MMSC Further Mathematical Methods HT2020 — Sheet 2

1. For each of the following problems (a)–(e), use the Fredholm alternative to state the conditions under which it has a unique solution, the conditions under which it has no solution, and the conditions under which it has multiple solutions. Here α, β and b are continuous, real-valued functions.

(a) $u''(x) - u(x) = b(x)$ on $0 < x < 1$, with $u(0) = u(1) = 0$.

(b) $u''(x) + u(x) = b(x)$ on $0 < x < L$, with $u(0) = u(L) = 0$, where $L > 0$ is a given constant (possibly $n\pi$).

(c) $u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = b(x)$ on $0 < x < 1$, with $u(0) = u'(0) = 0$.

(d) $u'(x) + \alpha(x)u(x) = b(x)$ on $0 < x < 1$, with $u(0) = u(1)$.

2. Consider the problem

$$\epsilon\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} - \sin f + \epsilon, \qquad f' \to 0 \text{ as } |x| \to \infty,$$

with $f(-\infty)$ close to 0 and $f(\infty)$ close to 2π. Show that there exists a travelling wave solution

$$f \sim \phi(x - ct)$$

where

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} = \sin\phi, \qquad \phi \to 0 \text{ as } x \to -\infty, \quad \phi \to 2\pi \text{ as } x \to \infty, \quad \phi(0) = \pi,$$

and determine the wavespeed c to leading order in ϵ. [In fact $\phi(x) = 4\tan^{-1}\mathrm{e}^x$.]

3. Consider the problem

$$\omega^2\ddot{x} + \epsilon(\lambda - x^2)\dot{x} + x = 0, \qquad x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi).$$

By expanding

$$x(t) \sim x_0(t) + \epsilon x_1(t) + \cdots, \qquad \omega \sim 1 + \epsilon\omega_1 + \cdots,$$

Show that there is a solution

$$x_0 = A\cos t,$$

providing A satisfies a solvability condition, which you should determine.

[*You may use the identities*

$$\int_0^{2\pi}\cos^2 t\,\mathrm{d}t = \int_0^{2\pi}\sin^2 t\,\mathrm{d}t = \pi, \quad \int_0^{2\pi}\cos^3 t\sin t\,\mathrm{d}t = 0, \quad \int_0^{2\pi}\cos^2 t\sin^2 t\,\mathrm{d}t = \frac{\pi}{4}$$

without proof.]