} - 2Rw$$

This is just as in $\mathbb{R}^{1,25}$ except that
$\alpha_0^{25} = \frac{1}{2}p_R$, $\tilde{\alpha}_0^{25} = \frac{1}{2}p_L$ $\left(\alpha_0^{25} + \tilde{\alpha}_0^{25} = p^{25} ;\ \alpha_0^{25} - \tilde{\alpha}_0^{25} = -2Rw\right)$

The compact-boson Hilbert space is now

$$\mathcal{H}_{compact} \cong \bigoplus_{(m,w) \in \mathbb{Z} \times \mathbb{Z}} \mathcal{H}_{Fock}^{(m,w)}$$

where

$$\mathcal{H}_{Fock}^{(m,w)} \cong \text{Span} \left\{ \prod_i \alpha_{-n_i}^{25} \, |0; m, w\rangle \right\}$$

The mode expansion of $X^{\hat{\imath}}$ $\hat{\imath} = 0, \ldots, 24$ remains unchanged.

Virasoro operators & constraints:

$$L_0 = \frac{1}{2}\left(\alpha_0 \cdot \alpha_0 + \left(\alpha_0^{25}\right)^2\right) + \underbrace{\sum_{n>0} \alpha_{-n} \cdot \alpha_n + \sum_{n>0} \alpha_{-n}^{25} \alpha_n^{25}}_{N}$$

(1+24)-dimensional inner product!

$$L_m = \frac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n + \frac{1}{2} \sum_n \alpha_{m-n}^{25} \alpha_n^{25}$$

similar for $\tilde{L}_m$

L_0 & $\tilde{L}_0$ conditions:

$$(L_0-1)|\phi\rangle = 0, \quad (\tilde{L}_0-1)|\phi\rangle = 0$$

$$L_0 - 1 = \frac{1}{8}\left(p_{25}^2 + p_R^2\right) + N - 1$$

$$\tilde{L}_0 - 1 = \frac{1}{8}\left(p_{25}^2 + p_L^2\right) + \tilde{N} - 1$$

Thus:

$$M_{(25)}^2 = \Big(\underbrace{\frac{m}{R} - 2Rw}_{p_R}\Big)^2 + 8(N-1) = \Big(\underbrace{\frac{m}{R} + 2Rw}_{p_L}\Big)^2 + 8(\tilde{N}-1)$$

$$M_{(25)}^2 = \frac{m^2}{R^2} + \underbrace{4R^2w^2} + 4(N+\tilde{N}-2)$$

mass shell condition

contribution to the mass from momentum along the compact direction

energy to wrap string around S^1_R w times

$$N - \tilde{N} = mw$$

level-(mis)matching condition

[For $w=0$ this matches results from EFT]

String state space similar to that of $\mathbb{R}^{1,25}$ but now with quantized KK-modes on the S^1_R and with string winding

In $\mathbb{R}^{1,25}$ we had $\qquad \prod \alpha^\mu_{-n} \prod \alpha^\nu_{-m} \underbrace{|0; k\rangle}_{\text{tachyon}}$

We have instead

$$\prod \alpha^i_{-n} \prod \tilde{\alpha}^i_{-m} |K; m; w\rangle$$

K: 25 dim momentum

m: $p_{25} = \frac{m}{R}$

w: $X^{25}(\tau, \sigma+\pi) = X^{25}(\tau, \sigma) + 2\pi R w$

Massless spectrum:

tachyon with $m=w=0$
$M^2_{(25)} = -8$

$$\underbrace{\hphantom{|0,k^\ell\rangle\otimes|0,0\rangle}}$$

- 25 dim graviton $\quad \gamma_{ij}\,\alpha^i_{-1}\tilde{\alpha}^j_{-1}\,|0,k^\ell\rangle\otimes|0,0\rangle$
- 25 dim B-field $\quad B_{ij}\,\alpha^i_{-1}\tilde{\alpha}^j_{-1}\,|0,k^\ell\rangle\otimes|0,0\rangle$
- scalar from the trace part of γ: $\Phi_{(25)}$
- 2x 25 dim gauge fields $\quad (\zeta\cdot\alpha_{-1}\tilde{\alpha}^{25}_{-1} \pm \xi\cdot\tilde{\alpha}_{-1}\alpha^{25}_{-1})\,|0,k^\ell\rangle\otimes|0,0\rangle$

(graviphoton from the 26 dim metric + another photon from the 26 dim KR field)

- scalar ("radion") $\quad \alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}\,|0,k^\ell\rangle\otimes|0,0\rangle$

identified with the scalar σ

massless string spectrum $\longleftrightarrow$ massless spectrum from KK reduction of EFT

States with non-trivial m, w are obtained by acting with oscillators on the state

$$|0, k'\rangle \otimes |m, w\rangle$$

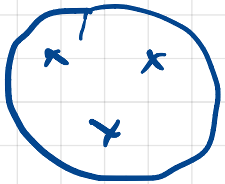
$N = \tilde{N} = 0$

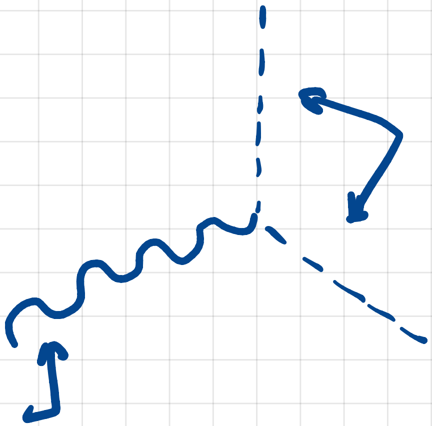
so $mw = 0$

$$M^2_{(25)} = \frac{m^2}{R^2} + 4R^2 w^2 - 8$$

When $m = 0$ $\quad M^2_{(25)} = 4R^2w^2 - 8$ winding tachyon

Interesting effect: consider the 3-amplitude





winding tachyon

$|0,k\rangle \otimes |0,0\rangle$

$M_{25}^2 = -8 + 4R^2 w^2$

Kalb-Ramond photon

$(\xi \cdot \alpha_{-1} \tilde{\alpha}_{-1}^{25} - \xi \cdot \tilde{\alpha}_{-1} \alpha_{-1}^{25}) |0,k\rangle \otimes |0,0\rangle$

Vertex operator for the KR photon

$$V_-(k) \sim \int d\tau\, d\sigma\; \xi \cdot (\partial_+ X \partial_- X^{25} - \partial_- X \partial_+ X^{25}) e^{ik\cdot X}$$

Vertex operator for the tachyon:

$$V_{m,w}(p) \sim \int d\tau\, d\sigma\; e^{ip\cdot X} \quad e^{ip_L X^{25} + i p_R X^{25}}$$

Compute the amplitude:

$$A = \langle 0, -k_3; 0, w | \left(\xi\cdot\partial_+ X\, \partial_- X^{25} - \xi\cdot\partial_- X\, \partial_+ X^{25} \right) e^{i k_2\cdot X} | 0, k_1; 0, 0\rangle$$

$$= \langle 0, -k_3; 0, w | \left(\xi\cdot\tilde{\alpha}_0\, \alpha_0^{25} - \xi\cdot\alpha_0\, \tilde{\alpha}_0^{25} \right) | 0, k_1+k_2; 0, 0\rangle$$

$$= \xi\cdot(k_1+k_2)\, \langle 0, -k_3; 0, w | \left(\alpha_0^{25} - \tilde{\alpha}_0^{25} \right) | 0, k_1+k_2; 0, 0\rangle$$

$$= \underbrace{\xi\cdot k_3 \quad (2R\, w)} \, \delta^{(25)}(k_1+k_2+k_3)$$

But this reproduces the vertex from a term

$$\tilde{A}_\mu\, \phi\, \partial^\mu \phi$$

in the spacetime Lagrangian which implies the winding number is the KR-photon charge

(same for a graviphoton → momentum $\frac{w}{R}$ consistent with KK reduction)

Remark: we have introduced a new scale R

In fact, we have a one parameter family of compactifications with $R \in (0, \infty)$

Or do we?

T- duality

Returning to the mass formula (with α' restored)

$$M^2_{(25)} = \frac{m^2}{R^2} + \frac{1}{(\alpha')^2} w^2 R^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

$$N - \tilde{N} = mw$$

Limiting cases :

- $R \to \infty$: continuum of KK modes $\to$ sign of 25th dimension
- $R \to 0$: continuum of winding modes ?

Symmetry of the spectrum:

observe that the formulas

$$M^2_{(25)} = \frac{m^2}{R^2} + \frac{1}{(\alpha')^2} w^2 R^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

$$N - \tilde{N} = mw$$

are invariant under

$$m \leftrightarrow w \qquad R \leftrightarrow \frac{\alpha'}{R}$$

This is in fact an **exact** symmetry of the CFT

$$\boxed{\text{T-duality}}$$

What is being said is that the theory on S_R and the theory on $S_{1/R}$ are the same as physical theories.

Next: more on T-duality.