Further Partial Differential Equations
Problem Sheet 3

1. **Similarity solutions in the two-phase Stefan problem**

   Consider the two-phase Stefan problem (2.15) in the limit $t \to 0$. Show that the leading-order behaviour is given by

$$u(x,t) \sim \begin{cases} f(\eta) & 0 < \eta < \beta, \\ g(\eta) & \beta < \eta < \infty, \end{cases} \qquad s(t) \sim \beta\sqrt{t}, \qquad \eta = \frac{x}{\sqrt{t}},$$

   where

$$g(\eta) = \theta\left(\frac{\operatorname{erfc}\left(\eta\sqrt{\mathrm{St}}/2\sqrt{\kappa}\right)}{\operatorname{erfc}\left(\beta\sqrt{\mathrm{St}}/2\sqrt{\kappa}\right)} - 1\right), \qquad f(\eta) = \left(1 - \frac{\operatorname{erf}\left(\eta\sqrt{\mathrm{St}}/2\right)}{\operatorname{erf}\left(\beta\sqrt{\mathrm{St}}/2\right)}\right),$$

   and β satisfies the transcendental equation

$$\frac{\beta\sqrt{\pi}}{2\sqrt{\mathrm{St}}} = \frac{\mathrm{e}^{-\beta^2\mathrm{St}/4}}{\operatorname{erf}\left(\beta\sqrt{\mathrm{St}}/2\right)} - \frac{K\theta\mathrm{e}^{-\beta^2\mathrm{St}/4\kappa}}{\sqrt{\kappa}\operatorname{erfc}\left(\beta\sqrt{\mathrm{St}}/2\sqrt{\kappa}\right)}.$$

2. **A solid–liquid interface with a density change**
Consider the one-dimensional Stefan problem for melting of a solid considered in lectures. The full system behaviour may be described by equations expressing conservation of mass, momentum and total energy, which are given respectively by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}\left(\rho v^2 + p\right) = 0, \tag{2}$$

$$\frac{\partial}{\partial t}\left(\rho h + \frac{1}{2}\rho v^2\right) + \frac{\partial}{\partial x}\left(pv - k\frac{\partial T}{\partial x} + \rho\left(h + \frac{1}{2}v^2\right)v\right) = 0, \tag{3}$$

where ρ is the density, v the velocity, p the pressure, T the temperature and

$$h = \begin{cases} c(T - T_\mathrm{m}) + L & T > T_\mathrm{m} \\ c(T - T_\mathrm{m}) & T < T_\mathrm{m}. \end{cases}$$

is the *enthalpy* of the system, which is the total energy per unit mass, including heat. Here, c is the specific heat and L the latent heat.

Suppose that liquid occupies a region $0 \le x \le s(t)$ and solid occupies a region $x > s(t)$.

(a) Show that when the density of the fluid and the solid are the same and a constant then $v = 0$ and the temperature in the liquid and the solid is described by the one-dimensional heat equation

$$\frac{\partial}{\partial t}(\rho c T) - \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = 0. \tag{4}$$

(b) Now suppose that the densities in the solid and the liquid phases are different. Integrate (1) over a domain $x_1 < x < x_2$ that contains the interface (so $x_1 < s(t)$ and $x_2 > s(t)$). Divide the integral into $x_1 \le x \le s(t)$ and $s(t) \le x \le x_2$ and take the limit as $x_1 \to s(t)^-$ and $x_2 \to s(t)^+$ to show that the following jump condition is satisfied by the density:

$$[\rho]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = [\rho v]_-^+ . \tag{5}$$

(c) By performing an identical process for (2) and (3) obtain the jump conditions

$$[\rho v]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = \left[\rho v^2 + p\right]_-^+ , \tag{6}$$

$$\left[\rho h + \frac{1}{2}\rho v^2\right]_-^+ \frac{\mathrm{d}s}{\mathrm{d}t} = \left[pv - k\frac{\partial T}{\partial x} + \rho\left(h + \frac{1}{2}v^2\right)v\right]_-^+ . \tag{7}$$

(d) Explain how these reduce to the Stefan condition presented in lectures when the solid and liquid densities are equal.

3. **Linear stability of a two-dimensional Stefan problem**
Consider the linear stability of the free boundary problem depicted in Figure 2.2 in the limit $\mathrm{St} \to 0$. Assume that the free boundary is moving at constant speed V under a constant temperature gradient $-\lambda_{1,2}$ in each phase before being perturbed, so the solutions take the form

$$u_1(x,y,t) = -\lambda_1(x - Vt) + \tilde{u}_1(x,y,t), \qquad u_2(x,y,t) = -\lambda_2(x - Vt) + \tilde{u}_2(x,y,t)$$

and the position of the free boundary is given by

$$x = Vt + \xi(y,t).$$

By linearising the problem with respect to $\tilde{u}_1$, $\tilde{u}_2$ and ξ, show that perturbations with wavenumber $k > 0$ and growth rate σ are possible provided

$$\frac{\sigma}{Vk} = -\frac{\lambda_1 + K\lambda_2}{\lambda_1 - K\lambda_2}.$$

4. **One-dimensional welding**

(a) Derive the dimensionless one-dimensional welding problem (2.31).

(b) Show that the normalised heating coefficient is given by

$$q = \frac{a^2 J^2}{\sigma k (T_\mathrm{m} - T_0)} = \frac{\sigma V^2}{k(T_\mathrm{m} - T_0)},$$

where V is the applied voltage. Assuming that we require $q = O(1)$ to melt the plate, roughly how high must the voltage be to achieve melting?