Introduction to Cryptology

11.1 - RSA and other factoring-based schemes

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RSA Encryption Scheme

Designed by R. Rivest, A. Shamir and L. Adleman in 1977.

Variants of the original scheme still used nowadays, for both public-key encryption and digital signatures.

Security is based on the RSA problem.

Plain RSA encryption algorithm

The RSA encryption scheme $E_{\mathrm{RSA}} = (\mathrm{KeyGen}, \mathrm{Enc}, \mathrm{Dec})$ is composed by three PPT algorithms defined as follows.

- $(\mathrm{PK}, \mathrm{SK}) \leftarrow \mathrm{KeyGen}(n)$: it runs a GenRSA algorithm on input a security parameter n. Then PK is set to (N, e), while SK is set to (N, d)[1].
- $c \leftarrow \mathrm{Enc}(\mathrm{PK}, m \in \mathbb{Z}_N^*)$: on input a public key (N, e) and a message m, it outputs $c = m^e$.
- $m \leftarrow \mathrm{Dec}(\mathrm{SK}, c)$: on input a secret key (N, d) and a ciphertext c, it computes $m = c^d$.

[1] We recall that $N = pq$, where p and q are two distinct n-bit odd primes, while $[e]_{\varphi(N)}[d]_{\varphi(N)} = [1]_{\varphi(N)}$.

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Correctness: $c^d = (m^e)^d = m^{ed} = m^{\ell\varphi(N)+1} = m$.

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Plain RSA security

E_{RSA} is deterministic, so it does not have indistinguishable encryptions in the presence of an eavesdropper.

The factoring assumption implies that it is computationally infeasible to recover the private key from the public key.

The RSA assumption implies that an adversary $\mathcal{A}$ cannot recover m from (N, e) and c if m is uniform in $\mathbb{Z}_N^*$.

- What if m is not chosen uniformly from $\mathbb{Z}_N^*$?
- What if $\mathcal{A}$ learns partial information about m?

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- Enc chooses a uniform $r \in \{0,1\}^{\ell(n)}$, and sets $\tilde{m} = r||m$.
- The security of the padded variant depends on $\ell(n)$.
  - $\ell(n) = \mathcal{O}(\log n)$ is a bad choice;
  - provable security based on the RSA assumption when $m \in \{0,1\}$ and ℓ is as large as possible;
  - for other cases, no security proof, but no attacks are known either!

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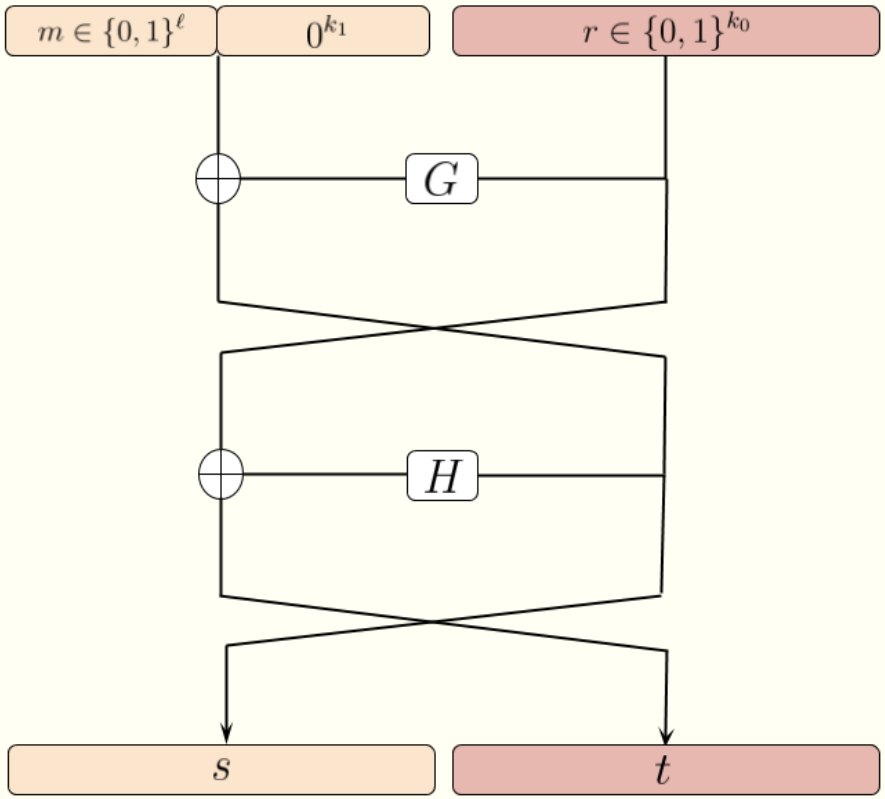
- It uses integer-valued functions $\ell(n), k_0(n), k_1(n)$ where $k_0(n)$, $k_1(n) = \Theta(n)$ and $\ell(n) + k_0(n) + k_1(n)$ smaller than the minimum bit-length of the moduli output by GenRSA(n).
- Two hash functions H and G are also employed. They are modelled as random oracles in the security proof.
- The transformation executed by OAEP is a two-round Feistel network (G and H are the round functions).

The OAEP mechanism



$$H : \{0,1\}^{\ell+k_1} \to \{0,1\}^{k_0}$$
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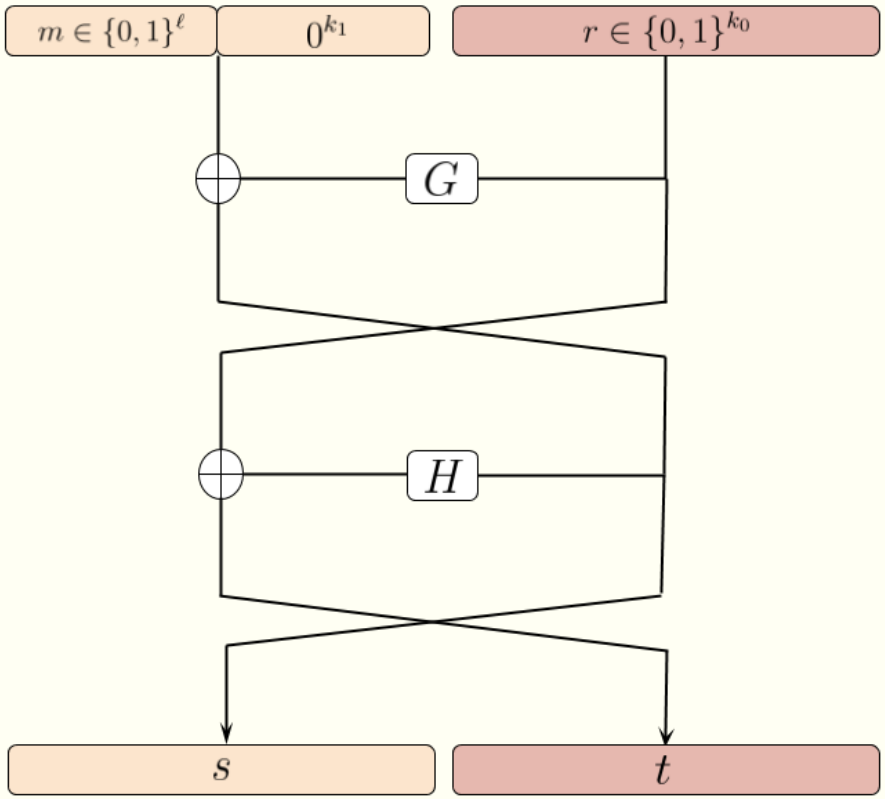

$H : \{0,1\}^{\ell+k_1} \to \{0,1\}^{k_0}$

$G : \{0,1\}^{k_0} \to \{0,1\}^{\ell+k_1}$

Input: $m \in \{0,1\}^{\ell}$

$m' := m||0^{k_1}$

$r \leftarrow \{0,1\}^{k_0}$

$s := m' \oplus G(r) \in \{0,1\}^{\ell+k_1}$

$t := r \oplus H(s) \in \{0,1\}^{k_0}$

$\tilde{m} := s||t$

return $\tilde{m}$

RSA-OAEP

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- $c \leftarrow \mathrm{Enc}(\mathrm{PK}, m \in \{0,1\}^{\ell(n)})$: m is padded with the OAEP mechanism, obtaining $\tilde{m}$. The ciphertext c is set to $([\tilde{m}]_N)^e$, where $\mathrm{PK} = (N, e)$.
- $m \leftarrow \mathrm{Dec}(\mathrm{SK}, c)$: on input a secret key $\mathrm{SK} = (N, d)$ and a ciphertext c, $\tilde{m}$ is set to c^d. If $|\tilde{m}| \neq \ell + k_0 + k_1$, the algorithm outputs $\perp$. Otherwise:
  - it parses $\tilde{m}$ as $(s||t)$, where $s \in \{0,1\}^{\ell+k_1}, t \in \{0,1\}^{k_0}$;
  - it computes $r := H(s) \oplus t$;
  - it computes $m' := G(r) \oplus s$. If the most-significant k_1 bits of m' are not all 0, it outputs $\perp$. Otherwise, it outputs the ℓ least-significant bits of m'.

Security of RSA-OAEP

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In 2001, James Manger showed an attack on a variant of RSA-OAEP specified in PKCS#1 v2.0.

- It exploited implementation weaknesses, prone to side-channel attack.
- Two different conditions make Dec output $\perp$. The times to return the message errors were not identical.
- The attack recovered the plaintext m with $||N||$ queries to an oracle leaking the error.

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Cryptographic implementations should adhere to the theoretical formalisations.

RSA - Weak-key Generator Attack

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A concrete attack in 2012:

Ron was wrong, *Whit is right*, Lenstra *et al.*

It showed that 2/1000 of the gathered RSA keys were weak.

A CCA-secure KEM in the ROM

Consider a KEM $(\mathrm{KeyGen}, \mathrm{Encaps}, \mathrm{Decaps})$ defined as follows:

- $(\mathrm{PK}, \mathrm{SK}) \leftarrow \mathrm{KeyGen}(n)$: given a security parameter n, it runs a GenRSA algorithm on n and specifies a hash function $H : \mathbb{Z}_N^* \to \{0,1\}^n$. Then it sets PK to (N, e, H) and SK to (N, d, H).

- $(c, k) \leftarrow \mathrm{Encaps}(\mathrm{PK}, n)$: on input a public key $\mathrm{PK} = (N, e, H)$ and n, it picks a random $r \in \mathbb{Z}_N^*$ and outputs the ciphertext $c := r^e$ and the key $k := H(r)$.

- $k \leftarrow \mathrm{Decaps}(\mathrm{SK}, c)$: on input a secret key $\mathrm{SK} = (N, d, H)$ and a ciphertext c, it outputs $k := H(c^d)$.

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Its security relies on the RSA assumption. Part of the ISO/IEC18033-2 standard for public-key encryption.

Rabin Encryption Scheme

The Rabin encryption scheme $(\mathrm{KeyGen}, \mathrm{Enc}, \mathrm{Dec})$ consists of three PPT algorithms:

- $(\mathrm{PK}, \mathrm{SK}) \leftarrow \mathrm{KeyGen}(n)$: it runs GenModulus on input a security parameter n, obtaining (N, p, q), where $N = pq$ and p, q are n-bit primes with $[p]_4 = [q]_4 = [3]_4$.
  PK is set to N, SK is set to (p, q).
- $c \leftarrow \mathrm{Enc}(\mathrm{PK}, m \in \{0,1\})$: on input a public key N and a message m, it chooses a uniform $r \in QR(N)$, where $lsb(r) = m$, and outputs $c := r^2$.
- $m \leftarrow \mathrm{Dec}(\mathrm{SK}, c)$: given a secret key (p, q) and a ciphertext c, it computes the unique $r \in QR(N)$ s.t. $r^2 = c$, and outputs $lsb(r)$.

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Theorem

If the factoring problem is hard relative to GenModulus*, then the encryption scheme is CPA-secure.*

Further Reading I

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Further Reading II


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