Further Mathematical Methods

Andrew Krause
based on notes by Jon Chapman

February 15, 2021

Contents

1 Introduction

In the Supplementary Applied Mathematics course you were introduced to several approaches for understanding and solving boundary value problems given by ordinary differential equations.

The first half of this course provides further tools and perspectives on related problems in integral equations and the theory of linear operators, filling in some of the theoretical gaps in boundary value problem theory. The second half of this course will then explore the calculus of variations, and optimal control theory. Throughout, by way of example, we will also introduce you to some aspects of perturbation theory which arise naturally in the examples. We will not have time to cover any of these topics in detail, but you should be able to come away from this course having a basic understanding of the ideas, and the ability to continue learning about any of these methods on your own. Good resources for the material on integral equations and the Fredholm Alternative can be found in Chapters 1, 3, and 4 of *Principles Of Applied Mathematics: Transformation And Approximation* by James Keener. Further material on the calculus of variations and optimal control can be found in *Calculus of Variations and Optimal Control Theory: A Concise Introduction* by Daniel Liberzon. You should review some aspects of linear algebra, particularly the rank-nullity Theorem, the kernel and nullspace of a matrix, and how these ideas relate to eigenvalues and diagonalization.

1.1 Integral Equations

There are many different formulations of integral equations, but the following are four common nontrivial examples.

Volterra non-homogeneous

$$y(x) = f(x) + \int_a^x K(x,t)\, y(t)\, \mathrm{d}t, \qquad x \in [a,b].$$

Volterra homogeneous

$$y(x) = \int_a^x K(x,t)\, y(t)\, \mathrm{d}t, \qquad x \in [a,b].$$

Fredholm non-homogeneous

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\, y(t)\, \mathrm{d}t, \qquad x \in [a,b].$$

Fredholm homogeneous

$$y(x) = \lambda \int_a^b K(x,t)\, y(t)\, \mathrm{d}t, \qquad x \in [a,b].$$

The function $K(x,t)$ is the *kernel* of the integral equation.

A value of λ for which the homogeneous Fredholm equation has a solution which is not identically zero is called an *eigenvalue*, and the corresponding non-zero solution $y(x)$ is an *eigenfunction*.

As with boundary value problems, one can develop a spectral theory of the eigenvalues and eigenfunctions in order to express the solution to the inhomogeneous problem.

We remark that throughout we *always* consider $\lambda \neq 0$. Note that some literature will put the λ on the $y(x)$ on the left-hand side, or will relate the eigenvalues written each way via $\lambda = 1/\mu$; these conventions are unimportant as long as $\lambda \neq 0$ and one is careful where the λ appears. We will also only consider real solutions in this course, but the theory generalizes easily to the complex case. Finally, unless otherwise stated, all functions considered will be continuous. Rather than work out or present the general theory, we will focus on Fredholm equations of a particular type. First, we relate these operators to familiar boundary value problems.

1.1.1 Relationship with differential equations

Example 1. Consider the differential equation

$$y''(x) + \lambda y(x) = g(x),$$

where $\lambda > 0$ is constant and g is contonuous on $[a, b]$. Integrating from a to $x \in [a, b]$ gives

$$y'(x) - y'(a) + \lambda \int_a^x y(t)\,\mathrm{d}t = \int_a^x g(t)\,\mathrm{d}t.$$

Integrating again gives

$$y(x) - y(a) - y'(a)(x-a) + \lambda \int_a^x \int_a^u y(t)\,\mathrm{d}t\,\mathrm{d}u = \int_a^x \int_a^u g(t)\,\mathrm{d}t\,\mathrm{d}u.$$

Switching the order of integration gives

$$y(x) - y(a) - y'(a)(x-a) + \lambda \int_a^x (x-t)y(t)\,\mathrm{d}t = \int_a^x (x-t)g(t)\,\mathrm{d}t. \tag{1}$$

Initial conditions Suppose $y(a)$ and $y'(a)$ are given. Then we have a Volterra non-homogeneous integral equation with

$$K(x,t) = \lambda(t-x), \qquad f(x) = y(a) + y'(a)(x-a) + \int_a^x (x-t)g(t)\,\mathrm{d}t.$$

Boundary conditions Suppose $y(a)$ and $y(b)$ are given. Then, putting $x = b$ in (1)

$$y(b) - y(a) - y'(a)(b-a) + \lambda \int_a^b (b-t)y(t)\,\mathrm{d}t = \int_a^b (b-t)g(t)\,\mathrm{d}t,$$

so that

$$y'(a) = \frac{1}{b-a}\left(y(b) - y(a) + \lambda \int_a^b (b-t)y(t)\,\mathrm{d}t - \int_a^b (b-t)g(t)\,\mathrm{d}t\right).$$

On substituting into (1) and simplifying this gives the non-homogeneous Fredholm equation

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\,y(t)\,\mathrm{d}t$$

where

$$f(x) = y(a) + \frac{(x-a)}{(b-a)}(y(b)-y(a)) + \frac{1}{b-a}\int_a^x (x-b)(t-a)g(t)\,\mathrm{d}t + \frac{1}{b-a}\int_x^b (x-a)(t-b)g(t)\,\mathrm{d}t$$

$$K(x,t) = \begin{cases} \dfrac{(t-a)(b-x)}{b-a} & a \le t \le x \le b, \\ \dfrac{(x-a)(b-t)}{b-a} & a \le x \le t \le b. \end{cases}$$

■

NB: This kernel should look familiar to you from the study of Green's functions for boundary value problems.

2 Fredholm Alternative

The Fredholm Alternative is often considered one of the most important Theorems in applied mathematics (competing with Taylor's Theorem, among others). It gives a notion of 'solvability criterion' for a wide range of linear operators, and has numerous applications in differential and integral equations. Here we will present it first in the finite-dimensional case of linear algebra, followed by the cases of integral and differential equations. For integral equations of a particular type, the proof of this Theorem demonstrates how to construct solutions in a manner analogous to the eigenfunction expansions for boundary value problems.

2.1 Matrices

Consider a linear equation of the form,

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{2}$$

where $\boldsymbol{A}$ is an $m \times n$ real matrix, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$. Let $\boldsymbol{a_i}$ be the ith column of $\boldsymbol{A}$. Then we have a solvability condition as follows.

Proposition 1. The Fredholm Alternative for general matrices
Either

1. The system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$;

or

2. The system $\boldsymbol{A}^T\boldsymbol{v} = \boldsymbol{0}$ has a solution $\boldsymbol{v}$ with $\boldsymbol{v}^T\boldsymbol{b} \neq 0$;

Thus $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution $\boldsymbol{x}$ if and only if $\boldsymbol{v}^T\boldsymbol{b} = 0$ for every $\boldsymbol{v}$ in $\mathbb{R}^m$ such that $\boldsymbol{A}^T\boldsymbol{v} = \boldsymbol{0}$.

Proof.

$$\begin{aligned}
\boldsymbol{Ax} = \boldsymbol{b} \text{ has a solution } \boldsymbol{x} \quad &\Leftrightarrow \quad \boldsymbol{b} \text{ is a linear combination of the columns of } \boldsymbol{A} \\
&\Leftrightarrow \quad \boldsymbol{b} \in \operatorname{span}(\{\boldsymbol{a_1}, \cdots, \boldsymbol{a_n}\}) \\
&\Leftrightarrow \quad \operatorname{span}(\boldsymbol{b}) \subseteq \operatorname{span}(\boldsymbol{a_1}, \cdots, \boldsymbol{a_n}) \\
&\Leftrightarrow \quad \operatorname{span}(\boldsymbol{b})^\perp \supseteq \operatorname{span}(\boldsymbol{a_1}, \cdots, \boldsymbol{a_n})^\perp \\
&\Leftrightarrow \quad \text{every vector } \boldsymbol{v} \text{ with each } \boldsymbol{a_i}^T\boldsymbol{v} = 0 \text{ also has } \boldsymbol{b}^T\boldsymbol{v} = 0.
\end{aligned}$$

Note that $B^\perp$ denotes all vectors perpendicular to every vector in the set B. □

To relate this to integral equations we need to consider square matrices. Then we can write

Proposition 2. The Fredholm Alternative for square matrices
Either

1. The system $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution $\boldsymbol{x}$;

or

2. There exist nonzero solutions to the system $\boldsymbol{A}^T\boldsymbol{v} = \boldsymbol{0}$. In this case $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{v}^T\boldsymbol{b} = 0$ for every $\boldsymbol{v}$ such that $\boldsymbol{A}^T\boldsymbol{v} = \boldsymbol{0}$. Such a solution (if it exists) is not unique, since any null vector of A may be added to it.

2.2 Integral equations

We consider the non-homogeneous Fredholm equation

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\, y(t)\, \mathrm{d}t, \qquad x \in [a,b].$$

2.2.1 A simple case

To motivate the statement and proof of the theorem we consider the simplest possible degenerate kernel, that is, we set

$$K(x,t) = g(x)h(t), \qquad x, t \in [a,b],$$

where g and h are continuous on $[a,b]$. Then

$$y(x) = f(x) + \lambda \int_a^b g(x)h(t)y(t)\, \mathrm{d}t = f(x) + \lambda X g(x), \tag{3}$$

where

$$X = \int_a^b h(t)y(t)\, \mathrm{d}t.$$

We also need to consider the non-homogeneous transpose (adjoint) equation

$$y(x) = f(x) + \lambda \int_a^b g(t)h(x)y(t)\,\mathrm{d}t = f(x) + \lambda Y h(x), \tag{4}$$

where

$$Y = \int_a^b g(t)y(t)\,\mathrm{d}t.$$

In addition, we have the two corresponding homogeneous equations

$$y(x) = \lambda \int_a^b g(x)h(t)y(t)\,\mathrm{d}t = \lambda X g(x), \tag{5}$$

and the transpose

$$y(x) = \lambda \int_a^b g(t)h(x)y(t)\,\mathrm{d}t = \lambda Y h(x), \tag{6}$$

Multiplying (3) by $h(x)$ and integrating with respect to x gives

$$X\left(1 - \lambda \int_a^b g(x)h(x)\,\mathrm{d}x\right) = \int_a^b f(x)h(x)\,\mathrm{d}x. \tag{7}$$

Multiplying (4) by $g(x)$ and integrating with respect to x gives

$$Y\left(1 - \lambda \int_a^b g(x)h(x)\,\mathrm{d}x\right) = \int_a^b f(x)g(x)\,\mathrm{d}x. \tag{8}$$

Now, if

$$1 - \lambda \int_a^b g(x)h(x)\,\mathrm{d}x \neq 0$$

then we can solve for X and Y and hence determine the solutions

$$y(x) = f(x) + \frac{\lambda \int_a^b f(x)h(x)\,\mathrm{d}x}{1 - \lambda \int_a^b g(x)h(x)\,\mathrm{d}x} g(x)$$

of (3) and

$$y(x) = f(x) + \frac{\lambda \int_a^b f(x)g(x)\,\mathrm{d}x}{1 - \lambda \int_a^b g(x)h(x)\,\mathrm{d}x} h(x)$$

of (4).

However, if

$$1 - \lambda \int_a^b g(x)h(x)\,\mathrm{d}x = 0$$

then neither (3) or (4) has a unique solution. Indeed, in this case from (7), equation (3) cannot have a solution unless

$$\int_a^b f(x)h(x)\,\mathrm{d}x = 0,$$

and, from (8), equation (4) cannot have a solution unless

$$\int_a^b f(x)g(x)\,\mathrm{d}x = 0,$$

Now every solution of (5) must be of the form $y(x) = cg(x)$ for some constant c. Furthermore, if

$$1 - \lambda \int_a^b g(x)h(x)\,\mathrm{d}x = 0$$

and $y(x) = cg(x)$ then

$$\lambda X = \lambda c \int_a^b h(t)g(t)\,\mathrm{d}t = c,$$

so that $y = cg(x)$ is a solution for any c. Similarly, if $\lambda \int_a^b g(x)h(x)\,\mathrm{d}x = 1$, then $y(x) = dh(x)$ solves (6) for any d.

Thus we arrive at the following conclusion.

Proposition 3. Fredholm Alternative (Degenerate Integral Kernel)
Either

1. There are unique solutions to (3) and (4);

or

2. The are nonzero solutions to (5) and (6). In this case there exists a solution to (3) if and only if the integral of f times the solution of (6) vanishes ($\int_a^b f(x)h(x)\,\mathrm{d}x = 0$). If a solution exists it is nonunique, since any nonzero solution of (5) can be added.

If the solvability condition $\int_a^b f(x)h(x)\,\mathrm{d}x = 0$ is met then the general solution of (3) is

$$y(x) = f(x) + cg(x),$$

for all $c \in \mathbb{R}$.

2.3 Integral equations: general case

We consider the Fredholm equation

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)\,\mathrm{d}t, \qquad x \in [a,b] \tag{F}$$

along with the adjoint and homogeneous equations

$$y(x) = f(x) + \lambda \int_a^b K(t,x)y(t)\,\mathrm{d}t, \qquad x \in [a,b] \tag{F^T}$$

$$y(x) = \lambda \int_a^b K(x,t)y(t)\,\mathrm{d}t, \qquad x \in [a,b] \tag{H}$$

$$y(x) = \lambda \int_a^b K(t,x)y(t)\,\mathrm{d}t, \qquad x \in [a,b] \tag{H^T}$$

where $f : [a, b] \to \mathbb{R}$ and the kernel $K : [a, b]^2 \to \mathbb{R}$ are continuous and λ is constant.

Theorem 1. The Fredholm Alternative For each fixed λ exactly one of the following two statements is true. Either

1. The equation (F) has a unique continuous solution. In particular if $f \equiv 0$ on $[a, b]$ then $y \equiv 0$ on $[a, b]$. In this case (F^T) also has a unique continuous solution.

or

2. The equation (H) has a finite maximal linearly independent set of, say, r continuous solutions $y_1, \dots, y_r$ ($r > 0$). In this case (H^T) also has a maximal linearly independent set of r continuous solutions $z_1, \dots, z_r$ and (F) has a solution if and only if the solvability conditions

$$\int_a^b f(x) z_k(x) \, \mathrm{d}x = 0, \qquad k = 1, \dots, r,$$

are all satisfied. When they are, the complete solution to (F) is given by

$$y(x) = g(x) + \sum_{i=1}^{r} c_i y_i(x), \qquad x \in [a, b],$$

where $c_1, \dots, c_r$ are arbitrary constants and $g : [a, b] \to \mathbb{R}$ is any continuous solution to (F).

We sketch the proof of the theorem for the degenerate kernel

$$K(x, t) = \sum_{j=1}^{n} g_j(x) h_j(t), \qquad x, t \in [a, b].$$

Proof. We may assume that each of the sets $\{g_1, g_2, \cdots, g_n\}$ and $\{h_1, h_2, \cdots, h_n\}$ are linearly independent (otherwise express each element in terms of a linearly independent subset). Then we have

$$y(x) = f(x) + \lambda \sum_{j=1}^{n} X_j g_j(x), \qquad \text{where} \qquad X_j = \int_a^b h_j(t) y(t) \, \mathrm{d}t, \tag{F$_1$}$$

$$y(x) = f(x) + \lambda \sum_{j=1}^{n} Y_j h_j(x), \qquad \text{where} \qquad Y_j = \int_a^b g_j(t) y(t) \, \mathrm{d}t, \tag{F$_1^T$}$$

$$y(x) = \lambda \sum_{j=1}^{n} X_j g_j(x), \tag{H$_1$}$$

$$y(x) = \lambda \sum_{j=1}^{n} Y_j h_j(x). \tag{H$_1^T$}$$

Multiply (F_1) by $h_i(x)$ and integrate over x to give

$$\mu X_i - \sum_{j=1}^{n} a_{ij} X_j = b_i,$$

where

$$\mu = \frac{1}{\lambda}, \qquad a_{ij} = \int_a^b g_j(x) h_i(x)\,\mathrm{d}x, \qquad b_i = \mu \int_a^b f(x) h_i(x)\,\mathrm{d}x.$$

We may write this as

$$(\mu I - A)\,\mathbf{X} = \mathbf{b} \tag{F$_2$}$$

where $\mathbf{X} = (X_j)$ and $\mathbf{b} = (b_j)$ are column vectors, $A = (a_{ij})$ is a matrix, and I is the identity matrix. Similarly (F_1^T) becomes

$$(\mu I - A)^T\,\mathbf{Y} = \left(\mu I - A^T\right)\mathbf{Y} = \mathbf{c} \tag{F$_2^T$}$$

where A^T is the transpose of A and $\mathbf{c} = (c_j)$ with

$$c_j = \mu \int_a^b f(x) g_i(x)\,\mathrm{d}x.$$

Similarly (H_1) and (H_1^T) become

$$(\mu I - A)\,\mathbf{X} = \mathbf{0} \tag{H$_2$}$$

$$(\mu I - A)^T\,\mathbf{Y} = \mathbf{0} \tag{H$_2^T$}$$

Now we are back in the case of linear algebra. So, suppose that there are no nontrivial solutions to (H_2), i.e., that μ is not an eigenvector of A. Then, since $\mu I - A$ is nonsingular, there are unique solutions to (F_2) and (F_2^T), thus (1) holds.

On the other hand, suppose μ is an eigenvalue of A with eigenspace of dimension r spanned by eigenvectors $\mathbf{X}^k$, $k = 1, \ldots, r$. Then the corresponding eigenspace of A^T is also of dimension r and spanned by $\mathbf{Y}^k$, $k = 1, \ldots, r$, say. Then

$$y_k(x) = \lambda \sum_{j=1}^{n} X_j^k g_j(x), \tag{9}$$

$$z_k(x) = \lambda \sum_{j=1}^{n} Y_j^k h_j(x), \tag{10}$$

form a maximal set of linearly independent solutions of (H) and (H^T) respectively. We know (F_2) has a solution if and only if

$$\mathbf{b}^T \mathbf{Y}^k = 0, \qquad k = 1, \ldots, r,$$

which is, noting from (10) that $\mathbf{Y}^k$ corresponds to the solution $z_k(x)$ of (H^T),

$$\sum_{j=1}^{n} \left(\mu \int_a^b f(x) h_j(x)\,\mathrm{d}x \right) \left(\int_a^b g_j(t) z_k(t)\,\mathrm{d}t \right) = 0.$$

Rearranging, this is

$$\int_a^b \left(\int_a^b \left(\sum_{j=1}^n g_j(t) h_j(x) \right) z_k(t)\, \mathrm{d}t \right) f(x)\, \mathrm{d}x = 0,$$

i.e.

$$\int_a^b \left(\int_a^b K(t,x) z_k(t)\, \mathrm{d}t \right) f(x)\, \mathrm{d}x = 0,$$

which gives

$$\int_a^b z_k(x) f(x)\, \mathrm{d}x = 0,$$

since z_k is a solution of (H^T).

$\square$

This method of proof can be used to solve (F) for degenerate kernels.

Example 2. Solve the integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} \sin(x+t) y(t)\, \mathrm{d}t,$$

in the two cases

(a) $f(x) = 1$;

(b) $f(x) = x$.

The equation may be written

$$\begin{aligned} y(x) &= f(x) + \lambda \int_0^{2\pi} (\sin x \cos t + \cos x \sin t)\, y(t)\, \mathrm{d}t \\ &= f(x) + \lambda X_1 \sin x + \lambda X_2 \cos x \end{aligned} \tag{11}$$

where

$$X_1 = \int_0^{2\pi} y(t) \cos t\, \mathrm{d}t, \qquad X_2 = \int_0^{2\pi} y(t) \sin t\, \mathrm{d}t.$$

Note that it is self-adjoint. Multiplying (11) by $\cos x$ (and $\sin x$) and integrating with respect to x gives

$$X_1 - \lambda\pi X_2 = \int_0^{2\pi} f(x) \cos x\, \mathrm{d}x, \tag{12}$$

$$X_2 - \lambda\pi X_1 = \int_0^{2\pi} f(x) \sin x\, \mathrm{d}x \tag{13}$$

since
$$\int_0^{2\pi} \cos^2 x \, \mathrm{d}x = \int_0^{2\pi} \sin^2 x \, \mathrm{d}x = \pi, \qquad \int_0^{2\pi} \cos x \sin x \, \mathrm{d}x = 0.$$
This system is invertible if the determinant of the coefficient matrix
$$\begin{vmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{vmatrix} = 1 - \lambda^2\pi^2 \neq 0.$$
In this case the (unique) solution is
$$X_1 = \frac{1}{1-\lambda^2\pi^2} \int_0^{2\pi} f(x) \left(\cos x + \lambda\pi \sin x\right) \, \mathrm{d}x,$$
$$X_2 = \frac{1}{1-\lambda^2\pi^2} \int_0^{2\pi} f(x) \left(\sin x + \lambda\pi \cos x\right) \, \mathrm{d}x.$$
Since
$$\int_0^{2\pi} x \sin x \, \mathrm{d}x = -2\pi, \qquad \int_0^{2\pi} \cos x \, \mathrm{d}x = \int_0^{2\pi} \sin x \, \mathrm{d}x = \int_0^{2\pi} x \cos x \, \mathrm{d}x = 0,$$
in case (a) we have $X_1 = X_2 = 0$ and therefore
$$y(x) = 1, \qquad x \in [0, 2\pi],$$
while in case (b) we have
$$y(x) = x - \frac{2\pi\lambda}{1-\lambda^2\pi^2} \left(\lambda\pi \sin x + \cos x\right),$$
provided $\lambda^2\pi^2 \neq 1$.

When $\lambda = 1/\pi$ the homogeneous version of (12)-(13) has solutions $X_1 = X_2$, while when $\lambda = -1/\pi$ it has solutions $X_1 = -X_2$. Thus the homogeneous version of (11) has solutions
$$y(x) = c(\sin x + \cos x) \qquad \text{when } \lambda = 1/\pi,$$
$$y(x) = d(\sin x - \cos x) \qquad \text{when } \lambda = -1/\pi,$$
where c and d are constants. Thus in order for solutions to exist we have the solvability conditions
$$\int_0^{2\pi} f(x)(\sin x + \cos x) \, \mathrm{d}x = 0, \qquad \text{when } \lambda = 1/\pi,$$
and
$$\int_0^{2\pi} f(x)(\sin x - \cos x) \, \mathrm{d}x = 0, \qquad \text{when } \lambda = -1/\pi.$$
In case (a) both conditions are met. Since $y = 1$ is a particular solution when $\lambda = \pm 1/\pi$, the general solution in this case is
$$y(x) = 1 + c(\sin x + \cos x) \qquad \text{when } \lambda = 1/\pi,$$
$$y(x) = 1 + d(\sin x - \cos x) \qquad \text{when } \lambda = -1/\pi,$$
where c and d are arbitrary constants.

In case (b) neither condition is met and there are no solutions when $\lambda = \pm 1/\pi$.

■

2.4 Linear ordinary differential equations

We are going to describe solvability conditions for linear ODE's analogous to those for linear algebraic equations. We will do this for the 2nd order real scalar case, and give the general version later.

Consider a differential operator

$$L[u] = \frac{d^2u}{dx^2} + \alpha(x)\frac{du}{dx} + \beta(x)u = u'' + \alpha u' + \beta u,$$

where $\alpha(x)$, $\beta(x)$ are continuous real-valued functions on $[0,1]$. We are going to consider:

Primary problem

$$L[u] = b(x) \qquad \text{on } 0 \le x \le 1,$$

with 2 linear homogeneous boundary conditions on u and u' at $x = 0$, 1.

Adjoint problem

$$L^*[v] = 0 \qquad \text{on } 0 \le x \le 1,$$

with 2 linear homogeneous boundary conditions on v and v' at $x = 0$, 1.

The solvability result is that

$$\boxed{\text{Primary has a solution } u \Leftrightarrow \int_0^1 v(x)b(x)\,dx = 0 \text{ for every solution } v \text{ of the Adjoint problem}}$$

The adjoint differential operator is

$$L^*[v] = v'' - (\alpha v)' + \beta v.$$

This obeys the fundamental identity

$$\begin{aligned}\int_0^1 (vL[u] - uL^*[v])\,dx &= \int_0^1 v(u'' + \alpha u' + \beta u) - u(v'' - (\alpha v)' + \beta v)\,dx \\ &= \left[vu' - uv' + \alpha uv\right]_0^1 \\ &= B(u,v),\end{aligned}$$

a bilinear form in the *boundary values* of u and v. This bilinear form is non-singular if $B(u,v) = 0$ for all v implies $u = 0$. Equivalently

$$B(u,v) = \begin{pmatrix} v(1) & v'(1) & v(0) & v'(0) \end{pmatrix} \begin{pmatrix} \alpha(1) & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -\alpha(0) & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u(1) \\ u'(1) \\ u(0) \\ u'(0) \end{pmatrix}$$

and $B(u,v)$ is non-singular if the central matrix is non-singular. Then if $u(1)$, $u'(1)$, $u(0)$, $u'(0)$ obey 2 linear homogeneous equations (the primary boundary conditions) then we shall need 2 linear homogeneous equations on $v(1)$, $v'(1)$, $v(0)$, $v'(0)$ to force $B(u,v) = 0$ (there are 2 degrees of freedom left). These conditions on v are the adjoint boundary conditions.

2.4.1 Examples of adjoints

Example 3. Suppose the primary boundary conditions are

$$u(0) = 0, \qquad u'(0) = 0 \qquad\qquad \text{Primary boundary conditions.}$$

(an initial value problem, IVP).Then

$$B(u, v) = v(1)u'(1) - v'(1)u(1) + \alpha(1)v(1)u(1).$$

To force this to vanish (for arbitrary $u(1)$, $u'(1)$) we must have

$$v(1) = 0, \qquad v'(1) = 0 \qquad\qquad \text{Adjoint boundary conditions.}$$

■

Example 4. Suppose the primary boundary conditions are

$$u(0) = 0, \qquad u(1) = 0 \qquad\qquad \text{Primary boundary conditions.}$$

(a boundary value problem, BVP).Then

$$B(u, v) = v(1)u'(1) - v(0)u'(0).$$

To force this to vanish (for arbitrary $u'(0)$, $u'(1)$) we must have

$$v(0) = 0, \qquad v(1) = 0 \qquad\qquad \text{Adjoint boundary conditions.}$$

■

Example 5. Suppose the primary boundary conditions are

$$u(0) = u(1), \qquad u'(1) = 0 \qquad\qquad \text{Primary boundary conditions.}$$

(a generalised boundary value problem).Then

$$B(u, v) = -v'(1)u(0) + \alpha(1)v(1)u(0) - v(0)u'(0) + v'(0)u(0) - \alpha(0)v(0)u(0).$$

To force this to vanish (for arbitrary $u(0)$, $u'(0)$) we must have

$$v(0) = 0, \qquad v'(1) - \alpha(1)v(1) = v'(0) \qquad\qquad \text{Adjoint boundary conditions.}$$

■

Easy part of proof If a solution u of the primary problem exists, and v is any solution of the adjoint problem, then

$$\int_0^1 (vL[u] - uL^*[v]) \; dx = B(u,v) = 0.$$

We then multiply the primary problem by v and integrate, and multiply the adjoint problem by u and integrate, then subtract the first from the second to get,

$$\int_0^1 (vL[u] - uL^*[v]) \; dx = \int_0^1 vb \, dx = 0.$$

The harder part (if adjoint condition holds then a solution exists) requires 2 steps:

1. Convert the ode problem to an integral equation by using a Green's function.
2. Use the "Fredholm Alternative" theory of integral equations to write down solvability conditions for the integral equation.

This is why the solvability condition for ODE's is sometimes called the Fredholm Alternative.

2.4.2 Applications

Example 6. Primary:

$$u'' = b(x), \qquad u'(0) = u'(1) = 0.$$

Adjoint:

$$v'' = 0, \qquad v'(0) = v'(1) = 0.$$

There is a nontrivial solution of the adjoint, namely

$$v = 1.$$

Hence there is a solution of the primary if and only if

$$\int_0^1 b(x) \, dx = 0.$$

■

Example 7. Find the asymptotic solution of the equation

$$\ddot{x} + (1+\epsilon)x = \cos t, \qquad x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi), \tag{14}$$

as $\epsilon \to 0$. Suppose we try a perturbation expansion

$$x(t) \sim x_0(t) + \epsilon x_1(t) + \cdots . \tag{15}$$

Substituting into the equation gives

$$(\ddot{x}_0 + \epsilon \ddot{x}_1 + \cdots) + (1+\epsilon)(x_0 + \epsilon x_1 + \cdots) = \cos t.$$

Expanding the brackets gives

$$\ddot{x}_0 + x_0 + \epsilon(\ddot{x}_1 + x_0 + x_1) + \cdots = \cos t.$$

Equating coefficients of powers of ϵ gives

$$\ddot{x}_0 + x_0 = \cos t, \qquad \ddot{x}_1 + x_0 + x_1 = 0, \qquad \cdots$$

Thus the leading-order problem is

$$\ddot{x}_0 + x_0 = \cos t, \qquad x_0(0) = x_0(2\pi), \quad \dot{x}_0(0) = \dot{x}_0(2\pi). \tag{16}$$

Note that this is self-adjoint. Is there a solution? The homogeneous version

$$\ddot{x}_0 + x_0 = 0, \qquad x_0(0) = x_0(2\pi), \quad \dot{x}_0(0) = \dot{x}_0(2\pi),$$

has solutions

$$x_0 = \cos t, \qquad \text{and} \qquad x_0 = \sin t.$$

Since

$$\int_0^{2\pi} \cos^2 t \, \mathrm{d}t \neq 0$$

we conclude that (16) has no solution. This does not mean that (14) has no solution: it means that our expansion (15) was incorrect. In (14) we are forcing with a term that is almost resonant (it is resonant when $\epsilon = 0$). Thus we expect the response to be large. Let us try instead

$$x(t) \sim \frac{1}{\epsilon} x_0(t) + x_1(t) + \cdots . \tag{17}$$

Substituting into the equation gives

$$\ddot{x}_0 + x_0 + \epsilon(\ddot{x}_1 + x_0 + x_1) + \cdots = \epsilon \cos t.$$

Equating coefficients of powers of ϵ now gives

$$\ddot{x}_0 + x_0 = 0, \qquad \ddot{x}_1 + x_0 + x_1 = \cos t, \qquad \cdots$$

This time the leading-order problem is

$$\ddot{x}_0 + x_0 = 0, \qquad x_0(0) = x_0(2\pi), \quad \dot{x}_0(0) = \dot{x}_0(2\pi), \tag{18}$$

with solution

$$x_0 = A\cos t + B \sin t,$$

where A and B are arbitrary constants, undetermined at this stage. To determine A and B we need to consider the equation at next order. This is

$$\ddot{x}_1 + x_0 + x_1 = \cos t,$$

or, using our expression for x_0,

$$\ddot{x}_1 + x_1 = (1 - A)\cos t - B\sin t, \qquad x_1(0) = x_1(2\pi), \quad \dot{x}_1(0) = \dot{x}_1(2\pi).$$

Now we use the Fredholm alternative again. There is a solution for x_1 if and only if the right-hand side is orthogonal to the solutions $\cos t$ and $\sin t$ of the homogeneous problem. Multiplying by $\cos t$ and integrating gives

$$1 - A = 0 \qquad \Rightarrow \quad A = 1.$$

Multiplying by $\sin t$ and integrating gives

$$B = 0.$$

Thus the leading order solution is

$$x \sim \frac{1}{\epsilon}\cos t.$$

In fact, this leading-order solution *is* the exact solution of the original problem, and we can continue looking at higher-order terms to see they are all zero. While this is a linear problem (and hence we could have solved the problem directly), this example illustrates a powerful combined use of asymptotic methods and solvability conditions which is widely applicable for nonlinear systems, especially oscillators. The solvability theory always tells us something important in the case that $\epsilon = 0$ – namely, that there is no solution, as we saw above in the regular perturbation expansion.

■

Example 8. Consider the equation

$$\epsilon\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = u - u^3 + \epsilon, \qquad u'(-\infty) = 0, \quad u'(\infty) = 0,$$

(with $u(-\infty)$ close to -1 and $u(\infty)$ close to 1). Consider an expansion

$$u \sim u_0 + \epsilon u_1 + \cdots .$$

Then, at leading order (equating coefficents of ϵ^0)

$$-\frac{\partial^2 u_0}{\partial x^2} = u_0 - u_0^3, \qquad u_0(-\infty) = -1, \quad u_0(\infty) = 1.$$

The solution is

$$u_0 = \tanh\left(\frac{x - x_0(t)}{\sqrt{2}}\right),$$

where $x_0(t)$ is arbitrary. This is the solution to the steady problem with $\epsilon = 0$, but it can be translated arbitrarily. To determine x_0 we need to go to the next order. At first order (equating coefficients of ϵ^1)

$$\frac{\partial u_0}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = u_1 - 3u_0^2 u_1 + 1, \qquad u_1'(-\infty) = 0, \quad u_1'(\infty) = 0.$$

Rearranging
$$-\frac{\partial^2 u_1}{\partial x^2} - u_1 + 3u_0^2 u_1 = 1 - \frac{\partial u_0}{\partial t} = 1 + \frac{\mathrm{d}x_0}{\mathrm{d}t}\frac{\partial u_0}{\partial x}. \tag{19}$$
Now, since
$$-\frac{\partial^2 u_0}{\partial x^2} - u_0 + u_0^3 = 0,$$
differentiating gives
$$-\frac{\partial^3 u_0}{\partial x^3} - \frac{\partial u_0}{\partial x} + 3u_0^2\frac{\partial u_0}{\partial x} = 0.$$
Thus $u_1 = \partial u_0/\partial x$ satisfies the homogeneous version of (19). Therefore, by the Fredholm Alternative, the right-hand side must be orthogonal to $\partial u_0/\partial x$:
$$0 = \int_{-\infty}^{\infty}\left(1 + \frac{\mathrm{d}x_0}{\mathrm{d}t}\frac{\partial u_0}{\partial x}\right)\frac{\partial u_0}{\partial x} = [u_0]_{-\infty}^{\infty} + \frac{\mathrm{d}x_0}{\mathrm{d}t}\int_{-\infty}^{\infty}\left(\frac{\partial u_0}{\partial x}\right)^2 \mathrm{d}x = 2 + \frac{\mathrm{d}x_0}{\mathrm{d}t}\int_{-\infty}^{\infty}\left(\frac{\partial u_0}{\partial x}\right)^2 \mathrm{d}x.$$
Thus
$$\frac{\mathrm{d}x_0}{\mathrm{d}t} = -\frac{2}{\int_{-\infty}^{\infty}(\partial u_0/\partial x)^2\,\mathrm{d}x}.$$

■

Example 9. Consider the equation for $y(x)$:
$$y'' + Ty + y^3 = 0, \qquad y(0) = 0, \quad y(1) = 0. \tag{20}$$
Let us first consider the linearised equation:
$$y'' + Ty = 0, \qquad y(0) = 0, \quad y(1) = 0.$$
This is an eigenvalue problem: there are solutions only for particular values of T. After imposing the boundary condition at $x = 0$ we have
$$y = \sin\sqrt{T}x.$$
The condition at $x = 1$ then implies
$$\sin\sqrt{T} = 0 \qquad \Rightarrow \qquad T = n^2\pi^2.$$
Then the solution is
$$y = A\sin n\pi x, \qquad T = n^2\pi^2,$$
where A is arbitrary.

Let us see how the nonlinear term affects this calculation when we are close to the bifurcation point $T = n^2\pi^2$. Returning to (20) let us pose an expansion
$$y = \epsilon y_0 + \epsilon^3 y_1 + \cdots, \qquad T = T_0 + \epsilon^2 T_1 + \cdots.$$

Then, equating coefficients of ϵ^1:

$$y_0'' + T_0 y_0 = 0, \qquad y_0(0) = 0, \quad y_0(1) = 0,$$

so that

$$T_0 = n^2\pi^2, \qquad y_0 = A\sin n\pi x,$$

as above. The coefficient A is determined by proceeding to next order. Equating coefficients of ϵ^3:

$$y_1'' + T_0 y_1 + T_1 y_0 + y_0^3 = 0, \qquad y_1(0) = 0, \quad y_1(1) = 0.$$

Substituting in for y_0, T_0 gives

$$y_1'' + n^2\pi^2 y_1 = -AT_1 \sin n\pi x - A^3 \sin^3 n\pi x. \tag{21}$$

Now the homogeneous equation is satisfied by $\sin n\pi x$. Thus in order for there to be a solution for y_1, by the Fredholm Alternative the right-hand side must be orthoginonal to $\sin n\pi x$. Thus

$$0 = \int_0^1 AT_1 \sin^2 n\pi x + A^3 \sin^4 n\pi x \,\mathrm{d}x = \frac{AT_1}{2} + \frac{3A^3}{8}.$$

since

$$\int_0^1 \sin^2 n\pi x \,\mathrm{d}x = \frac{1}{2}, \qquad \int_0^1 \sin^2 n\pi x \,\mathrm{d}x = \frac{3}{8}.$$

As an alternative to evaluating the integrals we observe

$$\begin{aligned} \sin^3 n\pi x &= \left(\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}n\pi x} - \mathrm{e}^{-\mathrm{i}n\pi x}\right)\right)^3 = -\frac{1}{8\mathrm{i}}\left(\mathrm{e}^{3\mathrm{i}n\pi x} - 3\mathrm{e}^{\mathrm{i}n\pi x} + 3\mathrm{e}^{\mathrm{i}n\pi x} - \mathrm{e}^{-3\mathrm{i}n\pi x}\right)^3 \\ &= -\frac{1}{4}\left(\sin 3n\pi x - 3\sin n\pi x\right). \end{aligned}$$

Thus the right-hand side of (21) is

$$-AT_1 \sin n\pi x + A^3 \frac{1}{4}\left(\sin 3n\pi x - 3\sin n\pi x\right).$$

We know that $\sin 3n\pi x$ is orthogonal to $\sin n\pi x$. Thus we need the coefficient of $\sin n\pi x$ to vanish, i.e.

$$-AT_1 - \frac{3A^3}{4} = 0.$$

Thus the amplitude is

$$A = \sqrt{-\frac{4T_1}{3}}.$$

Note that this means that the branch of solutions exists for $T_1 < 0$, i.e. for T slightly less than the critical value $n^2\pi^2$.

■

2.4.3 Generalisation

Suppose $\boldsymbol{u}$ is a vector of complex-valued functions, obeying a higher-order primary problem

Primary

$$L[\boldsymbol{u}] = \boldsymbol{b}(x) \qquad \text{on } 0 \leq x \leq 1,$$

with primary boundary conditions on $\boldsymbol{u}$ at $x = 0$, 1, where

$$L[\boldsymbol{u}] = \sum_{r=0}^{k} \boldsymbol{A_r}(x) \frac{d^r \boldsymbol{u}}{dx^r} = \sum_{r=0}^{k} \boldsymbol{A_r}(x) \boldsymbol{u}^{(r)},$$

where the $\boldsymbol{A_r}(x)$ are matrices, continuous in x, and $\boldsymbol{b}$ is a vector of continuous functions. To state the adjoint problem we introduce some notation.

1. $\boldsymbol{A}^*$ = conjugate of transpose of A [like A' in Matlab].
2. If $\boldsymbol{v}$ is a vector of continuous functions (same order as $\boldsymbol{b}$) then define an inner product

$$\langle \boldsymbol{v}, \boldsymbol{b} \rangle = \int_0^1 \boldsymbol{v}(x)^* \boldsymbol{b}(x)\, dx = \sum_i \int_0^1 \overline{v_i(x)} b_i(x)\, dx.$$

Then

$$\text{Primary has a solution } \boldsymbol{u} \Leftrightarrow \langle \boldsymbol{v}, \boldsymbol{b} \rangle = 0 \quad \text{for every solution } \boldsymbol{v} \text{ of the Adjoint problem}$$

Adjoint

$$L^*[\boldsymbol{v}] = 0,$$

with adjoint boundary conditions on v at $x = 0$, 1. The adjoint differential operator is

$$L^*[\boldsymbol{v}] = \sum_{r=0}^{k} (-1)^r \left(\boldsymbol{A_r^*} \boldsymbol{v} \right)^{(r)}.$$

The fundamental identity is

$$\begin{aligned} \langle \boldsymbol{v}, L[\boldsymbol{u}] \rangle - \langle L^*[\boldsymbol{v}], \boldsymbol{u} \rangle &= \int_0^1 \sum_r \left(\boldsymbol{v}^* \boldsymbol{A_r} \boldsymbol{u}^{(r)} - (-1)^r (\boldsymbol{v}^* \boldsymbol{A_r^*})^{(r)} \boldsymbol{u} \right) dx \\ &= \left[\sum_r \boldsymbol{v}^* \boldsymbol{A_r} \boldsymbol{u}^{(r-1)} - (\boldsymbol{v}^* \boldsymbol{A_r})' \boldsymbol{u}^{(r-2)} + \cdots + (-1)^{r-1} (\boldsymbol{v}^* \boldsymbol{A_r})^{(r-1)} \boldsymbol{u} \right]_0^1 \\ &= B(\boldsymbol{u}, \boldsymbol{v}). \end{aligned}$$

This B is used to construct the adjoint boundary conditions exactly as in the basic case considered earlier (B is a Hermitian form now). The easy part of the proof is just as before.

3 Calculus of variations

Start with a **simple example**: consider a plane curve joining two points (a, c) and (b, d) and given by the smooth graph $y = y(x)$.

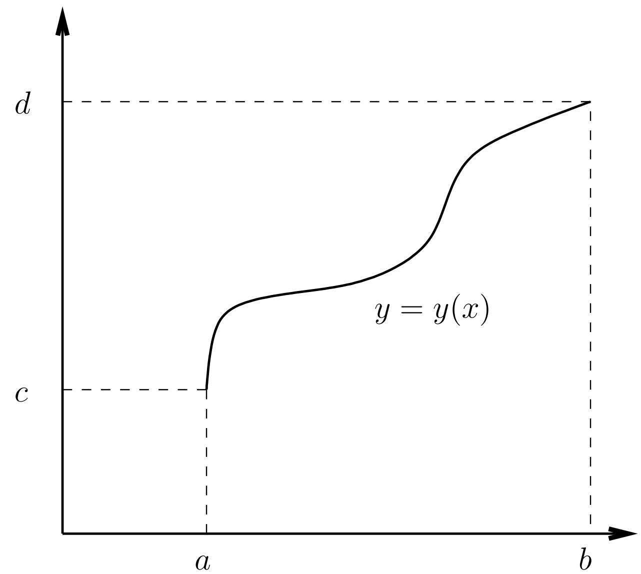


Figure 1:

NB this disallows some slopes.

Define the **functional**

$$J[y] = \int_a^b (y'(x))^2 \, \mathrm{d}x.$$

NB $J : V \to \mathbb{R}$, where V is a suitable **function space**, e.g. the set $C^2[a, b]$ of twice continuously differentiable functions $y(x)$ defined on $[a, b]$, satisfying $y(a) = c$ and $y(b) = d$ [we won't dwell much on the strict conditions on $y(x)$].

Now we ask: which function $y(x) \in V$ **minimises** the functional $J[y]$?

To answer this, let $y(x)$ be the desired *extremal* function which minimises $J[y]$. Then any admissable perturbation about $y(x)$ should *increase* J. So consider $J[y + \epsilon\eta]$, where $\eta \in C^2[a, b]$ with $\eta(a) = \eta(b) = 0$. Now

$$\begin{aligned} J[y + \epsilon\eta] &= \int_a^b (y'(x) + \epsilon\eta'(x))^2 \, \mathrm{d}x \\ &= J[y] + 2\epsilon \int_a^b y'(x)\eta'(x) \, \mathrm{d}x + \epsilon^2 \int_a^b (\eta'(x))^2 \, \mathrm{d}x. \end{aligned}$$

We want this to have a **minimum** when $\epsilon = 0$, and a necessary condition is

$$\int_a^b y'(x)\eta'(x)\,\mathrm{d}x = 0.$$

[Then the coefficient of $\epsilon^2 \geq 0$ so it *is* a minimum not a maximum.]

Now integrate by parts to give

$$\underbrace{\big[y'(x)\eta(x)\big]_a^b}_{=0 \text{ since } \eta(a)=\eta(b)=0} - \int_a^b \eta(x)y''(x)\,\mathrm{d}x = 0.$$

We deduce that

$$\int_a^b \eta(x)y''(x)\,\mathrm{d}x = 0$$

for all $\eta \in C^2[a,b]$ with $\eta(a) = \eta(b) = 0$.

Fundamental Lemma of Calculus of Variations (FLCV)

If

$$\int_a^b \eta(x)\phi(x)\,\mathrm{d}x = 0 \qquad \forall \eta \in C^2[a,b] \text{ with } \eta(a) = \eta(b) = 0,$$

and ϕ is **continuous**, then

$$\phi(x) \equiv 0 \qquad \text{on } [a,b].$$

Hence we find that the function $y(x)$ that minimises $J[y]$ satisfies

$$y''(x) \equiv 0,$$

i.e.

$$y = Ax + B = c + \frac{(d-c)}{(b-a)}(x-a),$$

which is a **straight line** from (a,c) to (b,d).

Possible motivations

(i) 1-d flow of electricity through a semiconductor. $\phi(x) =$ electric potential (voltage).

The energy dissipated (as heat) in the medium is given by

$$J[\phi] = \int_0^l \sigma(x)(\phi'(x))^2\,\mathrm{d}x,$$

where $\sigma(x)$ is the ocnductivity of the medium. So dissipation is minimised when ϕ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sigma(x)\phi'(x)) = 0.$$

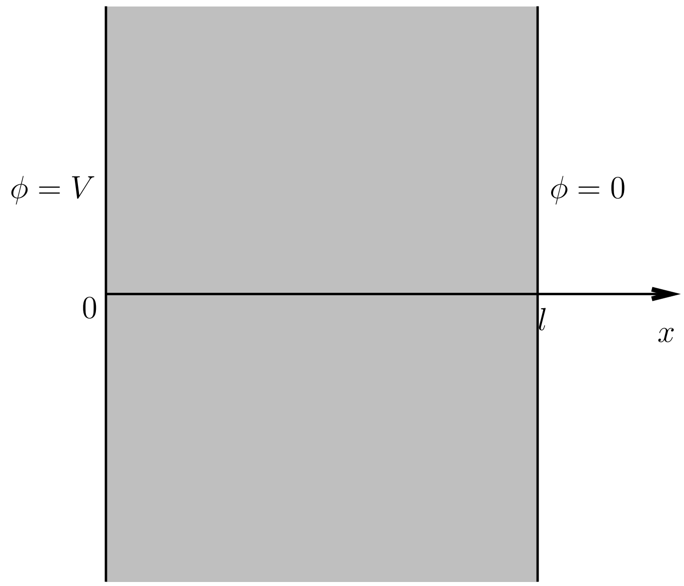


Figure 2:

(ii) Drive from A to B in a given time T. Let your position at time t be $x(t)$. Then $x(0) = a$, $x(T) = b$. Suppose there is a frictional resistance $k\dot{x}(t)$. Then the work done against friction during the journey is

$$J[x] = \int_0^T k\dot{x}(t)^2 \, \mathrm{d}t.$$

This suggests that driving at **constant speed** ($\ddot{x} = 0$) is the most efficient.

Class of problems

This simple example falls into a class of problems: to minimise or maximise a functional

$$J[y] = \int_a^b F(x, y(x), y'(x)) \, \mathrm{d}x$$

(where $F(x, y, y')$ is given) over all $y \in C^2[a, b]$ satisfying $y(a) = c$, $y(b) = d$.

Let $y(x)$ be an extremal function and perturb:

$$J[y + \epsilon\eta] = \int_a^b F(x, y + \epsilon\eta, y' + \epsilon\eta') \, \mathrm{d}x,$$

where $\eta \in C^2[a, b]$ with $\eta(a) = \eta(b) = 0$. Expand using Taylor's theorem:

$$J[y + \epsilon\eta] \quad = \quad J[y] + \epsilon \int_a^b \left(\eta \frac{\partial F}{\partial y}(x, y, y') + \eta' \frac{\partial F}{\partial y'}(x, y, y') \right) \mathrm{d}x + O(\epsilon^2).$$

NB here we treat x, y and y' as independent variables.

At an extremal we must have

$$\int_a^b \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} \right) \mathrm{d}x = 0.$$

Integrate by parts:

$$\int_a^b \eta \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) \right) \mathrm{d}x + \underbrace{\left[\eta \frac{\partial F}{\partial y'} \right]_a^b}_{=0 \text{ since } \eta(a)=\eta(b)=0} = 0.$$

Since this is true for all $\eta \in C^2[a, b]$ with $\eta(a) = \eta(b) = 0$ by the FLCV we have Euler's equation (basic equation of Calculus of Variations):

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) - \frac{\partial F}{\partial y} = 0}$$

NB $\mathrm{d}/\mathrm{d}x$ not $\partial/\partial x$.

Examples

(i) In our previous example $F(x, y, y') = (y')^2$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(2y') = 0, \qquad \text{i.e.} \qquad y'' = 0.$$

(ii) Curve of minimum length joining (a, c) to (b, d). Length

$$J[y] = \int_a^b \sqrt{1 + (y')^2}\, \mathrm{d}x,$$

subject to $y(a) = c$, $y(b) = d$. Then

$$F = \sqrt{1 + (y')^2}, \qquad \frac{\partial F}{\partial y} = 0, \qquad \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}}.$$

So Euler's equation is

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) = \frac{y''}{(1 + (y')^2)^{3/2}} = 0.$$

Thus $y'' = 0$ so $y = Ax + B$. Linear (again). Thus

$$y(x) = c + \frac{(d - c)}{(b - a)}(x - a),$$

a straight line, as expected.

Extensions

Natural boundary conditions

This time let

$$J[y] = \int_a^b F(x, y, y')\,\mathrm{d}x$$

where $y(a) = c$ but $y(b)$ is NOT prescribed. Again let $y(x)$ be an extremal of $J[y]$ and consider $y + \epsilon\eta$, where $\eta(a) = 0$ but $\eta(b)$ is *arbitrary*. Then

$$\begin{aligned} J[y + \epsilon\eta] &= \int_a^b F(x, y, y')\,\mathrm{d}x \\ &\sim J[y] + \epsilon \int_a^b \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'}\right)\mathrm{d}x + O(\epsilon^2). \end{aligned}$$

At an extremal, we must have

$$\begin{aligned} &\int_a^b \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'}\right)\mathrm{d}x = 0 \\ \Rightarrow \qquad &\int_a^b \eta \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right)\mathrm{d}x + \left[\eta \frac{\partial F}{\partial y'}\right]_b^a = 0. \end{aligned}$$

This is true for all $\eta \in C^2[a, b]$ satisfying $\eta(a) = 0$. *In particular* it is true for all $\eta \in C^2[a, b]$ satisfying $\eta(a) = \eta(b) = 0$, so

$$\int_a^b \eta \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right)\mathrm{d}x + \left[\eta \frac{\partial F}{\partial y'}\right]_b^a = 0, \qquad \forall \eta \in C^2[a, b] \text{ such that } \eta(a) = 0.$$

Then FLCV $\Rightarrow$ Euler's equation again. Now we are left with

$$\left[\eta \frac{\partial F}{\partial y'}\right]_b^a = 0 = \eta(b)\left.\frac{\partial F}{\partial y'}\right|_{x=b}.$$

Since $\eta(b)$ is arbitrary we must have

$$\boxed{\frac{\partial F}{\partial y'} = 0 \qquad \text{at } x = b.}$$

This is the **natural boundary condition** applied at any boundary where no boundary conditions are prescribed in advance.

Trivial Example
Minimise the length

$$J[y] = \int_a^b \sqrt{1 + (y')^2}\,\mathrm{d}x$$

subject to $y(a) = c$ but $y(b)$ kept free.

Euler equation is

$$y'' = 0 \qquad \Rightarrow \qquad y = Ax + B.$$

Boundary conditions.

Imposed boundary condition

$$y(a) = c.$$

Natural boundary condition

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + (y')^2}} = 0 \qquad \text{at } x = b.$$

Thus $y'(b) = 0$. Thus $A = 0$ and $y' \equiv 0$, i.e.

$$y = c$$

as expected.

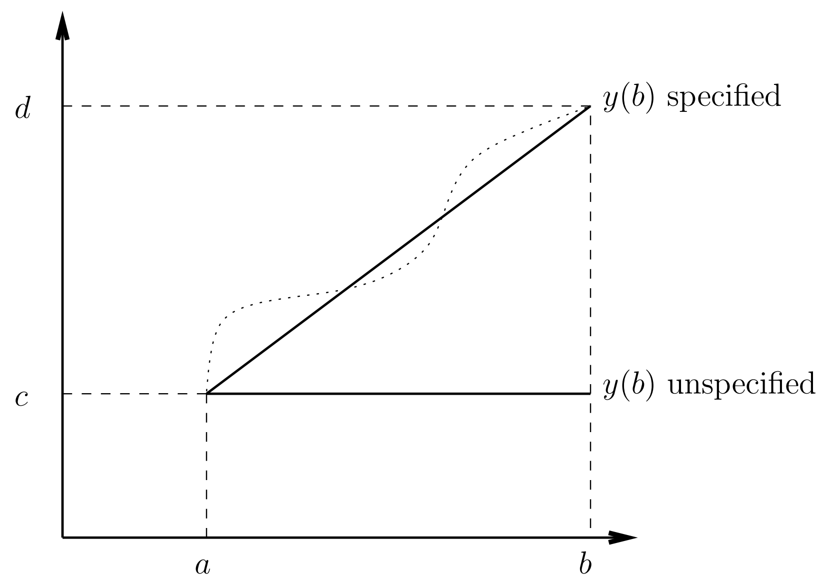


Figure 3:

Constraints

Suppose we have to minimise or maximise a functional

$$J[y] = \int_a^b F(x, y, y')\,\mathrm{d}x$$

subject to $y(a) = c$ and $y(b) = d$ [can easily generalise to natural boundary conditions] and y has to satisfy the constraint

$$K[y] = \int_a^b G(x, y, y') = C \qquad \text{(constant)}.$$

Example
The minimal length curve enclosing a given area

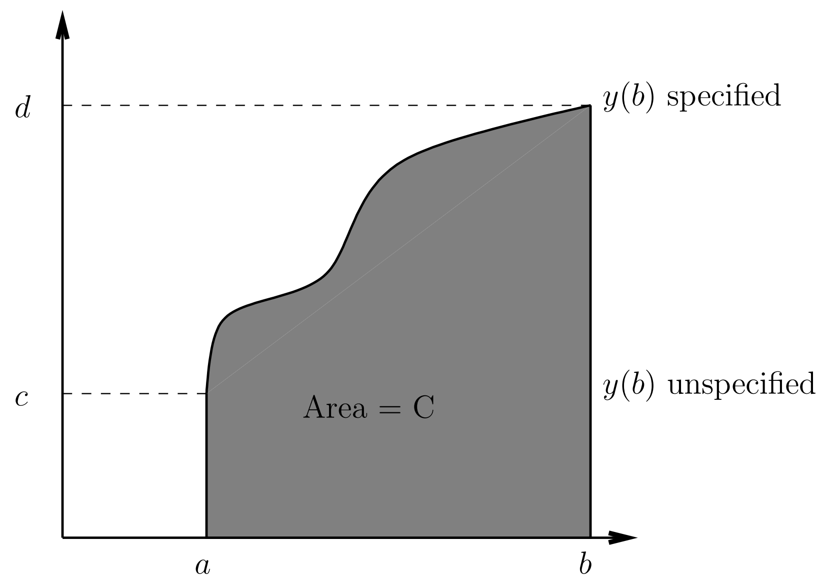


Figure 4:

$$\min J[y] = \int_a^b \sqrt{1 + (y')^2}\, \mathrm{d}x$$

subject to $y(a) = c$, $y(b) = d$ and

$$K[y] = \int_a^b y(x)\, \mathrm{d}x = C.$$

Now if we perturb about the extremal $y(x)$ then

$$\begin{aligned} K[y + \epsilon\eta] &\sim K[y] + \epsilon \int_a^b \left(\eta \frac{\partial G}{\partial y} + \eta' \frac{\partial G}{\partial y'} \right) \mathrm{d}x + O(\epsilon^2) \\ &\sim C + \epsilon \int_a^b \left(\eta \frac{\partial G}{\partial y} + \eta' \frac{\partial G}{\partial y'} \right) \mathrm{d}x + O(\epsilon^2) \\ &= C, \end{aligned}$$

so η is **not** arbitrary. It has to satisfy

$$\int_a^b \left(\eta \frac{\partial G}{\partial y} + \eta' \frac{\partial G}{\partial y'} \right) \mathrm{d}x = 0.$$

A trick to get around this problem is to add **two** perturbation functions, ξ and η satisfying $\xi(a) = \eta(a) = \xi(b) = \eta(b) = 0$. Then

$$\begin{aligned} K[y + \epsilon\eta + \delta\xi] &\sim K[y] + \epsilon \int_a^b \left(\eta \frac{\partial G}{\partial y} + \eta' \frac{\partial G}{\partial y'} \right) \mathrm{d}x + \delta \int_a^b \left(\xi \frac{\partial G}{\partial y} + \xi' \frac{\partial G}{\partial y'} \right) \mathrm{d}x + O(\epsilon^2) \\ &= C. \end{aligned} \tag{22}$$

The idea now is to *fix* the function $\xi(x)$ and, for any subsequently chosen $\eta(x)$, then to determine δ as a function of ϵ in such a way that (22) is satisfied. Thus η will be arbitrary, but we have to choose $\delta(\epsilon)$ in the right way. In order to be able to choose such a δ we need

$$\left.\frac{\partial K}{\partial \delta}\right|_{\delta=0,\epsilon=0} = \int_a^b \left(\xi \frac{\partial G}{\partial y} + \xi' \frac{\partial G}{\partial y'} \right) \mathrm{d}x = \int_a^b \xi \left(\frac{\partial G}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial G}{\partial y'} \right) \right) \mathrm{d}x \neq 0.$$

Provided

$$\frac{\partial G}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial G}{\partial y'} \right) \not\equiv 0$$

(cases where this is zero are degenerate and uninteresting), we can certainly choose ξ so that this is true. Let us choose such a ξ. Then, for *any* subsequent choice of η, we can determine δ as a function of ϵ so that (22) is satisfied. Now

$$\begin{aligned} J[y + \epsilon\eta + \delta(\epsilon)\xi] &\sim J[y] + \epsilon \int_a^b \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} \right) \mathrm{d}x + \delta(\epsilon) \int_a^b \left(\xi \frac{\partial F}{\partial y} + \xi \frac{\partial F}{\partial y'} \right) \mathrm{d}x \\ &\quad + O(\epsilon^2) \\ &\sim J[y] + \epsilon \int_a^b \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} \right) \mathrm{d}x + \epsilon \frac{\mathrm{d}\delta}{\mathrm{d}\epsilon}(0) \int_a^b \left(\xi \frac{\partial F}{\partial y} + \xi \frac{\partial F}{\partial y'} \right) \mathrm{d}x \\ &\quad + O(\epsilon^2). \end{aligned}$$

Since y is an extremal we must have

$$\int_a^b \eta \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) \right) \mathrm{d}x + \frac{\mathrm{d}\delta}{\mathrm{d}\epsilon}(0) \int_a^b \xi \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) \right) \mathrm{d}x = 0. \tag{23}$$

Similarly, integrating by parts in (22) gives

$$\int_a^b \eta \left(\frac{\partial G}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial G}{\partial y'} \right) \right) \mathrm{d}x + \frac{\mathrm{d}\delta}{\mathrm{d}\epsilon}(0) \int_a^b \xi \left(\frac{\partial G}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial G}{\partial y'} \right) \right) \mathrm{d}x = 0. \tag{24}$$

Solving (24) for $\mathrm{d}\delta/\mathrm{d}\epsilon$ and substituting into (23) gives

$$\int_a^b \eta \left(\frac{\partial}{\partial y}(F - \lambda G) - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} - \lambda \frac{\partial G}{\partial y'} \right) \right) \mathrm{d}x = 0, \tag{25}$$

where λ is a constant, defined as the ratio of two definite integrals involving the arbitrary **fixed** function ξ:

$$\lambda = \frac{\displaystyle\int_a^b \xi \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'} \right) \right) \mathrm{d}x}{\displaystyle\int_a^b \xi \left(\frac{\partial G}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial G}{\partial y'} \right) \right) \mathrm{d}x}.$$

Since (25) is true for *any* $\eta \in C^2[a,b]$ satisfying $\eta(a) = \eta(b) = 0$ the FLVC implies that $F - \lambda G$ satisfies Euler's equation:

$$\boxed{\frac{\partial}{\partial y}(F - \lambda G) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial}{\partial y'}(F - \lambda G) \right) = 0.}$$

λ is called a **Lagrange multiplier** and is fixed by satisfying the constraint

$$\boxed{\int_a^b G(x, y, y')\,\mathrm{d}x = C.}$$

This can also be thought of (and is taught in many books as) introducing a new functional (e.g. for $C = 0$)

$$\hat{J} = \int_a^b F(x, y, y') - \lambda G(x, y, y')\,\mathrm{d}x$$

and minimising over y. Then λ is determined from the constraint

$$\int_a^b G(x, y, y')\,\mathrm{d}x = C.$$

Simple Example
Minimise

$$\int_0^1 (y'(x))^2\,dx$$

over all $C^2[0,1]$ functions satisfying $y(0) = y(1) = 0$ and

$$\int_0^1 y(x)\,\mathrm{d}x = 1.$$

So $F = (y')^2$, $G = y$, giving

$$-\lambda - \frac{\mathrm{d}}{\mathrm{d}x}(2y') = 0$$

so that

$$y(x) = -\frac{\lambda x^2}{4} + Ax + B = -\frac{\lambda x(x-1)}{4},$$

after imposing the boundary conditions. Then fix λ by imposing the constraint

$$\int_0^1 y(x)\,\mathrm{d}x = \frac{\lambda}{24} = 1.$$

Thus $\lambda = 24$ and

$$y(x) = 6x(1-x).$$

Generalisation to higher derivatives

Suppose we want to minimise

$$J[y] = \int_a^b F(x, y, y', y'')\,\mathrm{d}x$$

subject to $y(a) = c$, $y(b) = d$, $y'(a) = m$, $y'(b) = n$. Perturbing y to $y + \epsilon\eta$ and linearising in η gives

$$J[y + \epsilon\eta] \quad \sim \quad J[y] + \epsilon \int_a^b \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} + \eta'' \frac{\partial F}{\partial y''} \right) \mathrm{d}x + O(\epsilon^2).$$

At an extremal we have

$$\int_a^b \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} + \eta'' \frac{\partial F}{\partial y''} \right) \mathrm{d}x = 0$$

$$\Rightarrow \qquad \int_a^b \left(\eta \frac{\partial F}{\partial y} - \eta \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) + \eta \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(\frac{\partial F}{\partial y''} \right) \right) \mathrm{d}x + \left[\eta \frac{\partial F}{\partial y'} + \eta' \frac{\partial F}{\partial y''} - \eta \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y''} \right) \right]_a^b = 0$$

Thus the Euler equation is

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left(\frac{\partial F}{\partial y''} \right) = 0.$$

This generalises in the obvious way.

More dependent variables

Suppose we want to minimise

$$J[y, z] = \int_a^b F(x, y, y', z, z')\,\mathrm{d}x$$

subject to $y(a) = c$, $y(b) = d$, $z(a) = m$, $z(b) = n$. We perturb and consider $J[y + \epsilon\eta, z + \delta\xi]$. Since we can vary η and ξ independently we get an Euler equation for each variable:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial y'} \right) &= \frac{\partial F}{\partial y}, \\ \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial F}{\partial z'} \right) &= \frac{\partial F}{\partial z}. \end{aligned}$$

These will be coupled in general.

Pointwise constraints

Once we have more dependent variables we can consider pointwise constraints of the form

$$G(y,z)=0. \tag{26}$$

The condition for stationarity is

$$\int_a^b \left(\left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]\eta + \left[\frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial z'}\right)\right]\xi \right) \mathrm{d}x = 0. \tag{27}$$

However, now η and ξ cannot be assigned arbitrarily because of the constraint (26). Taylor expanding (26) gives

$$\frac{\partial G}{\partial y}\eta + \frac{\partial G}{\partial z}\xi = 0.$$

Multiply by a Lagrange multiplier λ (which in this case is a function of x) and integrate to give

$$\int_a^b \left(\lambda\frac{\partial G}{\partial y}\eta + \lambda\frac{\partial G}{\partial z}\xi \right) \mathrm{d}x = 0.$$

Subtract this from (27) to give

$$\int_a^b \left(\left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \lambda\frac{\partial G}{\partial y}\right]\eta + \left[\frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial z'}\right) - \lambda\frac{\partial G}{\partial z}\right]\xi \right) \mathrm{d}x.$$

Now suppose we choose λ so that the coefficient of η vanishes. Then since ξ can be chosen arbitrarily its coefficient must also vanish. Thus

$$\begin{aligned} \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) - \lambda\frac{\partial G}{\partial y} &= 0, \\ \frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial z'}\right) - \lambda\frac{\partial G}{\partial z} &= 0. \end{aligned}$$

These two equations and (26) form three equations for y, z and λ. Note that again this is the same as minimising $F-\lambda G$, as G does not depend on y' or z'.

More independent variables

Consider

$$J[\phi] = \iint_D F(x,\phi,\phi_x,\phi_y)\,\mathrm{d}x\,\mathrm{d}y$$

where $\phi=\phi(x,y)$, D is a region of the (x,y) plane, and ϕ satisfies $\phi=0$ on ∂D.

$$J[\phi+\epsilon\eta] = J[\phi] + \epsilon\iint_D \left(\eta\frac{\partial F}{\partial \phi} + \eta_x\frac{\partial F}{\partial \phi_x} + \eta_y\frac{\partial F}{\partial \phi_y}\right) \mathrm{d}x\,\mathrm{d}y$$

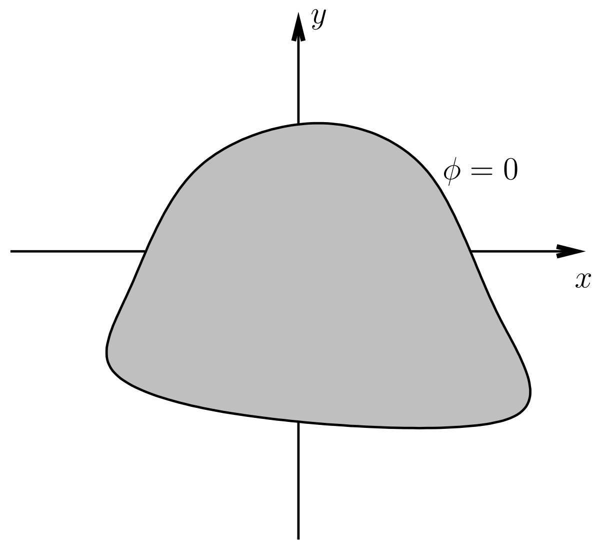


Figure 5:

Now instead of integration by parts we need to use Green's Theorem. From the identity

$$\nabla \cdot (\eta \mathbf{f}) = \nabla \eta \cdot \mathbf{f} + \eta \nabla \cdot \mathbf{f},$$

we find

$$\iint_D (\nabla \eta \cdot \mathbf{f} + \eta \nabla \cdot \mathbf{f}) \,\mathrm{d}x \,\mathrm{d}y = \int_{\partial D} \eta \,\mathbf{f} \cdot \mathbf{n} \,\mathrm{d}s.$$

Thus, with

$$\mathbf{f} = \left(\frac{\partial F}{\partial \phi_x}, \frac{\partial F}{\partial \phi_y} \right),$$

we find

$$\begin{aligned} \iint_D \left(\eta_x \frac{\partial F}{\partial \phi_x} + \eta_y \frac{\partial F}{\partial \phi_y} \right) \mathrm{d}x \,\mathrm{d}y \;&=\; -\iint_D \left(\eta \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial \phi_x} \right) + \eta \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial \phi_y} \right) \right) \mathrm{d}x \,\mathrm{d}y \\ &\quad + \int_{\partial D} \eta \left(\frac{\partial F}{\partial \phi_x} n_x + \frac{\partial F}{\partial \phi_y} n_y \right) \mathrm{d}s. \end{aligned}$$

Thus

$$\iint_D \left(\eta \frac{\partial F}{\partial \phi} - \eta \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial \phi_x} \right) - \eta \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial \phi_y} \right) \right) \mathrm{d}x \,\mathrm{d}y = 0$$

for all η so the Euler equation (or Euler-Lagrange equation) is

$$\frac{\partial}{\partial x} \left(\frac{\partial F}{\partial \phi_x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial \phi_y} \right) = \frac{\partial F}{\partial \phi}.$$

This is now a p.d.e.

Hamiltonian

Suppose a function $y(x)$ satisfies Euler's equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y}$$

for some function $F(x, y, y')$. Note that

$$\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}x} &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x} \\
&= \frac{\partial F}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\
&= \frac{\partial F}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial F}{\partial y'}\right).
\end{aligned}$$

Therefore, if we define the **Hamiltonian**

$$H = y'\frac{\partial F}{\partial y'} - F,$$

then

$$\frac{\mathrm{d}H}{\mathrm{d}x} = -\frac{\partial F}{\partial x}.$$

If F does not depend explicitly on x (the problem is *autonomous*) then

$$\frac{\partial F}{\partial x} = 0$$

and hence $H =$ constant. In this case H is a *conserved quantity* (often identifyable as energy).

Example
Suppose

$$F = \sqrt{1 + (y')^2} + y^2.$$

The Euler equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right) = 2y.$$

This is not very nice...but

$$H = \frac{(y')^2}{\sqrt{1 + (y')^2}} - \sqrt{1 + (y')^2} - y^2 = -\frac{1}{\sqrt{1 + (y')^2}} - y^2 = \text{constant},$$

gives a *first integral* of the o.d.e.

We can transform the Euler equation into *canonical form* by changing independent variables. Think of F and H as functions of (x, p, q) instead of (x, y, y'), where

$$q = y, \qquad p = \frac{\partial F}{\partial y'};$$

p is known as the generalised momentum. Then, then definition of H is

$$H = py' - F$$

(where y' is a function of x, p, q) and Euler's equation is

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{\partial F}{\partial y}.$$

So

$$\frac{\partial H}{\partial y'} = p + y' \frac{\partial p}{\partial y'} - \frac{\partial F}{\partial y'} = y' \frac{\partial p}{\partial y'}$$

by the Chain rule, since $p = \partial F / \partial y'$. But

$$\frac{\partial H}{\partial y'} = \frac{\partial H}{\partial q} \frac{\partial q}{\partial y'} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial y'} = \frac{\partial H}{\partial p} \frac{\partial p}{\partial y'}.$$

Thus

$$y' = \frac{\mathrm{d}q}{\mathrm{d}x} = \frac{\partial H}{\partial p}.$$

Also

$$\frac{\mathrm{d}p}{\mathrm{d}x} \quad = \quad \frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\big(py' - H\big) = y' \frac{\partial p}{\partial y} - \frac{\partial H}{\partial y}.$$

But

$$\frac{\partial H}{\partial y} = \frac{\partial H}{\partial p} \frac{\partial p}{\partial y} + \frac{\partial H}{\partial q} \frac{\partial q}{\partial y} = y' \frac{\partial p}{\partial y} + \frac{\partial H}{\partial q}.$$

Thus

$$\frac{\mathrm{d}p}{\mathrm{d}x} = y' \frac{\partial p}{\partial y} - y' \frac{\partial p}{\partial y} - \frac{\partial H}{\partial q} = -\frac{\partial H}{\partial q}.$$

Thus

$$\boxed{\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{\partial H}{\partial q} \qquad\qquad \frac{\mathrm{d}q}{\mathrm{d}x} = \frac{\partial H}{\partial p}.}$$

These are **Hamilton's equations**. Note that

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial p} \frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial H}{\partial q} \frac{\mathrm{d}q}{\mathrm{d}x} = \frac{\partial H}{\partial x} + \frac{\mathrm{d}q}{\mathrm{d}x} \frac{\mathrm{d}p}{\mathrm{d}x} - \frac{\mathrm{d}p}{\mathrm{d}x} \frac{\mathrm{d}q}{\mathrm{d}x} = \frac{\partial H}{\partial x}.$$

Thus if

$$\frac{\partial H}{\partial x} = 0$$

then H is conserved as expected.

Free boundaries

Minimise

$$J[y,b] = \int_a^b F(x,y,y')\,\mathrm{d}x$$

subject to $y(a) = c$, $y(b) = d$ where b is **unspecified**.

$$\begin{aligned} J[y+\epsilon\eta; b+\epsilon\beta] &= \int_a^{b+\epsilon\beta} F(x, y+\epsilon\eta, y'+\epsilon\eta')\,\mathrm{d}x \\ &= J[y,b] + \epsilon\left\{\int_a^b \left(\eta\frac{\partial F}{\partial y} + \eta'\frac{\partial F}{\partial y'}\right)\mathrm{d}x + \beta F(b, y(b), y'(b))\right\} + O(\epsilon^2). \end{aligned}$$

Taylor expanding the boundary condition

$$\begin{aligned} d &= y(b+\epsilon\beta) + \eta(b+\epsilon\beta) \\ &= y(b) + \epsilon\left[\beta y'(b) + \eta(b)\right] + O(\epsilon^2) \\ &= d + \epsilon\left[\beta y'(b) + \eta(b)\right] + O(\epsilon^2). \end{aligned}$$

Thus

$$\eta(a) = 0, \qquad \eta(b) = -\beta y'(b).$$

At an extremal

$$\int_a^b \left(\eta\frac{\partial F}{\partial y} + \eta'\frac{\partial F}{\partial y'}\right)\mathrm{d}x + \beta F(b, y(b), y'(b)) = 0.$$

Integrate by parts to give

$$\beta F(b, y(b), y'(b)) + \int_a^b \eta\left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]\mathrm{d}x + \left[\eta\frac{\partial F}{\partial y'}\right]_a^b = 0.$$

Hence

$$\beta\left[F - y'\frac{\partial F}{\partial y'}\right]_{x=b} + \int_a^b \eta\left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]\mathrm{d}x = 0.$$

Thus FLCV gives us Euler's equation and the extra free boundary condition

$$F = y'\frac{\partial F}{\partial y'} \qquad \text{at } x = b$$

(i.e. $H = 0$).

Example
minimise

$$J[y,b] = \int_0^b \left(\frac{1}{2}\left(y'\right)^2 + \frac{1}{2}y^2 + 1\right)\mathrm{d}x$$

subject to $y(0) = 0$, $y(b) = 1$. Euler's equation is

$$y'' = y.$$

Solving and applying the boundary conditions gives

$$y = \frac{\sinh x}{\sinh b}.$$

The extra free boundary condition is

$$\frac{1}{2}\left(y'\right)^2 - \frac{1}{2}y^2 - 1 = 0 \text{ at } x = b.$$

This gives

$$\frac{\cosh^2 b}{\sinh^2 b} = 3 \quad \Rightarrow \quad b = \tanh^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

CHECK

$$J\left[\frac{\sinh x}{\sinh b}, b\right] = \frac{1}{2}\coth b + b.$$

This is minimised when

$$-\frac{1}{2}\text{cosech}^2 b + 1 = 0 \quad \Rightarrow \quad b = \tanh^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

OR note that

$$H = \frac{1}{2}\left(y'\right)^2 - \frac{1}{2}y^2 - 1 = \text{constant (autonomous)} = 0$$

by the free boundary condition. Hence

$$y' = \sqrt{y^2 + 2}.$$

Thus

$$x = \int \frac{\mathrm{d}y}{\sqrt{y^2+2}} = \sinh^{-1}\left(\frac{y}{\sqrt{2}}\right).$$

Thus

$$y = \sqrt{2}\sinh x.$$

Then the boundary condition $y(b) = 1$ gives

$$b = \sinh^{-1}\left(\frac{1}{\sqrt{2}}\right) = \tanh^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

4 Optimal control

Example
Suppose $x(t)$ satisfies the differential equation

$$\dot{x} = u + x,$$

where $u(t)$ is our control variable. Suppose we want to vary u so as to control x. For example, starting from $x(0) = a$ we may wish to arrive at $x(T) = 0$.

Is this possible? Yes! Just choose *any* function $x(t)$ satisfying the initial and final condition and then read off the required control as

$$u = \dot{x} - x.$$

However, in practice there may be bounds on the achievable u, e.g. $-1 \leq u \leq 1$. This will leads to bounds on the initial condition for which the desired final condition is achievable. In the example if $u \leq 1$ then the maximum achievable value of $x(T) - x(0)$ occurs when

$$\dot{x} - x = 1 \quad \Rightarrow \quad x = -1 + A\mathrm{e}^{t} = -1 + (1+a)\mathrm{e}^{t}.$$

Then $x(T) = 0$ gives $a = -1 + \mathrm{e}^{-T}$. The problem is controllable only if a is greater than this value.

We may wish to find the control which *minimises* a cost function. For example, the work done agaist friction may be

$$\int_0^T u\dot{x}\,\mathrm{d}t.$$

Thus we may want to define the cost function as

$$C = \int_0^T u(u+x)\,\mathrm{d}t$$

and ask for the control which achieves the goal and minimises $C[x,u]$.

So, in general we may find the following optimal control problem:

$$\text{minimise } C[x,u] = \int_0^T h(t,x,u)\,\mathrm{d}t,$$

over all controls $u(t)$ satisfying the control problem

$$\dot{x} = f(t,x,u), \qquad x(0) = a, \quad x(T) = b.$$

This now resembles a variational problem, with the control problem acting as a **constraint**.

Let us approach it by perturbing about the extremal functions:

$$C[x+\epsilon\xi, u+\epsilon\eta] = C[x,u] + \epsilon\int_0^T \left(\xi\frac{\partial h}{\partial x} + \eta\frac{\partial h}{\partial u}\right)\,\mathrm{d}t + O(\epsilon^2),$$

while

$$\dot{x} + \epsilon\dot{\xi} = f(t,x,u) + \epsilon\left(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial u}\right) + O(\epsilon^2)$$

$\xi(0) = 0 = \xi(T)$. Since $\dot{x} = f(t,x,u)$ for an extermal function we need

$$\int_0^T \left(\xi\frac{\partial h}{\partial x} + \eta\frac{\partial h}{\partial u}\right)\,\mathrm{d}t = 0,$$

for all ξ and η satisfying

$$\dot{\xi} = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial u},$$

with $\xi(0) = \xi(T) = 0$. We require $\partial f / \partial u \neq 0$, otherwise the control u has no influence on the problem. Then we can solve for

$$\eta = \left(\dot{\xi} - \xi \frac{\partial f}{\partial x} \right) \Big/ \frac{\partial f}{\partial u},$$

and plug it into the integral

$$\int_0^T \left(\xi \frac{\partial h}{\partial x} + \left(\dot{\xi} - \xi \frac{\partial f}{\partial x} \right) \frac{\partial h}{\partial u} \Big/ \frac{\partial f}{\partial u} \right) \mathrm{d}t = 0.$$

As usual integrate by parts to give

$$\int_0^T \xi \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial x} \frac{\partial h}{\partial u} \Big/ \frac{\partial f}{\partial u} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial h}{\partial u} \Big/ \frac{\partial f}{\partial u} \right) \right) \mathrm{d}t + \left[\xi \frac{\partial h}{\partial u} \Big/ \frac{\partial f}{\partial u} \right]_0^T = 0$$

Since $\xi(0) = \xi(T) = 0$ the boundary term is zero. Hence we find that x and u have to satisfy the o.d.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial h}{\partial u} \Big/ \frac{\partial f}{\partial u} \right) = \frac{\partial h}{\partial x} - \frac{\partial f}{\partial x} \left(\frac{\partial h}{\partial u} \Big/ \frac{\partial f}{\partial u} \right).$$

This o.d.e. is coupled with the control problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f,$$

with $x(0) = a$, $x(T) = b$. In principle two coupled first order o.d.e.s with two boundary conditions gives a unique solution.

Return to the example

$$f = u + x, \qquad h = u(u + x),$$

so that we get

$$\frac{\mathrm{d}}{\mathrm{d}t} (2u + x) = u - (2u + x),$$

i.e.

$$2\dot{u} + \dot{x} = -(u + x),$$

along with the control problem

$$\dot{x} = u + x.$$

Adding gives

$$\dot{u} + \dot{x} = 0,$$

so that $u + x = A$ constant. Then $\dot{x} = A$, $x(0) = a$, $x(T) = 0$ gives

$$x = a + At, \qquad A = -\frac{a}{T},$$

so that

$$x = a\left(1 - \frac{t}{T}\right),$$

(constant velocity is the most efficient), and the optimal control is

$$u = a\left(\frac{t}{T} - 1 - \frac{1}{T}\right).$$

Note the existence of a first integral which facilitated the solution of this example. As in the calculus of variations this will be generally true for *autonomous* problems. To see this define the **Hamiltonian**

$$H(t, x, u) = f\left(\frac{\partial h}{\partial u}\bigg/\frac{\partial f}{\partial u}\right) - h.$$

Then direct differentiation and the chain rule leads to

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \left(\frac{\partial h}{\partial u}\bigg/\frac{\partial f}{\partial u}\right)\frac{\partial f}{\partial t} - \frac{\partial h}{\partial t},$$

so if the problem is autonomous, then

$$\frac{\partial f}{\partial t} = \frac{\partial h}{\partial t} = 0$$

and H is conserved. In the example $f = u + x$, $h = u(u + x)$ and

$$H = (u + x)(2u + x) - u(u + x) = (u + x)^2$$

which is conserved as we found before.

Example 2
Solve

$$\dot{x} = x + u, \qquad x(0) = 0, \quad x(1) = 1,$$

where u is chosen to minimise

$$\int_0^1 u^2\,\mathrm{d}t.$$

Now $f = x + u$, $h = u^2$, so the Hamiltonian is

$$H = (x + u) \times 2u - u^2 = u^2 + 2xu.$$

Completing the square

$$(u + x)^2 = H^2 + x^2 \quad \Rightarrow \quad u + x = \pm\sqrt{H + x^2}.$$

Choose the plus sign ($\dot{x}$ should be positive from the initial and final conditions) to give

$$\dot{x} = \sqrt{H + x^2}.$$

Therefore

$$t = \int \frac{\mathrm{d}x}{\sqrt{H + x^2}} = \sinh^{-1}\left(\frac{x}{\sqrt{H}}\right).$$

Therefore

$$x = \sqrt{H}\sinh t.$$

The final condition $x(1) = 1$ determines H, to gives

$$x = \frac{\sinh t}{\sinh 1}, \qquad H = \operatorname{cosech}^2 1, \qquad u = \frac{\mathrm{e}^{-t}}{\sinh 1}.$$

4.1 The Pontryagin Maximum Principle (Non-examinable)

Form of problem : The state vector x of a system obeys

$$\dot{x} = f(x, t, u), \qquad x, f \in \mathbb{R}^n,$$

where u is a **control** which we are free to choose subject to $u(t) \in U_f(x(t), t)$, the set of **feasible** controls which may depend on x and t. We have to choose u in such a way as to maximise (or minimise) some "gain" function

$$\int_0^T h(x, t, u)\, dt, \qquad h(x, t, u) \in \mathbb{R}.$$

Boundary conditions: typically $x(0)$ given; T and $x(T)$ may both be given, or T fixed but $x(T)$ free, or $x(T)$ fixed but T free, etc. E.g. if $x(T)$ is given, T is free and $h = 1$ we have a minumum time control, $\int_0^T h\, dt = T =$ time to get between specified end states.

Procedure (Pontryagin Maximum Principle) Introduce a vector $p \in \mathbb{R}^n$ and define

$$H_0(x, t, u, p) = h(x, t, u) + pf(x, t, u) \equiv h(x, t, u) + \sum_i p_i f_i(x, t, u).$$

(The "pre-Hamiltonian".) Let $u_0(x, t, p)$ be the value of u in $U_f(x, t)$ that maximises $H_0(x, t, u, p)$, and let the **Hamiltonian**

$$H(x, t, p) = \max\{H_0(x, t, u, p) : u \in U_f(x, t)\} = H_0(x, t, u_0(x, t, p), p). \tag{28}$$

Then the optimal trajectory is found by solving

$$\begin{aligned} \dot{x} &= f(x, y, u_0(x, t, p)) = \frac{\partial H}{\partial p} \qquad \text{(if max is attained)}, \\ \dot{p} &= -\frac{\partial H}{\partial x} \end{aligned}$$

(system of $2n$ ode's), subject to

(i) the given value of $x(0)$,

(ii) given value of $x(T)$, or $p(T) = 0$ if $x(T)$ is free,

(iii) given value of T, or $H = 0$ at T if T is free.

Notes

(i) Can replace max with min throughout.

(ii) p is called the dual variable vector, adjoint, co-state.

Example Suppose $x \in \mathbb{R}$, $\ddot{x} = u$, and u is restricted by $-1 \leq u \leq 1$, and from some initial state you have to reach $x(T) = 0$ in minimum time T. Take the start vector to be

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix},$$

so the differential equations are

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u, \end{aligned}$$

i.e.

$$f = \begin{pmatrix} x_2 \\ u \end{pmatrix}.$$

We want to minimise

$$T = \int_0^T 1\, dt, \qquad \text{so take } h = 1,$$

and we have

$$(x_1, x_2) = \begin{cases} (x_0, \dot{x}_0) & \text{as } t = 0, \\ (0, 0) & \text{as } t = T. \end{cases}$$

Then

$$H_0 = h + pf = 1 + p_1 x_2 + p_2 u,$$

where p_1,p_2 are conjugate to x_1 and x_2 respectively. Hence

$$H = \min H_0 = 1 + p_1 x_2 - |p_2|, \qquad u_0 = -\text{sign}(p_2).$$

Then

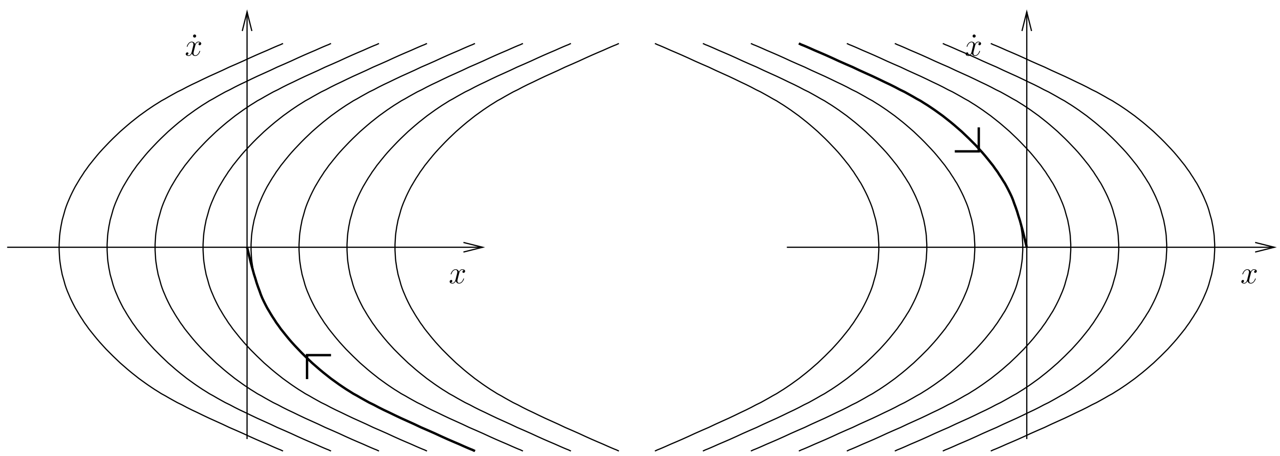
$$\begin{aligned} \dot{p}_1 &= -\frac{\partial H}{\partial x_1} = 0, \\ \dot{p}_2 &= -\frac{\partial H}{\partial x_2} = -p_1. \end{aligned}$$

We already see that

(a) the optimal trajectory will use $u = \pm 1$ only: "bang-bang" control.

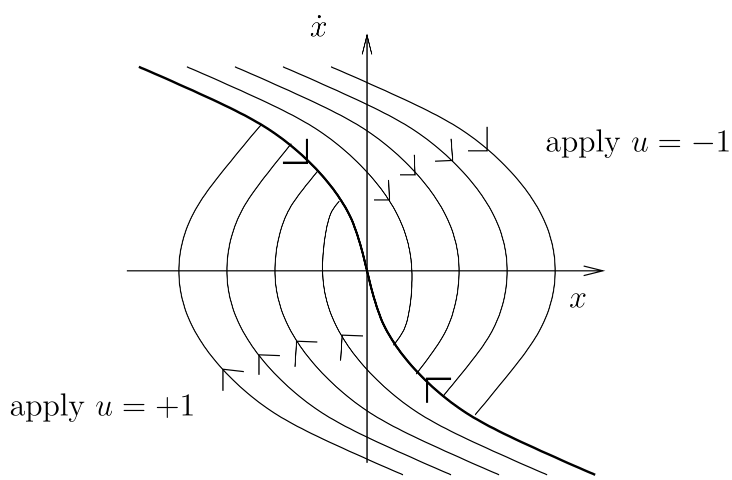
(b) u changes between ± 1 **at most once** on the optimal trajectory (since p_2 monotonic and $u_0 = -\text{sign}(p_2)$.)

What does this mean in the phase plane?



trajectories with $u = +1$ trajectories with $u = -1$

After the first switch, we must be on one of the dark paths. These are called the switching locus. Hence we must follow one family until we hit the switching locus and the the other until $x = \dot{x} = 0$.



This is the "time optimal" control. (Can also find p_1 and p_2 etc by using the boundary conditions, but using (a) and (b) and the phase plane is easier.)

Note u does not vary continuously on the optimal trajectory. There is a discontinuity in $\ddot{x}$ where u changes sign. In some applications u may not be a real variable or vector at all, e.g. sound insulation: a board is to be built of layers of different materials subject to constraints on weight, thickness, cost, so as to minimise sound coming through. This is optimal control

$$\begin{aligned} t &\to x, \\ u &\to \text{what material used at } x, \\ \text{state} &\to \text{displacement/stress at } x \end{aligned}$$

(all assumed $\propto e^{i\omega t}$). Consequently we should prove the Pontryagin Maximum Principle (P.M.P.) by a method not assuming and continuity in u.

"Proof" (Why the method usually works) First we prove the following
Lemma Suppose

$$g(y,z) = \max\{f(x,y,z) : x \in X(y)\} = f(x_0(y,z), y, z), \qquad x_0(y,z) \in X(y).$$

Then

$$\frac{\partial g}{\partial z}(y,z) = \frac{\partial f}{\partial z}(x_0(y,z), y, z).$$

Proof If f is differentiable

$$\frac{\partial g}{\partial z}(y,z) = \frac{\partial f}{\partial z}(x_0(y,z), y, z) + \frac{\partial f}{\partial x}(x_0(y,z), y, z)\frac{\partial x_0(y,z)}{\partial z}.$$

But x_0 defined to be maximum implies that

$$\frac{\partial f}{\partial x}(x_0(y,z), y, z) = 0.$$

But this inequality holds even if f is not differentiable in x. We have

$$f(x_0(y,z), y, z') \le g(y, z')$$

with equality at $z = z'$. Hence the z' derivatives are equal at $z = z'$, i.e. the required result. Depends on z being in the interior of the set over which f is defined and on f, g being differentiable in z. Does not depend on any differentiability in x or y. □

Now, to prove the Pontryagin maximum principle we have to show that there is a p defined on the optimal trajectory such that

(i) the optimal control u is the value maximising H.

(ii) $\dot{p} = -\dfrac{\partial H}{\partial x}$ on the optimal trajectory.

(ii) The boundary conditions hold.

Define

$$F(\xi,\tau)=\sup\int_{\tau}^{T}h(x,t,u)\,dt \qquad \text{starting from } x(\tau)=\xi,$$

(subject to $\dot{x}=f$, $u\in U_f(x,t)$ etc.) Then $F(x(0),0)$ is the required maximum. Assume f, f are continuous in (x,t) and F is C^1. We are going to show that $p=F_x$ (i.e. $p_i=\partial F/\partial x_i$) is the required function.

From the point (x,t) one possible control is to hold u constant (some value in $U_f(x,t)$) for small time δ, and then apply the optimal control from where you reach $(x_1,t+\delta)$. Here $x_1=x+f(x,t,u)\delta+o(\delta)$, so $h(x,t,u)\delta+o(\delta)+F(x_1,t+\delta)\le F(x,t)$. Subtract $F(x,t)$, divide by δ and let $\delta\to 0$:

$$h(x,t,u)+F_x(x,t)f(x,t,u)+F_t\le 0, \tag{29}$$

for all $u\in U_f(x,t)$. (i.e.

$$h(x,t,u)+\sum_i\frac{\partial F}{\partial x_i}f_i+F_t\le 0.$$

If we integrate this inequality along any feasible trajectory (optimal or not) we have

$$\int_0^T h(x,t,u)\,dt+F(x(T),T)-F(x(0),0)\le 0,$$

i.e.

$$\int_0^T h(x,t,u)\,dt\le F(x(0),0).$$

(remember $h\ge 0$.) This equation also clearly follows from the definition of F, since $F(x(0),0)$ is the supremum of the left-hand side over all possible controls. However, this definition of F means that there are controls that get arbitrarily close in this inequality. For simplicity, assume equality is **attained** for some optimal control. Then, for the optimal trajectory, equality holds in (29) for almost all t, and again for simplicity assume it holds everywhere. So (29) says

$$H_0(x,t,u,F_x(x,t))+F_t(x,t)\le 0,$$

for all u, with equality for the optimal trajectory. Hence the optimal control does maximise H_0 for $p=F_x$ [(i) is satisfied], and we also see that the maximised value is

$$H(x,t,F_x(x,t))=-F_t(x,t). \tag{30}$$

Now assume that H is C^1 and F is C^2. To derive the $\dot{p}$ equation, first note that by (28)

$$H_0(x,t,u_0(x,t,p),p')\le H(x,t,p'),$$

with equality at $p'=p$. Hence (by the previous Lemma) the p'-derivatives must agree at p, so

$$f(x,t,u_0(x,t,p))=H_p(x,t,p). \tag{31}$$

Then the derivative of p along the optimal trajectory is

$$\dot{p}=\frac{d}{dt}\left(F_x(x,t)\right)=F_{xt}(x,t)+F_{xx}f(x,t,u_0(x,t,F_x)).$$

But by (30)

$$\begin{aligned} F_{xt}(x,t) &= -\frac{\partial}{\partial x}\left(H(x,t,F_x)\right) \\ &= -H_x(x,t,F_x) - H_p(x,t,F_x)F_{xx} \\ &= -H_x(x,t,F_x) - f(x,t,F_x)F_{xx} \end{aligned}$$

by (31). So

$$\dot{p} = -H_x(x,t,F_x)$$

as required [(ii) is satisfied].

For the boundary conditions, note that if $x(T)$ is free then $F(x,T) \equiv 0$ for all x, so $p = \partial F/\partial x = 0$ at T. If T is free but $x(T) = x_T$ is fixed, then $F(x_T, T) \equiv 0$, so

$$0 = \left.\frac{\partial F}{\partial t}\right|_T = -H(T)$$

by (30). So (iii) is satisfied.

Note If max is replaced by min, all inequalities are reversed and the "proof" is still OK.