Further Partial Differential Equations
Problem Sheet 4

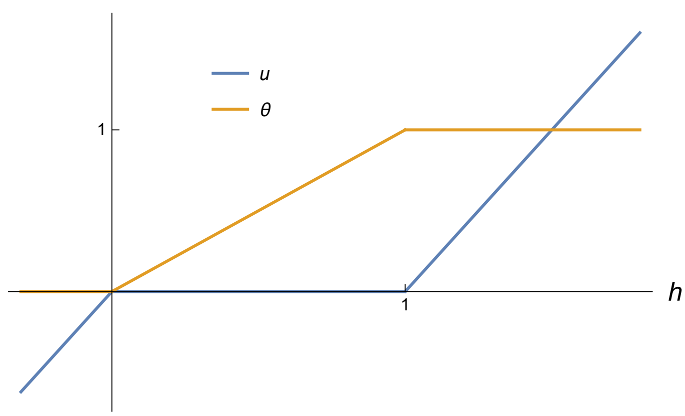

Figure 1: Normalised temperature u and liquid fraction θ versus enthalpy h.

1. **Enthalpy for mushy layers**

   Show that the free boundary problem (2.31) may be posed as

   $$\frac{\partial h}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q,$$

   where $h = \mathrm{St}u + \theta$ is the (dimensionless) *enthalpy*. Deduce that u is a piecewise linear function of h, as indicated in figure 1.

2. **Unsteady electropainting**
Consider the unsteady version of the model problem depicted in Figure 2.9, i.e., with the conditions on $y = 0$ replaced by

$$\frac{\partial \phi}{\partial y} = \frac{\phi}{h}, \quad \frac{\partial h}{\partial t} = \frac{\partial \phi}{\partial y} - \delta \qquad y = 0, \ |x| < c,$$
$$\phi = 0 \qquad y = 0, \ |x| > c,$$

where now $c = c(t)$. Find the small-time behaviour of the solution by expanding

$$\begin{aligned} \phi(x, y, t) &\sim \phi_0(x, y) + t\phi_1(x, y) + \cdots, \\ h(x, t) &\sim th_1(x) + t^2 h_2(x) + \cdots, \\ c(t) &\sim c_0 + tc_1 + \cdots. \end{aligned}$$

Show that painting commences provided $\delta < 1/\pi$, in which case the layer initially grows over a half-width $c_0 = \sqrt{1/(\delta\pi) - 1}$.

3. **One-dimensional welding**

(a) Consider the dimensionless one-dimensional welding problem (2.31). Show that, before melting occurs, the solution is given by

$$u(x,t) = -1 + \frac{q}{2}\left(1-x^2\right) + \sum_{n=0}^{\infty} c_n \cos\left[\left(n+\tfrac{1}{2}\right)\pi x\right] \mathrm{e}^{-\left(n+\frac{1}{2}\right)^2 \pi^2 t/\mathrm{St}}$$

and use Fourier series to evaluate the constants c_n.

(b) Deduce that the sample will eventually melt provided $q > 2$, at a time t_{m} that satisfies

$$q = \left(\frac{1}{2} - 2\sum_{n=0}^{\infty} \frac{(-1)^n \mathrm{e}^{-\left(n+\frac{1}{2}\right)^2 \pi^2 t_{\mathrm{m}}/\mathrm{St}}}{\left(n+\frac{1}{2}\right)^3 \pi^3}\right)^{-1}. \qquad (1)$$

(c) Show that the leading-order asymptotic dependence of equation (1) between $t_{\mathrm{m}}/\mathrm{St}$ and q is

$$\frac{t_{\mathrm{m}}}{\mathrm{St}} \sim \frac{1}{q} \qquad \text{as} \quad t_{\mathrm{m}}/\mathrm{St} \to 0,$$
$$\frac{t_{\mathrm{m}}}{\mathrm{St}} \sim \frac{4}{\pi^2} \log\left(\frac{64}{\pi^3(q-2)}\right) \qquad \text{as} \quad t_{\mathrm{m}}/\mathrm{St} \to \infty.$$

(d) For $t > t_{\mathrm{m}}$, consider the free boundary problem (2.31). Explain why $s_2(t) = 0$ until $t = t_{\mathrm{m}} + 1/q$.

(e) Now consider the limit $\mathrm{St} \to 0$. Show that the plate will have melted entirely to a depth $x = 1 - \sqrt{2/q}$ (so the mush has disappeared) after a time $t_{\mathrm{c}} \sim t_{\mathrm{m}} + 1/q + O(\mathrm{St})$.

(f) Show that the subsequent leading-order behaviour of the solid–liquid free boundary $x = s(t)$ is governed by

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{q}{2}(1+s) - \frac{1}{1-s}, \qquad s(t_{\mathrm{c}}) = 1 - \sqrt{\frac{2}{q}}.$$

(g) Deduce that the solid ahead of the free boundary is not superheated, and that the system approaches a steady state with the plate melted to a depth $x = \sqrt{1 - 2/q}$.