Conformal Field Theory: Problem Sheet #1

Heeyeon Kim

Mathematical Institute, Oxford University, Trinity 2020

1 Canonical scaling dimensions

Write down the canonical scaling dimensions of the following fields and determine weather the corresponding couplings are relevant, irrelevant or marginal.

(i) $\lambda_3\phi^3$ coupling in the d-dimensional free scalar field theory.

(ii) $\lambda_4\bar{\psi}\psi\bar{\psi}\psi$ in the d-dimensional free fermion theory.

(iii) Gauge coupling in 3d quantum electrodynamics (QED):

$$\mathcal{S} = \int d^3x \; F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\sigma^\mu(\partial_\mu - igA_\mu)\psi \; .$$

2 EM tensor, scale and conformal transformations

2.1 Free massless scalar theory

Consider the free massless scalar theory in d-dimensions (in Euclidean metric $\eta^{\mu\nu} = \text{diag}(1, \cdots, 1)$):

$$\mathcal{S} = \int d^dx \; \partial_\mu\phi\partial^\mu\phi \; .$$

(i) Compute the canonical energy momentum (EM) tensor $T_c^{\mu\nu}$, and show that it is conserved, i.e.,

$$\partial_\mu T_c^{\mu\nu} = 0 \; .$$

(ii) Show that there exists a vector j_μ such that the trace of the EM tensor can be written as

$$T^{\mu}_{c\;\mu} = \partial^\mu j_\mu \; .$$

As we discussed in Lecture 3, this implies that the action is scale-invariant. We called this vector the *virial current*.

(iii) Show that there exist a tensor $\sigma^{\mu\rho}$ such that the virial current can be written as

$$j^\mu = \partial_\rho\sigma^{\rho\mu} \; .$$

(iv) Consider the modified EM tensor

$$T^{\mu\nu} = T_c^{\mu\nu} + \frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu} \; ,$$

where

$$X^{\lambda\rho\mu\nu} = -\frac{2}{d-2}\left[\eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\mu}\sigma_+^{\rho\nu} - \eta^{\lambda\nu}\sigma_+^{\mu\rho} + \eta^{\mu\nu}\sigma_+^{\lambda\rho} - \frac{1}{d-1}(\eta^{\lambda\rho}\eta^{\mu\nu} - \eta^{\lambda\mu}\eta^{\rho\nu})\sigma^{\alpha}_{+\alpha}\right] \; ,$$

$$\sigma_+^{\mu\nu} = \frac{1}{2}(\sigma^{\mu\nu} + \sigma^{\nu\mu}) \; .$$

Show that the modified EM tensor $T^{\mu\nu}$ is conserved and traceless. This implies that the free massless scalar theory is conformal in any dimensions.

2.2 Free Maxwell theory

Consider the free Maxwell theory in d-dimensions:

$$\mathcal{S} = \frac{1}{2e^2} \int d^d x \; F_{\mu\nu} F^{\mu\nu} \; .$$

(i) Compute the canonical EM tensor $T_c^{\mu\nu}$.

(ii) Show that the trace of the symmetrised EM tensor $T^{\mu\nu}$ vanishes only for $d = 4$. For $d \neq 4$, show that it can be written as

$$T^{\mu}{}_{\mu} = \partial^{\mu} j_{\mu} \; ,$$

for certain j_μ.

(iii) Argue that the Maxwell theory is scale invariant but not conformal invariant for $d \neq 4$.

3 State-operator correspondence

In Lecture 2, we defined the (spin-zero) *quasi-primary field* as a field ϕ that transforms as

$$\phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} \phi(x) \; ,$$

under the (global) conformal transformations $x \to x'$.

(i) Show that this implies

$$\begin{aligned} &\lim_{x \to 0} \; [K_\mu, \phi(x)] = 0 \; , \\ &\lim_{x \to 0} \; [D, \phi(x)] = i\Delta\phi(0) \; , \\ &\lim_{x \to 0} \; [P_\mu, \phi(x)] = i\partial_\mu \phi(x)|_{x=0} \end{aligned}$$

(ii) In Lecture 4, we studied the state-operator correspondence in two-dimensions. This discussion can be generalised to CFTs in d-dimensions, by considering the radial quantization on $\mathbb{R} \times S^{d-1}$. For a quasi-primary field with conformal dimensions Δ, the correspondence implies

$$\lim_{x \to 0} \; \phi(x)|0\rangle = |\Delta\rangle \; ,$$

where $x = 0$ is the origin in the radial quantization. Show that the state $|\Delta\rangle$ satisfies the following properties:

$$\begin{aligned} &K_\mu |\Delta\rangle = 0 \; , \\ &D|\Delta\rangle = i\Delta|\Delta\rangle \; , \\ &P_\mu |\Delta\rangle \sim |\Delta + 1\rangle \; . \end{aligned}$$

Together with the conformal algebra, these relations imply that the operators K_μ and P_μ play roles of 'annihilation' and 'creation' operators of harmonic oscillators. The state $|\Delta\rangle$ corresponds to the ground state in harmonic oscillator which is defined by the state annihilated by the annihilation operator.

(iii) Identify the state that corresponds to a quasi-primary field $\phi(x)$ inserted at $x \neq 0$.