Introduction to Cryptology

7.3 - Hash functions: Generic Attacks

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The Birthday Problem

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- How many people are required to have a probability larger than 1/2 ?
  - The answer is 23:

$$\Pr(\text{all distinct}) = \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{365-22}{365} < \frac{1}{2}$$

The Birthday Problem

Suppose you choose q elements uniformly in a set of N elements.

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- How large should q be with respect to N to have a probability larger than $1/2$?

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A formal solution to these problems in the following slides.

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Upper bound: For the event Coll_i (the i-th ball falls into an already occupied bin), it holds that $\Pr(\text{Coll}_i) \leq (i-1)/N$ (at most $i-1$ bins are already occupied).

$$\Pr(\text{Coll}) = \Pr(\bigcup_{i=1}^{q} \text{Coll}_i) \leq \sum_{i=1}^{q} \Pr(\text{Coll}_i) \leq$$

$$\leq 0/N + \cdots + (q-1)/N = \frac{q(q-1)}{2N}$$

The Birthday Problem

Lower bound: For the event NoColl_i (no collisions after throwing the i-th ball), it holds

$$\Pr(\text{NoColl}_{i+1}|\text{NoColl}_i) = (N - (i))/N. \tag{1}$$

We note

$$\Pr(\text{NoColl}_q) = 1 - \Pr(\text{Coll}). \tag{2}$$

Now,

$$\Pr(\text{NoColl}_q) = \Pr(\text{NoColl}_q \cap \text{NoColl}_{q-1}) =$$

$$= \Pr(\text{NoColl}_q|\text{NoColl}_{q-1}) \cdot \Pr(\text{NoColl}_{q-1})$$

and, iterating the above reasoning, we obtain:

$$\Pr(\text{NoColl}_q) = \prod_{i=1}^{q-1} \Pr(\text{NoColl}_{i+1}|\text{NoColl}_i). \tag{3}$$

The Birthday Problem

From equations (1), and (3) we obtain

$$\Pr(\text{NoColl}_q) = \prod_{i=1}^{q-1} \left(1 - \frac{i}{N}\right) \tag{4}$$

Since $1 - x \leq e^{-x}$ when $x \leq 1$ and $i/N < 1$, it holds

$$\Pr(\text{NoColl}_q) \leq e^{-\sum_{i=1}^{q-1}(i/N)} = e^{-q(q-1)/2N}. \tag{5}$$

Thanks to equation (2)

$$\Pr(\text{Coll}) \geq 1 - e^{-q(q-1)/2N}$$

where

$$1 - e^{-q(q-1)/2N} \geq q(q-1)/4N$$

since $q < \sqrt{2N}$ and $e^{-x} \leq 1 - x/2$ when $|x| \leq 1$.

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How does the birthday problem relate to hash functions?

Hash functions: The Birthday attack

Consider the function $F : \{0,1\}^* \to \{0,1\}^\ell$. Its range has size $N = 2^\ell$.

When $q \approx 2^{\ell/2}$ evaluations are executed, the probability of finding a collision is $\approx 1/2$.

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There are no generic attacks for preimage and second-preimage resistance.

Hash Functions: A Better Birthday Attack

The original birthday attack uses lots of memory storage, as it has to store $\mathcal{O}(q) = \mathcal{O}\left(2^{\ell/2}\right)$ digests.

Managing storage for 2^{60} bytes is often more difficult than executing 2^{60} CPU instructions.

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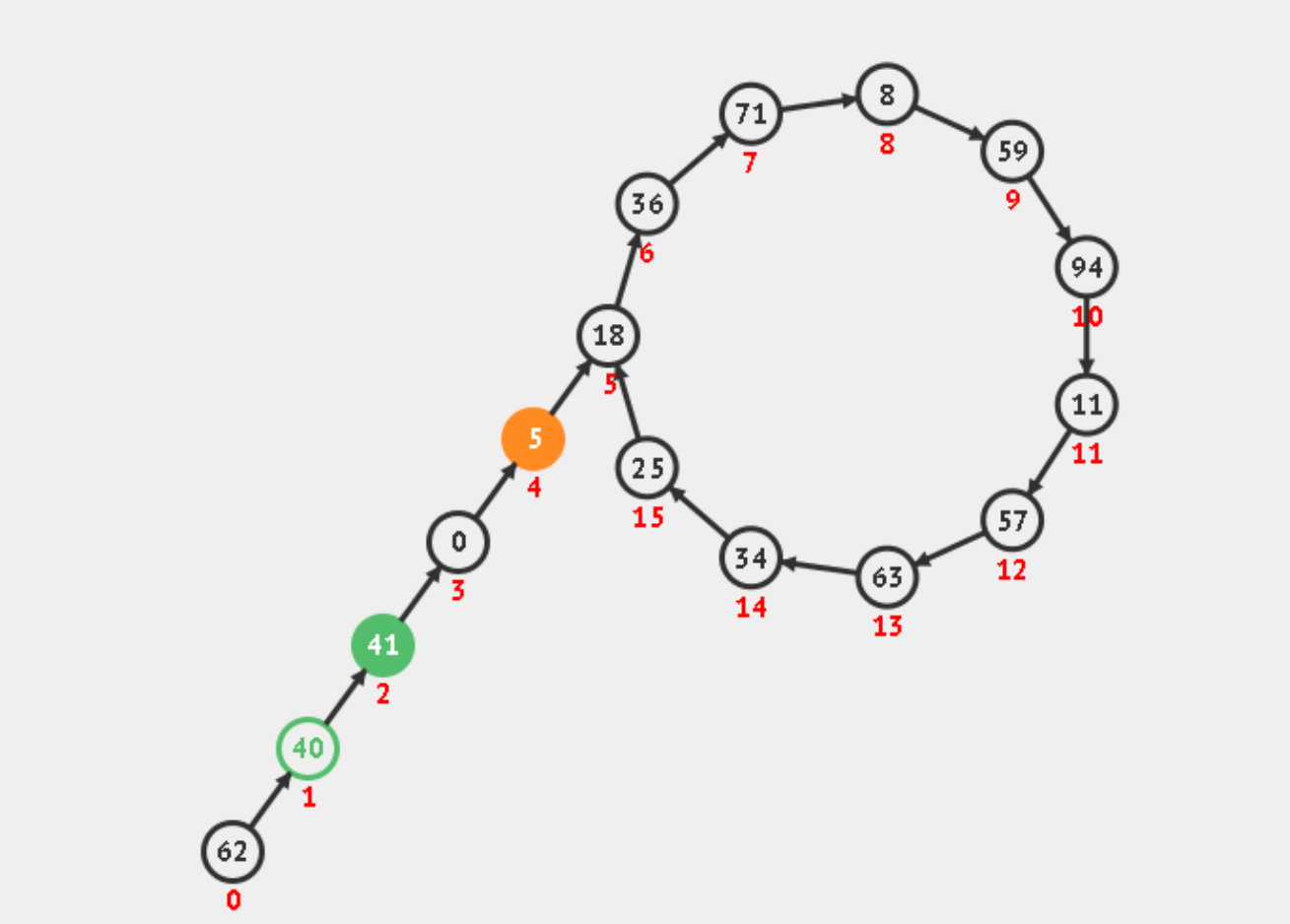
Anything better?

Hash Functions: A Better Birthday Attack

The Floyd's cycle finding idea can be exploited!



https://visualgo.net/bn/cyclefinding

Hash Functions: A Better Birthday Attack

Let x_0 be a uniform input of the H and denote $H^i(x_0)$ by x_i.

Suppose $x_I = x_J$ for some distinct $I, J \in [1, q]$.

Then, there is $i < J$ s.t. $x_i = x_{2i}$.

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Then, there is $i < J$ s.t. $x_i = x_{2i}$.

- From x_I, the sequence is periodic with a period Δ that divides $J - I$;
- let $i < J$ be the smallest multiple of Δ greater than or equal to I ($x_I, x_{I+1}, \ldots, x_{J-1}$ contains Δ consecutive indices);
- $x_i = x_{2i}$ since $i > I$ and $2i - i = i$ is a multiple of Δ.

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The index i is found after an exhaustive search (in the i-th iteration $H(x_{i-1})$ and $H(H(x_{2(i-1)}))$ are computed).

Hash Functions: A Better Birthday Attack

Starting from x_0 and after a maximum of i-steps, a cycle is entered.

The goal is finding the *first* element of the cycle. To this end, two starting points, $x := x_0$ and $x' := x_i$, are considered.

- At each step, x and x' are updated with an evaluation of H;
- while x is out of the cycle, $x \neq x'$, since x' is moving within the cycle;
- as soon x enters the cycle, it *meets* x', i.e. $x = x'$. Indeed, if j is the smallest index for which x_j is in the cycle, then $x' = x_{i+j}$ where $i \equiv 0 \pmod{\Delta}$.

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Also j is found after an exhaustive search.

We note that x_{j-1} and x_{i+j-1} form a collision (one point out of the cycle, the other inside).

A Better Birthday Attack

Let $H : \{0,1\}^* \to \{0,1\}^\ell$ be an unkeyed hash function. The goal is finding x, x' s.t. $H(x) = H(x')$.

$x_0 \leftarrow\!\!\$\ \{0,1\}^{\ell+1}$

$x' := x_0, x := x_0, \ell := 0$

for $i = 1, 2, \cdots$ do

  $\ell = \ell + 1$

  $x = H(x) = H^{(i)}(x_0)$

  $x' = H(H(x')) = H^{(2i)}(x_0)$

  if $x = x'$ break

$x' := x, x := x_0, i := \ell$

for $j = 1 \cdots, i$

  if $H(x) = H(x')$ return x, x'

  else $x = H(x) = H^{(j)}(x_0)$

    $x = H(x') = H^{(i+j)}(x_0)$

Hash Functions: A Better Birthday Attack

The algorithm has approximately the same time complexity of the birthday attack. The success probability is the same.

It only requires $\mathcal{O}(1)$ memory, namely,
storage of two digests at each step!

Further Reading I


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