Problem Sheet 1

(Questions with a $*$ are optional and not required for homework completion.)

1. Show that Maxwell's equations are conformally invariant under $g_{ab} \to \Omega^2 g_{ab}$. [*Hint: show that Hodge duality on 2-forms is conformally invariant.*]

   Show that the stress-energy tensor is trace-free for a conformally invariant theory.

   In d-dimensions, find the power of the conformal factor required to rescale a scalar field so that the action for the conformally invariant wave equation is scale invariant.

   Harder (optional *): show that the wave equation with scalar curvature term $R/6$ is conformally invariant.

   [*Hint: prove invariance of the action up to boundary terms. For the wave equation, use the conformal variation of R from the trace of that for the Schouten tensor,* (53) *in the notes.*]

2. What is the Domain of dependence of the space-like hypersurface $t = 0$, $x > 0$ in Minkowski space.

   Consider the metric

   $$ds^2 = a^2\xi^2 d\tau^2 - d\xi^2 - dy^2 - dz^2\,, \qquad \xi > 0\,.$$

   Find a coordinate transformation to the flat metric and explain its relation to the domain of dependence above. What are the trajectories $(\xi, y, z) =$ constant and the interpretation of the parameter a? [These are Rindler coordinates on which there is a good wikipedia entry.]

   The Milne universe is somewhat analogous with metric

   $$ds^2 = d\tau^2 - \tau^2 d\chi^2 - dy^2 - dz^2\,, \qquad \tau > 0.$$

   Give its relation to flat space. Is it globally hyperbolic? Why?

3. Show that the metric of de Sitter space can also be put in the form

   $$ds^2 = dT^2 - a^2 d\mathbf{x}^2\,, \qquad a(T) = \mathrm{e}^{HT}\,.$$

What region does this represent on the full Penrose diagram of de Sitter represented on the Einstein cylinder? Is it complete?

Show that in de Sitter space, the light cone of the south pole of S^3 at $\mathscr{I}^-$ refocuses at the north pole at $\mathscr{I}^+$. How does this relate to the above picture? What region of de Sitter space-time can an observe at the north pole see?

4. Prove the Sachs equation and geodesic deviation equation in terms of both $(\zeta, \bar{\zeta})$ and in terms of (ρ, σ) starting from the definitions in section §5.3. Show that in flat space with vanishing twist and shear, the expansion is given in terms of an affine parameter s by

$$\rho = \frac{\rho_0}{1 - \rho_0 s} \,,$$

where $\rho_0 = \rho(0)$. What is the interpretation of the blow up ρ at $s = 1/\rho_0$? More generally, show that if the dominant energy condition is satisfied in curved space, and $\rho_0 > 0$ then ρ blows up in finite time.

5. * To provide a model for a collapsing star, consider the $k = 1$ dust FRW universe

$$ds^2 = dT^2 - a(T)^2(d\psi^2 + \sin^2\psi d\Omega^2) \,,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the round sphere metric, and the Friedmann equations give for dust in terms of cosmological time T

$$\left(\frac{da}{dT}\right)^2 + 1 = \frac{8\pi\rho_0}{3a} \,.$$

Show that it is possible to glue the hypersurface $\psi = \psi_0$ to Schwarzschild

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} - r^2 d\Omega^2 \,,$$

along a timelike 3-surface $(t, r) = (t(T), R(T))$ such that the curves $(\theta, \phi) =$ constant are radial time-like geodesics and the metric is continuous.