Problem sheet 2

(Questions with a $*$ are optional and not required for homework completion.)

1. [25] Let $\mathcal{H}$ be a bifurcate Killing horizon associated to Killing vector k_a and let the surface gravity κ be defined by $k^a\nabla_a k_b = \kappa k_b$. Show that $\nabla_a k^b k_b = -2\kappa k_a$. Use the fact that on $\mathcal{H}$, k_a is hypersurface-orthogonal, i.e., $k_{[a}\nabla_b k_{c]} = 0$, and Killing to show that

$$\kappa^2 = -\tfrac{1}{2}(\nabla_a k_b)(\nabla^a k^b)$$

   Show that if the Killing horizon is bifurcate, then κ is constant. [*Hint: κ is constant up the generators, so work on the bifurcation surface where $k_a = 0$. First prove that for any Killing vector $\nabla_a\nabla_b k_c = -R_{bcad}k^d$.*]

2. [25] Compute the surface gravity for the static spherically symmetric metric

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2\,,$$

   in terms of f at the horizon $r = r_0$, where $f(r_0) = 0$. Hence find the surface gravity of the Reissner-Nordstrom metric

$$f(r) = \frac{\Delta}{r^2}\,, \qquad \Delta = (r - r_+)(r - r_-) = r^2 - 2mr + Q^2,$$

   at the horizon $r = r_+ > r_-$. What is special about $Q = m$?

   [*Note that the original coordinates are singular at the horizon; define a smooth coordinate v in place of t by $dv = dt + dr/f$.*]

3. [25] Show that if a Killing vector k^a is thought of as a 1-form, $k_a dx^a$, then the 2-form ${}^*dk$ satisfies the identity

$$d^*dk = R_{ab}k^{a*}dx^b\,.$$

   Assuming Einstein's equations, deduce that

$$\nabla^b J_b = 0\,, \qquad J_a = T_{ab}k^b - \tfrac{1}{2}T k_a\,, \quad T = T^a_a\,.$$

   Evaluate the integral of ${}^*dk$ on a sphere of constant r in Reissner-Nordstrom. Interpret the answer.

4. [25] Extend the calculation in the lecture of the (Casimir) vacuum energy from the one dimensional spatial slices S^1 to flat spatial $\Sigma = \mathbb{R}^2 \times S^1$. Use modes $\exp\left(i(\frac{2\pi n x}{L} + k_y y + k_z z)\right)$. Discuss the appropriate infrared regulation and use ζ-function regularization in the ultraviolet (you may find helpful that $\zeta(-3) = 1/120$).

5. [10] * Prove lemma 6.1 in the lecture notes on the Bogoliubov coefficients:

Lemma B.1 *The* $(P_n^{\pm}, N_n^{\pm})$ *are both orthonormal bases for the pseudo-Hermitian form* $\langle \phi, \phi \rangle := i\Omega(\bar{\phi}, \phi)$ *and so we obtain (pseudo-unitarity) relations*

$$\alpha^\dagger \alpha - \beta^T \bar{\beta} = 1, , \qquad \alpha^\dagger \beta - \beta^T \bar{\alpha} = 0 \, .$$