Problem sheet 3

(Questions with a $*$ are optional and not required for homework completion.)

1. [25] For an arbitrary mode expansion of a scalar field on a spatially flat universe with scale factor depending on conformal time η we have

$$\hat{\phi} = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}a(\eta)} \left(a_{\mathbf{k}}\bar{v}_k(\eta)\mathrm{e}^{i\mathbf{k}\cdot x} + a^\dagger_{\mathbf{k}} v_k(\eta)\mathrm{e}^{-i\mathbf{k}\cdot x} \right) ,$$

with $dt = a(\eta)d\eta$, $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta^3(\mathbf{k} - \mathbf{k}')$ etc., and where $v_k(\eta)$ are arbitrary complex solutions to $\ddot{v}_k + \omega_k^2(\eta)v_k = 0$ where $\omega_k^2 = k^2 + m^2_{\text{eff}}$ and $m^2_{\text{eff}} = m^2a^2 - \ddot{a}/a$ is the time-dependent effective mass. Show that the Hamiltonian is

$$\hat{H} = \frac{1}{4} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(\bar{F}_k a_{\mathbf{k}} a_{-\mathbf{k}} + F_k a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + (a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + a_{\mathbf{k}} a^\dagger_{\mathbf{k}}) E_k \right)$$

where $E_k = |\dot{v}_k|^2 + \omega_k^2 |v_k|^2$ and $F_k = \dot{v}_k^2 + \omega_k^2 v_k^2$.

[You may assume that $T_{00} = \frac{1}{2}[(\partial_\eta(a\phi))^2 + a^2(\nabla\phi)^2 + m^2_{\text{eff}}a^2\phi^2]$*.]*

Show that at fixed $\eta = \eta_0$, if E_k is minimized subject to the Wronskian $\bar{v}_k\dot{v}_k - v_k\dot{\bar{v}}_k = 2i$, we obtain $\dot{v}_k = \pm i\omega_k(\eta_0)v_k$ and that $F_k(\eta_0) = 0$.

What is the significance the instantaneous vanishing of $F_k(\eta_0)$ as far as instantaneous particle creation is concerned.

2. [25] By considering an imaginary shift in U, show that

$$\int_{-\infty}^{\infty} \mathrm{e}^{i\Omega U + i\frac{\omega}{a}\mathrm{e}^{-aU}} dU = \mathrm{e}^{\pi\Omega/a} \int_{-\infty}^{\infty} \mathrm{e}^{i\Omega U - i\frac{\omega}{a}\mathrm{e}^{-aU}} dU$$

hence show that for the Rindler-Minkowski Bogoliubov coefficients we have

$$|\alpha_{\Omega\omega}|^2 = \mathrm{e}^{2\pi\Omega/a}|\beta_{\Omega\omega}|^2 .$$

Hence show that that the particle number density per unit volume of frequency Ω is given by $1/(\mathrm{e}^{2\pi\Omega/a} - 1)$.

3. $*$ [25] Prove the quoted identity for the Bogoliubov coefficients in (198)

$$\int_{-\infty}^{0} \begin{pmatrix} \mathrm{e}^{-i\omega u} \\ -\mathrm{e}^{i\omega u} \end{pmatrix} (-au)^{-i\frac{\Omega}{a}-1} du = \frac{1}{a} \begin{pmatrix} \mathrm{e}^{\frac{\pi\omega}{2a}} \\ -\mathrm{e}^{-\frac{\pi\omega}{2a}} \end{pmatrix} (\omega/a)^{i\frac{\Omega}{a}} \Gamma(-i\Omega/a) .$$

[Not for hand-in or homework completion. Recall the definition of the Gamma function $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}dt$ but care is needed in its analytic continuation. See appendix 1.3 of Mukhanov & Winitzki for details.]

4. [25] For the conformally invariant wave equation $(\Box + \frac{R}{6})\phi = 0$ on the spatially flat FRW universe, $ds^2 = a(\eta)^2(d\eta^2 - d\mathbf{x}\cdot d\mathbf{x})$ explain why

$$D(x, x') = \frac{1}{4\pi a(\eta)a(\eta')((\eta - \eta' + i\epsilon)^2 - |\mathbf{x} - \mathbf{x}'|^2)}$$

should be a Feynman propagator. What is the implicit choice of vacuum and positive frequency modes.

For de Sitter, $a(\eta) = -1/H_\lambda\eta$, by expressing this in terms of the proper time for a co-moving observer, explain why this is a thermal Green's function. For what temperature?

Relate this temperature to the surface gravity of the cosmological event horizon for the observer at $\mathbf{x} = 0$. What is the heat capacity?

[See 'Cosmological event horizons, thermodynamics and particle creation', Gibbons & Hawking, (1976), Phys Rev D, **15**, 10, 2738.]

5. [25] This question concerns the stability of a Schwarzschild black hole confined to a box. It is radiating at the Hawking temperature $T_H = 1/8\pi m$ (in Planck units) in a reservoir of radiation of finite volume.

Assume that a finite reservoir R of radiation of volume V at temperature T has an energy, E_R and entropy, S_R given by

$$E_R = \sigma V T^4\,, \qquad S_R = \frac{4\sigma}{3} V T^3$$

where σ is a constant. A Schwarzschild black hole of mass m is placed in the reservoir. Assuming that the black hole has entropy $S_{BH} = 4\pi m^2$, show that the total entropy $S = S_{BH} + S_R$ is extremized for fixed total energy $E = m + E_R$, when $T = T_H$. Show that the extremum is a maximum if and only if $V < V_c$ where the critical value of V is

$$V_c = \frac{2^{20}\pi^4 E^5}{5^5\sigma}\,.$$

What happens as V passes from $V < V_c$ to $V > V_c$, or vice-versa?