MMSC Further Mathematical Methods HT2020 — Sheet 1

1. Suppose A is a square matrix, $n \times n$. State (without proof) the Fredholm Alternative that gives necessary and sufficient conditions under which the system $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$.

Now consider the system $(A - \mu I)\mathbf{x} = \mathbf{b}$, where I is the $n \times n$ identity matrix, and μ is a constant. For what values of μ is there a unique solution? When μ is such that there is not a unique solution, what condition(s) must $\mathbf{b}$ satisfy in order for a solution to exist? When those conditions do hold, what is the most general solution $\mathbf{x}$?

2. Suppose A is a square symmetric matrix and λ is a simple eigenvalue of A with corresponding normalised eigenvector $\boldsymbol{v}$. We wish to solve

$$(A - (\lambda + \epsilon)I)\,\boldsymbol{x} = \boldsymbol{b},$$

where I is the identity matrix and the vector $\boldsymbol{b}$ is such that $\boldsymbol{v} \cdot \boldsymbol{b} \neq 0$. Show that

$$\boldsymbol{x} \sim \frac{c}{\epsilon}\boldsymbol{v} + \boldsymbol{x}_1 + \cdots ,$$

for some constant c which you should determine.

3. Find the eigenvalues and eigenfunctions of the integral equation

$$y(x) = \lambda \int_0^1 (g(x)h(t) + g(t)h(x))\, y(t)\, \mathrm{d}t, \qquad x \in [0, 1],$$

where g and h are continuous functions satisfying

$$\int_0^1 g(x)^2\, \mathrm{d}x = \int_0^1 h(x)^2\, \mathrm{d}x = 1, \qquad \int_0^1 g(x)h(x)\, \mathrm{d}x = 0.$$

4. Solve the equation

$$y(x) = 1 - x^2 + \lambda \int_0^1 (1 - 5x^2t^2)y(t)\, \mathrm{d}t.$$