String Theory II: Assignment 4

1. Spin Connections and Killing Spinors

   [Conventions for this questions are as in BLT Section 14.8.].

   The spin connection ω is a connection for the local Lorentz symmetry in a given representation and can be expanded in terms of 1-forms

   $$\omega = \omega_\mu(x) dx^\mu \,. \tag{1}$$

   As for gauge connections in Yang-Mills theory we can define the curvature 2-form as

   $$\mathcal{R} = d\omega + \omega \wedge \omega \,. \tag{2}$$

   Infinitesimal local Lorentz transformations map

   $$\delta_\Lambda \omega = d\Lambda + [\omega, \Lambda] \tag{3}$$

   and so $\delta_\Lambda \mathcal{R} = [\mathcal{R}, \Lambda]$.

   Let e_μ^a be the viel-bein where $a, b, \cdots$ are the flat indices and $\mu, \nu \cdots$ the curved indices. Let ∇_μ be the covariant derivative with Christoffel symbols

   $$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \,.$$

   The spin connection is then the 1-form valued in the local Lorentz algebra (ω has a μ curved index, and is a matrix valued object with a, b flat indices) that satisfies

   $$\nabla_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma^\rho_{\mu\nu} e^a_\rho + \omega^a_{\mu b} e^b_\nu = 0 \tag{4}$$

   In components it is given by

   $$\omega_\mu^{ab} = \frac{1}{2}(\Omega_{\mu\nu\rho} - \Omega_{\nu\rho\mu} + \Omega_{\rho\mu\nu}) e^{\nu a} e^{\rho b} \qquad \text{where} \qquad \Omega_{\mu\nu\rho} = (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu) e_{a\rho} \,. \tag{5}$$

   (a) Determine the components of the curvature 2-tensor $R_{\mu\nu}{}^{ab}$ in terms of ω.

   (b) Consider 10d spacetime $\mathbb{R}^{1,3} \times M_6$, with local Lorentz group $SO(1,3) \times SO(6)$ in the spinor representation as in the lecture (i.e. the generators are Γ^{ab}). Let ϵ be a **16** component spinor of $SO(1,9)$. Compute $[\nabla_\mu, \nabla_\nu]\epsilon$.

2. Kähler Manifolds, Projective Space

(a) Let X_n be a complex n-dimensional Kähler manifold. Determine the non-trivial Christoffel symbols and the Ricci tensor for X_n.

(b) Consider $\mathbb{P}^n = \mathbb{C}^{n+1}/\sim$, n-dimensional projective space, where the equivalence relation is $(z^0, \cdots, z^n) \sim (w^0, \cdots, w^n)$ if $\exists \lambda \neq 0$ such that $(z^0, \cdots, z^n) = \lambda(w^0, \cdots, w^n)$. An open cover is given by $U_r = \{z^r \neq 0\}$, with local coordinates in U_r given by $z^i_{(r)}$, $i = 1, \cdots, n$

   i. Determine charts ϕ_r and transition functions $\phi_r \circ \phi_s^{-1}$ which make this into a complex manifold.

   ii. Determine the metric that follows from the Kähler potential in the open patch U_r

$$K_{(r)} = \log(1 + \sum_{i=1}^{n} |z^i_{(r)}|^2)$$

   This is the Fubini–Study metric for $\mathbb{P}^n$.

   iii. Compute the Ricci form for this metric.

   iv. Show that for $n = 1$ the space is (as a real manifold) a 2-sphere S^2.

3. Calabi-Yau Manifolds [Bonus]

(a) Consider the Type IIB supergravity in 10d compactified on a Calabi-Yau three-fold W_6 – with Hodge numbers $h^{p,q}(W_6)$, in particular $h^{1,1}(W_6)$, $h^{1,2}(W_6)$. Determine the bosonic field content by expanding the 10d fields into harmonic forms along W_6. Confirm that the massless spectrum agrees with this of IIA on the mirror Calabi-Yau M_6, where $h^{1,1}(W_6) = h^{1,2}(M_6)$ and $h^{1,2}(W_6) = h^{1,1}(M_6)$.

(b) Consider now a Calabi-Yau 2-fold (real 4d space), also called a K3-surface. The Hodge diamond is completely fixed in this case and has non-trivial entries

$$h^{2,2} = h^{0,0} = h^{2,0} = h^{0,2} = 1, \qquad h^{1,1} = 20\,. \tag{6}$$

By Hodge duality, the (1,1) forms are self-dual.

   i. Determine the degrees of freedom that the following bosonic fields have in 6d: scalar, anti-symmetric tensor, graviton, vector.

   ii. IIB has two spinors ϵ, ϵ' in the **16**. Decompose the spinors appropriate for a compactification to $\mathbb{R}^{1,5} \times M_4$, where M_4 is (1) a generic 4d manifold and (2) a K3 surface. What is the supersymmetry of the 6d theory obtained from IIB on K3?

   iii. By expanding the IIB bosonic supergravity fields determine the massless spectrum of IIB on K3 (note: be careful about the self-duality of F_5.)