MMSC Further Mathematical Methods HT2020 — Sheet 3

1. Suppose that the function y satisfies the Euler equation for the functional

$$J[y] = \int_a^b F\left(x, y, y'\right) \mathrm{d}x$$

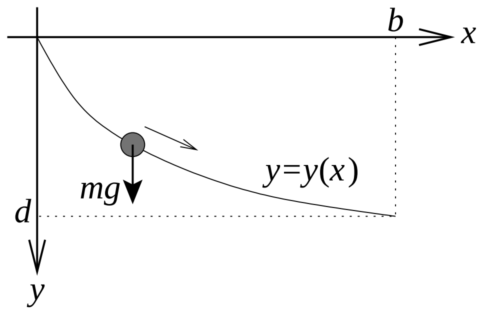
subject to the boundary conditions $y(a) = c$, $y(b) = d$. Show that, if $y(x)$ satisfies the inequalities

$$\frac{\partial^2 F}{\partial y^2}\left(x, y, y'\right) > \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial^2 F}{\partial y \partial y'}\left(x, y, y'\right)\right) \qquad \text{and} \qquad \frac{\partial^2 F}{\partial y'^2}\left(x, y, y'\right) > 0$$

for all $x \in [a, b]$, then y is a local minimiser of the functional J.

2. A bead slides under gravity along a smooth plane wire $y = y(x)$ joining the origin $(x, y) = (0, 0)$ to the point (b, d), where the y-axis is chosen to point vertically downwards. Show that the time taken to travel along the wire from rest is given by

$$J[y] = \frac{1}{\sqrt{2g}} \int_0^b \sqrt{\frac{1 + y'(x)^2}{y(x)}}\, \mathrm{d}x.$$



Deduce that a minimising function y for $J[y]$ must satisfy

$$y\left(1 + y'^2\right) = \text{constant}.$$

What happens near $x = 0$? Show that the solution can be written in parametric form as

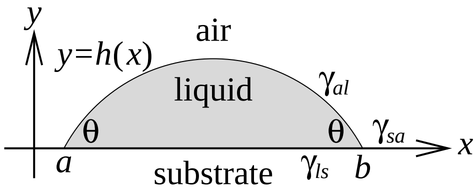
$$x = c^2\left(\tau - \sin\tau\cos\tau\right), \qquad y = c^2 \sin^2\tau.$$

How is the constant c determined?

[This minimising curve is called the *brachistochrone*.]

3. *Equilibrium of a capillary drop in two dimensions*

A two-dimensional drop of liquid sits on a flat substrate $y = 0$ between two contact points $x = a$ and $x = b$, with the air-liquid interface given by $y = h(x)$. The surface energy per unit length on the air-liquid, liquid-solid and solid-air interfaces are given by γ_{al}, γ_{ls} and γ_{sa} respectively.



Show that the net surface energy due to the drop is given by

$$J[h; a, b] = \int_a^b \left(\gamma_{ls} - \gamma_{sa} + \gamma_{al}\sqrt{1 + h'(x)^2}\right) \mathrm{d}x.$$

Show that minimisation of J over all drops of given area A leads to the Euler equation

$$-\gamma_{al}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{h'(x)}{\sqrt{1 + h'(x)^2}}\right) = \lambda,$$

where λ is a Lagrange multiplier (corresponding to the *pressure* in the drop). Deduce that the air-liquid interface forms a circular arc of radius γ_{al}/λ.

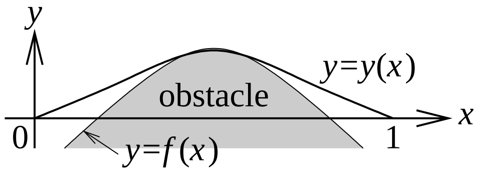
If the contact points are allowed to move, show that J is minimised when the *contact angle* θ at $x = a$ and $x = b$ satisfies *Young's equation*

$$\cos\theta = \frac{\gamma_{sa} - \gamma_{ls}}{\gamma_{al}}.$$

4. In the contact between an elastic string $y = y(x)$ and a given obstacle $y = f(x)$, the elastic energy

$$J[y] = \int_0^1 y'(x)^2 \, \mathrm{d}x$$

is minimised subject to $y(0) = y(1) = 0$ and the constraint $y(x) \geq f(x)$.



Show that

either $\quad y''(x) = 0$

or $\quad y(x) = f(x)$,

for all $x \in [0, 1]$, and that y and y' are continuous.

For the case where $f(x) = \delta - (x - 1/2)^2$, with $\delta \in (0, 1/4)$, show that the string makes contact with the obstacle in the interval $x \in (1/2 - s, 1/2 + s)$, where

$$s = \frac{1}{2}\left(1 - \sqrt{1 - 4\delta}\right).$$