Homework 2: interpolation and least-squares fitting

While implementing the functions below, you need to account for the singularities of the type 0/0.

Forward exchange curve obtained by the linear interpolation of cost-of-carry rates

Input:

S_0 : the current spot exchange rate.

$(D^d_i)_{i=1,\dots,M}$: the discount factors in domestic currency.

$(D^f_i)_{i=1,\dots,M}$: the discount factors in foreign currency.

$(t_i)_{i=1,\dots,M}$: the maturities of the discount factors, $t_0 < t_1$.

t_0 : the initial time given as year fraction.

Output: the forward curve

$$F(t) = S_0 \exp(q(t)(t - t_0)), \quad t \in [t_0, t_M],$$

where the cost-of-carry rate curve $q = (q(t))_{t\in[t_0,t_M]}$ is obtained by the linear interpolation of the market cost-of-carry rates $(q_i)_{i=1,\dots,M}$. We assume that on the interval $[t_0, t_1]$ the function q is constant:

$$q(t) = q_1, \quad t \in [t_0, t_1].$$

Discount curve obtained by log interpolation

Input:

$(t_i)_{i=1,\dots,M}$: the maturities, $t_i < t_{i+1}$,

$(d_i)_{i=1,\dots,M}$: the discount factors,

t_0 : the initial time, $t_0 < t_1$,

$\mathcal{I}$: an interpolation method.

Output: the discount curve

$$d(t) = \exp(l(t)), \quad t \in [t_0, t_M],$$

where the function $l = l(t)$ is the $\mathcal{I}$-interpolation of the logs of the market discount factors:

$$l(t) = \mathcal{I}((t_i)_{i=0,1,\dots,M}, (\log d_i)_{i=0,\dots,M}), \quad d_0 = 1.$$

Least-squares fitting of data curves

Fit of stationary volatility curve of Black model

Input:

$(t_i)_{i=1,\dots,M}$: the maturities, $t_i < t_{i+1}$.

$(\sigma_i > 0)_{i=1,\dots,M}$: the implied volatilities.

$\lambda \geq 0$: the mean-reversion rate.

t_0 : the initial time, $t_0 < t_1$.

Output:

$\sigma = \sigma(t)$: the fitted volatility curve.

$\epsilon = \epsilon(t)$: the error function of the fit for the volatility curve.

(κ, Γ, χ^2) : the fitted constant, its variance, and the total fitting error.

The stationary implied volatility curve in the Black model for commodities has the form:

$$\sigma(t; \kappa) = \kappa \sqrt{\frac{1 - \exp(-2\lambda(t - t_0))}{\lambda(t - t_0)}}, \quad t \geq t_0,$$

where constant κ is the result of the least-squares fit of the market volatilities:

$$\chi^2 = \min_{\kappa} \sum_{i=1}^{M} (\sigma(t_i, \kappa) - \sigma_i)^2.$$

Discount curve by the Svensson fit of yields

Input:

$(t_i)_{i=1,\dots,M}$: the maturities, $t_i < t_{i+1}$.

$(d_i)_{i=1,\dots,M}$: the discount factors.

$\lambda_1 \geq 0$: the first mean-reversion rate.

$\lambda_2 \geq 0$: the second mean-reversion rate, $\lambda_2 \neq \lambda_1$.

t_0 : the initial time, $t_0 < t_1$.

Output:

$d = d(t)$: the fitted discount curve.

$\epsilon = \epsilon(t)$: the error function of the fit for the discount curve.

$((c_i), \Gamma, \chi^2)$: the fitted constants, their covariance matrix, and the total fitting error.

The discount curve is given by

$$d(t) = \exp(-\gamma(t)(t - t_0)), \quad t \geq t_0,$$

where yield curve $\gamma = \gamma(t)$ has the Svensson form:

$$\begin{aligned}\gamma(t; c_0, c_1, c_2, c_3) = c_0 + c_1 \frac{1 - e^{-\lambda_1(t-t_0)}}{\lambda_1(t - t_0)} + c_2 \left(\frac{1 - e^{-\lambda_1(t-t_0)}}{\lambda_1(t - t_0)} - e^{-\lambda_1(t-t_0)} \right) \\ + c_3 \left(\frac{1 - e^{-\lambda_2(t-t_0)}}{\lambda_2(t - t_0)} - e^{-\lambda_2(t-t_0)} \right), \quad t \geq t_0,\end{aligned}$$

and constants c_0, c_1, c_2, c_3 are the result of the least-squares fit of the market yields:

$$\chi^2 = \min_{c_0,c_1,c_2,c_3} \sum_{i=1}^{M} (\gamma(t_i; c_0, c_1, c_2, c_3) - \gamma_i)^2,$$

$$\gamma_i = -\frac{\log d_i}{t_i - t_0}, \quad i = 1, \dots, M.$$