Sheet 2 - Chapters 3 and 4

1. Consider the canonical energy-momentum tensor for the free boson in $d > 2$. Find an improvement term which makes it classically traceless without spoiling classical conservation.

Hint: Note that the index structure suggests a general ansatz of the form:

$$(\alpha \eta^{\mu\nu} \partial_\rho \partial^\rho + \beta \partial^\mu \partial^\nu) f(x)$$

2. Consider two dimensional Liouville theory with the following Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 e^\phi$$

write down the canonical energy momentum tensor and verify that it is conserved (on the equations of motion). Add a term such that it is also traceless, without spoiling conservation.

3. Prove the following property under special conformal transformations

$$|x_i' - x_j'| = \frac{|x_i - x_j|}{\gamma_i^{1/2} \gamma_j^{1/2}}$$

where $\gamma_i = 1 - 2b \cdot x_i + b^2 x_i^2$.

4. Consider the inversion tensor $I_{\mu\nu}(x) = \eta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$. Show that under inversions

$$I_{\mu\alpha}(x) I^{\alpha\beta}(x - y) I_{\beta\nu}(y) = I_{\mu\nu}(x' - y')$$

where $(x')^\mu = x^\mu / x^2$ and $(y')^\mu = y^\mu / y^2$.

5. Consider the OPE $\phi_1(x)\phi_2(0)$ and the contribution to this OPE from a single primary $\phi_\Delta(0)$ plus all its tower of descendants:

$$\phi_1(x)\phi_2(0)|0\rangle = \frac{const}{|x|^{\Delta_1 + \Delta_2 - \Delta}} \left(\phi_\Delta(0) + \alpha x^\mu \partial_\mu \phi_\Delta(0) + \cdots\right)|0\rangle$$

In the lectures it has been shown how to fix α by acting on both sides with K_μ. By following the same idea compute the next orders in the above expansion, quadratic in x.

6. Rederive the results above by considering appropriate two and three point functions in a conformal field theory, following the "practical method" described in the lecture notes.

Sheet 3 - CFT in two dimensions

1. a.- Given four point in the complex plane $z_1, \cdots z_4$ show that the cross-ratio η defined in the lectures is invariant under global conformal transformations.

b.- Find a global transformation that maps the points $(0, i, 2)$ to the points $(0, 1, \infty)$.

2. Consider a free scalar field in two dimensions $\varphi(x)$ and the operator $\mathcal{O}_\alpha =: e^{i\alpha\varphi(z)} :$, where α is a real constant. Focusing only in its holomorphic dependence, compute the OPE of this operator with the stress tensor and verify that it is a primary operator of a given weight that you should compute. [*Note: The normal ordering symbol is meant to remind us not to Wick contract two scalar fields within the operator*]. Show that the two point function of such operators behaves as it should.

3. a.- Calculate the four-point function $\langle \partial\varphi\partial\varphi\partial\varphi\partial\varphi \rangle$ for the free two-dimensional boson, using Wick contraction. Compare it with the general expression given in the lectures and determine the function $g(\eta)$ in this case.

b.- Calculate now the correlator $\langle T(z)\partial\varphi\partial\varphi\partial\varphi\partial\varphi \rangle$, where $T(z)$ is the holomorphic stress tensor given in the lectures, using Wick contraction. Verify the conformal ward identities for this case.

4. Show that the Schwartzian derivative vanishes when restricted to global conformal transformations.

5. Given a Virasoro primary $|h\rangle$ such that

$$L_0|h\rangle = h|h\rangle, \quad \langle h|h\rangle = 1$$

Compute the inner products between all level two descendants and their conjugates.

6. Identity Virasoro conformal block

Consider two identical operators of conformal weight $(h, \bar{h})$ such that they are canonically normalized

$$\langle \phi_{h,\bar{h}}(z, \bar{z})\phi_{h,\bar{h}}(0) \rangle = \frac{1}{z^{2h}\bar{z}^{2\bar{h}}}$$

Consider the OPE expansion (6.25) in the lecture notes, and focus in the identity operator plus its Virasoro descendants.

a.-Compute the OPE coefficients $C_{12}^{Id,(k,\bar{k})}$ up to level two.

b.- Use the result of part a to compute the small z expansion of the Virasoro conformal block for the identity operator.

Sheet 4 - Minimal models

1. Given a Virasoro primary $|h\rangle$ determine the conditions for $|h\rangle$ to have level three null descendants. Write explicitly the expression for the descendants in each case.

2. In the lectures we have derived a differential equation for a correlator involving $\phi(z)$

$$\langle \phi(z)\phi_{h_1}(z_1)\cdots\rangle$$

where $\phi(z)$ is a primary field with a level two null descendant.

a.- Verify that the equation is automatically satisfied for two point functions of primary operators.

b.- Consider now a three point function and derive the selection rules stated in the lectures.

3. Consider the critical Ising model introduced in the lectures, and the four point correlator of four identical operators $\epsilon(z,\bar{z})$, of conformal dimension $h=\bar{h}=1/2$.

a.- Explain why conformal symmetry implies:

$$\langle \epsilon(z_1,\bar{z}_1)\cdots\epsilon(z_4,\bar{z}_4)\rangle = \frac{g(\eta,\bar{\eta})}{|z_{12}|^{4h}|z_{34}|^{4h}}$$

b.- Given that $\epsilon(z,\bar{z})$ admits a level two null descendant, write down a differential equation for $g(\eta,\bar{\eta})$. [you may focus in the holomorphic dependence only]

c.- Write down the full correlator as a linear combination of solutions to the equation above [reintroducing the anti-holomorphic dependence].

d.- What form do the crossing relations take? Find the most general expression for $g(\eta,\bar{\eta})$ consistent with the crossing relations and with part (c). Can you think how to fix $g(\eta,\bar{\eta})$ completely?

4. Write down the schematic decomposition of the result of problem 3 in terms of Virasoro conformal blocks. Verify that the small $z,\bar{z}$ behaviour for the identity conformal block of problem 3, and the one computed in the lecture notes,agree with the small $z,\bar{z}$ behaviour you obtained in problem six of the previous sheet.