Topological Groups, 2020–2021

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We begin with the course overview as described on `https://courses.maths.ox.ac.uk/node/51770`.

Course Overview:

Groups like the integers, the circle, and general linear groups (over $\mathbb{R}$ or $\mathbb{C}$) share a number of properties naturally captured by the notion of a topological group. Providing a unified framework for these groups and properties was an important achievement of 20th century mathematics, and in this course we shall develop this framework.

Highlights will include the existence and uniqueness of Haar integrals for locally compact topological groups, the topology of dual groups, and the existence of characters in locally compact Hausdorff Abelian topological groups. Throughout, the course will use the tools of analysis to tie together the topology and algebra, getting at superficially more algebraic facts by analytic means.

References

There are some references which may be of use: [Fol95, Kör08, Kra17, Meg17] and [Rud90].

Teaching

The lectures and these notes will appear online as they are produced. They will be supplemented by some tutorial-style teaching where we can discuss the course and also exercises from the sheets. Once I have a list of the MFoCS students attending I shall be in touch to arrange these.

Contact details and feedback

The current circumstances mean this course is appearing in a different way to normal. In particular, there will inevitably be less audience response so I encourage you to get in touch at tom.sanders@maths.ox.ac.uk if you have any questions or feedback.

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General prerequisites

The course is designed to be pretty self-contained. We assume basic familiarity with groups as covered in Prelims Groups and Group Actions (see *e.g.* [Ear14]). We shall also assume familiarity with Prelims Linear Algebra (see *e.g.* [May20]), though it is only in §7 that we use anything of substance; and Part A: Metric Spaces and Complex Analysis (see *e.g.* [McG20]) for material on normed spaces.

Familiarity with topology is essential, though not much is required. What we use (and more) is covered in Part A: Topology (see *e.g.* [DL18]), with the exception of Tychonoff's Theorem. This can be informally summarised as saying that a non-empty product of non-empty compact spaces is compact, and there is no harm in taking it as a black box for the course. Those interested in more detail may wish to consult Part C: Analytic Topology (see *e.g.* [Kni18]).

The Axiom of Choice is sometimes formulated as saying that an arbitrary product of non-empty sets is non-empty, and this perhaps makes it less surprising that it can be used to prove Tychonoff's Theorem. It turns out that the converse is also true, *i.e.* Tychonoff's Theorem and the other axioms of Zermelo–Fraenkel set theory can be used to prove the Axiom of Choice, and both are equivalent to Zorn's Lemma[1]. We shall make use of Zorn's Lemma once at the very end of the course (literally the last lemma – Lemma 7.22).

Finally no familiarity with functional analysis is assumed, though there are clear similarities and parallels for those who do have some. Those interested may consult [Pri17] and [Whi19].

1 Introduction

In this course we are interested in the interaction between group structure and a 'compatible' topological structure.

Suppose that G is a group written multiplicatively, by which we mean the binary operation of the group is denoted $G^2 \to G; (x, y) \mapsto xy$; with inversion is denoted $G \to G; x \mapsto x^{-1}$; and the identity is denoted 1_G. Suppose additionally that G is a topological space with topology τ. We say that the group operation is **jointly continuous** if the map $G^2 \to G; (x, y) \mapsto xy$ is continuous, where G^2 is equipped with the product topology (from the topology τ on each factor of G). If inversion is continuous and the group operation is jointly continuous then G is said to be a **topological group**.

Example 1.1 (Indiscrete groups)**.** Any group G endowed with the indiscrete topology is a topological group since any map into an indiscrete space is continuous.

[1] Those unfamiliar and looking for a reference may wish to consult the notes [Pil20] or [Con].

Example 1.2 (Discrete groups)**.** Any group G endowed with the discrete topology is a topological group since the product of two copies of the discrete topology is discrete – so both the topological spaces G and G^2 are discrete – and any map from a discrete space is continuous.

The reals under addition may be endowed with the discrete or indiscrete topologies to make them into a topological group as above, however, there are other topologies on $\mathbb{R}$ which are of interest.

Example 1.3 (The real line)**.** The group $\mathbb{R}$ (the operation is addition) endowed with its usual topology is a topological group. The relevant continuity is just the algebra of limits: in particular, if $x_n \to x_0$ then $-(x_n) = (-1)x_n \to (-1)x_0 = -x_0$; and if additionally $y_n \to y_0$, then $x_n + y_n \to x_0 + y_0$.

Remark 1.4. The reals illustrate some important general features for which we need a little terminology. A **neighbourhood base** of a point x in a topological space X is a family $B = (B_i)_{i \in I}$ of neighbourhoods of x such that if U is an open set containing x then there is some $i \in I$ such that $B_i \subset U$.

A topological space is said to be **first countable** if every element of the space has a countable neighbourhood base (meaning every x has a a neighbourhood base $B(x) = (B_n(x))_{n=1}^{\infty})$.

First countable spaces are important because matters of convergence may be resolved just by considering sequences and not more general nets. In particular, a function f is **sequentially continuous** if $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$, and it is easy enough to check that a continuous function is sequentially continuous (see *e.g.* [DL18, Proposition 1.24]); if the domain is first countable then[2] sequential continuity implies continuity.

Example 1.5 (The real line, revisited)**.** The group $\mathbb{R}$ endowed with its usual topology has $(x + (-1/n, 1/n))_{n=1}^{\infty}$ as a countable neighbourhood base for each $x \in \mathbb{R}$. In fact this is in some sense just the neighbourhood base $((-1/n, 1/n))_{n=1}^{\infty}$ for 0 translated around the group. This will be a general phenomenon of topological groups – they all have a (not necessarily countable) neighbourhood basis of the identity which can be translated to give a neighbourhood basis for any other point.

Example 1.6 (Normed spaces)**.** The additive group of a normed space X with the topology induced by the norm is a topological group.

The topology induced by the norm is the weakest topology such that $x \mapsto \|x\|$ is continuous. For each $x \in X$, $(x + \{y \in X : \|y\| < 1/n\})_{n=1}^{\infty}$ is a countable neighbourhood base for

[2] In general this is proved using the Axiom of Countable Choice, though there are interesting cases where this is not necessary.

this topology so X is first countable. Moreover, the product of two first countable spaces is first countable and hence to show that X is a topological group it is enough to note from homogeneity that if $x_n \to x_0$ then $-x_n \to -x_0$; and from the triangle inequality that if $x_n \to x_0$ and $y_n \to y_0$ then $x_n + y_n \to x_0 + y_0$.

Remark 1.7. The above is essentially the same argument as in Example 1.3, but in particular gives the more general fact that $\mathbb{R}^n$ and $\mathbb{C}^n$ are topological groups under addition.

Remark 1.8. A **topological vector space** over $\mathbb{R}$ (resp. $\mathbb{C}$) is a vector space over $\mathbb{R}$ (resp. $\mathbb{C}$) with a topology such that addition of vectors is jointly continuous, and scalar multiplication is jointly continuous. It can be shown similarly to the above that any normed space is a topological vector space, and a theory can be developed as in, for example, [Bou87]. This parallels many of our developments here but we shall not say more about it.

Given a normed space X we write $B(X)$ for the set of continuous linear maps $X \to X$, and $\mathrm{GL}(X)$ for the set of linear homeomorphisms. $\mathrm{GL}(X)$ is a group under composition and it can be profitably viewed as inheriting topologies from $B(X)$.

Example 1.9 ($\mathrm{GL}(X)$ with the operator norm topology)**.** $\mathrm{GL}(X)$ may be endowed with the subspace topology inherited from $B(X)$ with the operator norm topology. With this topology $\mathrm{GL}(X)$ is a topological group.

If $S_n \to S_0$ and $T_n \to T_0$ then $\|T_n\| \leqslant 2\|T_0\|$ for all sufficiently large n and hence

$$\|S_nT_n - S_0T_0\| \leqslant \|S_n - S_0\|\|T_n\| + \|S_0\|\|T_n - T_0\| \to 0$$

since the operator norm is sub-multiplicative; hence $S_nT_n \to S_0T_0$. $B(X)$ is a normed space so as in Example 1.6 the topology is first countable, whence so is the topology on $\mathrm{GL}(X)$ and on $\mathrm{GL}(X) \times \mathrm{GL}(X)$. Hence multiplication is continuous.

Now suppose that $T_n \to T_0$, and let $N \in \mathbb{N}$ be such that for $n \geqslant N$ we have $\|T_n - T_0\| \leqslant 1/(2\|T_0^{-1}\|)$. Now for all n we have

$$\|T_n^{-1} - T_0^{-1}\| = \|T_n^{-1}(T_0 - T_n)T_0^{-1}\| \leqslant \|T_n^{-1}\|\|T_n - T_0\|\|T_0^{-1}\|, \tag{1.1}$$

and so if $n \geqslant N$ then $\|T_n^{-1}\| \leqslant 2\|T_0^{-1}\|$ by the triangle inequality. Inserting this bound back into (1.1) we have

$$\|T_n^{-1} - T_0^{-1}\| \leqslant \|T_n - T_0\|2\|T_0^{-1}\|^2,$$

for $n \geqslant N$, and so $T_n^{-1} \to T_0^{-1}$ as required.

Remark 1.10. In particular $\mathbb{C}^*$ (which we may identify with $\mathrm{GL}(\mathbb{C})$) is a topological group under multiplication.

The above example is our first example where showing continuity of inversion is a little more involved. The multiplicative group $\mathbb{C}^*$ already gives a glimpse as to why: despite being continuous, the inversion map $\mathbb{C}^* \to \mathbb{C}^*; z \mapsto z^{-1}$ is not *uniformly* continuous.

Remark 1.11. There are other natural topologies on $\mathrm{GL}(X)$, for example the strong and weak operator topology, which are the same as the operator norm topology when X is finite dimensional but different for infinite dimensional normed spaces. Though they are important we shall not dwell on these.

Remark 1.12. In particular the matrix groups $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ are topological groups.

2 Some basics of the interplay of algebra and topology

Suppose that G is a group (written multiplicative) and $S, T \subset G$. We write

$$S^{-1} := \{s^{-1} : s \in S\} \text{ and } ST := \{st : s \in S, t \in T\}.$$

For $n \in \mathbb{N}_0$ we define S^n inductively by $S^0 := \{1_G\}$ and $S^{n+1} := S^n S$, and $S^{-n} := (S^{-1})^n$. It will also be convenient to write $xS := \{x\}S$ and $Sx := S\{x\}$ for $x \in G$.

Remark 2.1. The notation xS (and Sx) generalises the usual coset notation for when $S \leqslant G$.

⚠In general $SS^{-1} \neq S^0$ and $S^2 \neq \{s^2 : s \in S\}$.

⚠G^n denotes the n-fold Cartesian product $G \times \cdots \times G$ *not* the product defined above which is just G.

We write $\langle S \rangle$ for the group generated by S, that is $\bigcap\{H \leqslant G : S \subset H\}$, the intersection of all the subgroups of G containing S.

We call $S \subset G$ **symmetric** if $S = S^{-1}$.

Remark 2.2. If S and T are symmetric then $S \cap T$ is symmetric.

Remark 2.3. If S is symmetric then $\langle S \rangle = \bigcup_{n \in \mathbb{N}_0} S^n$ by the subgroup test.

We say that an Abelian group G is written additively to mean that the binary operation of the group is denoted $G^2 \to G; (x, y) \mapsto x + y$; the inversion is denoted $G \to G; x \mapsto -x$; and the identity is denoted 0_G. The preceding notation changes in the obvious way, so we write $S + T$ instead of ST *etc.* above.

Remark 2.4. The reals in Example 1.3, and more generally the additive group of normed spaces (Example 1.6) are examples of Abelian groups written additively. While, for example, $\mathbb{C}^*$ (as described in Remark 1.10), is an Abelian group written multiplicatively.

Suppose that G is a group written multiplicatively. To understand the interplay of algebra and topology in topological groups it is useful to have a further definition: We say that the group operation on G is **separately continuous** if the maps $G \to G; x \mapsto xy$ and $G \to G; x \mapsto yx$ are continuous for all $y \in G$.

Remark 2.5. Separate continuity of the group operation is exactly equivalent to saying that xU and Ux are open (resp. closed) whenever U is open (resp. closed) and $x \in G$.

Remark 2.6. The maps $G \to G^2; x \mapsto (x, y)$ (and $G \to G^2; x \mapsto (y, x)$) are continuous for all $y \in G$ and so joint continuity of the group operation implies separate continuity.

A group G with a separately continuous group operation is called a **semitopological group**; and if, additionally, inversion is continuous then we call it a **quasitopological group**.

These two definitions will only really be important to us in the present section, and then primarily for illustrating how powerful joint continuity is compared with separate continuity.

Remark 2.7. In view of Remark 2.6, every topological group is a quasitopological group, and of course every quasitopological group is a semitopological group.

Example 2.8 (Reals with the right order topology)**.** The reals $\mathbb{R}$ (with the operation of addition) and with topology $\{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ is a semitopological group since the translate of any interval of the form (a, ∞) is also of this form; we denote it $\mathbb{R}_{\mathrm{RO}}$. This is not a quasitopological group since $(-\infty, -a)$ is *not* open (whatever a).

Example 2.9. A group G endowed with the cofinite topology, that is the topology in which the closed sets are the finite sets (and the whole of G), is a quasitopological group since U^{-1} is finite if U is finite (so inversion is continuous), and xU and Ux are finite if U is finite (so multiplication is separately continuous).

If G is finite then the cofinite topology is the same as the discrete topology and G is a topological group (as in Example 1.2). On the other hand, we shall see in Remark 2.26 that it is *not* always a topological group.

Example 2.10. A group G endowed with the cocountable topology, that is the topology in which the closed sets are the countable sets (and the whole of G), is a quasitopological group by the same argument as in Example 2.9 with finite replaced by countable.

Now, if G is countable again we recover a discrete group; we shall be interested in the case when G is uncountable.

Remark 2.11. There is also a notion of **paratopological group**, which is a group with a jointly continuous group operation, but no continuity of the inverse is assumed. This notion will not even be of illustrative importance to us. Exercise I.1 gives an example of a paratopological group that is *not* a topological group.

There are a few key lemmas (Lemmas 2.12, 2.14, 2.18,2.23, 2.25, and 2.28) which capture how the group operations interact with the topology of a (semi)topological group and while these are not the main results we use later on, we highlight them in red because they each capture a crucial technique or idea.

Lemma 2.12 (Key Lemma I)**.** *Suppose that G is a semitopological group, U is open and V is any set. Then UV and VU are open, and U is a neighbourhood of x if and only if $x^{-1}U$ (or Ux^{-1}) is a neighbourhood of the identity.*

Proof. First, $UV = \bigcup_{v\in V} Uv$ which is a union of open sets by the first part and hence open. Similarly VU is a union of open sets and so open. Finally, if U is a neighbourhood of x then there is an open set $U_x \subset U$ containing x. Hence $x^{-1}U_x$ is an open set containing 1_G and contained in $x^{-1}U$, which is to say $x^{-1}U$ is a neighbourhood of the identity. Similarly if $x^{-1}U$ is a neighbourhood of the identity then U is a neighbourhood of x, and the same two arguments also work for Ux^{-1}. □

In a similar vein we can say something about separation in semitopological groups: A topological space X is **Fréchet** (or T_1) if every singleton in X is closed. (The other separation axioms we shall touch on in increasing order of strength are T_2, just before Remark 2.32, T_3 in Remark 2.34,$T_{3^1\!/_2}$ in Remark 4.8, and T_4 in Remark 4.9.)

Lemma 2.13. *Suppose that G is a semitopological group. Then G is Fréchet if and only if $\{1_G\}$ is closed.*

Proof. Immediate from separate continuity. □

Lemma 2.14 (Key Lemma II)**.** *Suppose that G is a quasitopological group. If U is a neighbourhood of 1_G then U contains a symmetric open neighbourhood of the identity. If K is a compact set then K is contained in a compact symmetric set.*

Proof. If U is a neighbourhood of 1_G then U contains an open neighbourhood V of 1_G. Put $S := V \cap V^{-1}$ which is open and contains 1_G (since $1_G^{-1} = 1_G$) and moreover $S = S^{-1}$ so that S is a symmetric open neighbourhood of 1_G contained in U. Since inversion is continuous and K is compact, K^{-1} is compact and since the union of compact sets is compact we conclude that $K \cup K^{-1}$ is a compact symmetric set. □

Remark 2.15. We did not actually use separate continuity of multiplication above.

Remark 2.16. ⚠Intersections of compact sets in topological groups are not necessarily compact. See Exercise I.6.

Remark 2.17. A symmetric set S in a quasitopological group has symmetric closure: Inversion is continuous and self-inverse so $\overline{S}^{-1}$ is closed and contains $S^{-1} = S$. It follows that $\overline{S} \subset \overline{S}^{-1}$. But inversion is an involution[3] so this tells us $\overline{S}^{-1} \subset (\overline{S}^{-1})^{-1} = \overline{S}$, and we conclude that $\overline{S}^{-1} = \overline{S}$.

The next lemma captures an important nesting of open and closed sets.

Lemma 2.18 (Key Lemma III)**.** *Suppose that G is a semitopological group, and S is a set and V is an open neighbourhood of the identity. Then $\overline{SV} \subset SVV^{-1}$.*

[3] An **involution** is a map $f : X \to X$ on a set that is self-inverse *i.e.* such that $f^2(x) = x$ for all $x \in X$.

Proof. Let $A := G \backslash (SVV^{-1})$, which is closed since V is open and so SVV^{-1} is open by Lemma 2.12; and $B := G \backslash (AV)$ which is closed since AV is open again by Lemma 2.12. If $x \in SV$ and $x \in AV$ then there is some $v \in V$ such that $xv^{-1} \in A$, so $xv^{-1} \notin SVV^{-1}$, a contradiction. Hence $SV \subset B$ and since B is closed $\overline{SV} \subset B$. Now if $x \in B$ then $x \notin AV$ and so in particular $x \notin A$ (since $1_G \in V$) and hence $x \in SVV^{-1}$ as claimed. □

The next result is, perhaps, a little surprising.

Corollary 2.19. *Suppose that G is a semitopological group and $H \leqslant G$. If H is a neighbourhood in G then H is open in G; and if H is open in G then H is closed in G.*

Proof. If H is a neighbourhood of some $x \in G$ then by Lemma 2.12 there is an open set U such that $x^{-1}U$ is an open set containing the identity. Now $H = HU$ is open, again by Lemma 2.12.

For the second part, if H is open then by Lemma 2.18 $\overline{H} \subset HH^{-1} = H$ and so H is closed. □

Remark 2.20. If U is a neighbourhood in a semitopological group G then by Corollary 2.19 $\langle U \rangle$ is closed so $\overline{U} \subset \langle U \rangle$ and hence $\langle \overline{U} \rangle = \langle U \rangle$. ⚠ This need not be true if U is not a neighbourhood, for example $U = \mathbb{Q}$ in $\mathbb{R}$ with its usual topology.

As it happens subgroup of a semitopological (resp. quasitopological) group with the subspace topology is itself a semitopological (resp. quasitopological) group, but more important to us is the following:

Proposition 2.21. *Suppose that G is a topological group and $H \leqslant G$. Then H is a topological group when endowed with the subspace topology.*

Proof. Suppose U is an open set in H, and let W be an open subset of G such that $U = W \cap H$. Then $U^{-1} = (W \cap H)^{-1} = W^{-1} \cap H^{-1} = W^{-1} \cap H$, but W^{-1} is open in G and so U^{-1} is open in H *i.e.* inversion is continuous.

For multiplication, let $V := \{(x, y) \in G^2 : xy \in W\}$ so that $V \cap H^2 = \{(x, y) \in H^2 : xy \in U\}$. Since multiplication on G is continuous, by definition of the product topology there are sets $\mathcal{S}$ and $\mathcal{T}$ of open subsets of G such that

$$V = \bigcup \{S \times T : S \in \mathcal{S}, T \in \mathcal{T}\}.$$

Now $(S \times T) \cap H^2 = (S \cap H) \times (T \cap H)$, and so the preimage of U under multiplication on H is open in the product of the subspace topology on H with itself. That is to say, multiplication is continuous on H and the result is proved. □

Example 2.22. $S^1 := \{z \in \mathbb{C}^* : |z| = 1\}$ is a subgroup of $\mathbb{C}^*$ and so it is a topological group. In this case it is closed, but in general we are not making the assumption that any subgroups we are considering are (topologically) closed.

We now turn to a couple of key lemmas which (like Proposition 2.21) make essential use of *joint* continuity in topological groups.

Lemma 2.23 (Key Lemma IV)**.** *Suppose that G is a topological group and $K_1, \dots, K_n$ are compact subsets of G. Then $K_1 \cdots K_n$ is compact. In particular, if K is compact then K^n is compact for all*[4] *$n \in \mathbb{N}_0$.*

Proof. The (topological) product of two compact sets is compact so if $K_1 \cdots K_{n-1}$ is compact and K_n is compact then $(K_1 \cdots K_{n-1}) \times K_n$ is compact. But then the continuous image of a compact set is compact and so $K_1 \cdots K_n = (K_1 \cdots K_{n-1})K_n$ is compact and the result follows by induction on n. □

Remark 2.24. Exercise I.3 gives an example of a quasitopological group where the conclusion above does not hold.

Lemma 2.25 (Key Lemma V)**.** *Suppose that G is a topological group and X is a neighbourhood of z. Then there is a symmetric open neighbourhood of the identity V such that $zV^2 \subset X$.*

Proof. Let $U \subset X$ be an open neighbourhood of z. The map $(x, y) \mapsto xy$ is continuous and so $\{(x, y) : xy \in U\}$ is an open subset of $G \times G$. By definition of the product topology there are sets $\mathcal{S}$ and $\mathcal{T}$ of open subsets of G such that

$$\{(x, y) : xy \in U\} = \bigcup \{S \times T : S \in \mathcal{S}, T \in \mathcal{T}\}.$$

Since $z1_G = z \in U$, there is some $S \in \mathcal{S}$ and $T \in \mathcal{T}$ such that $(z, 1_G) \in S \times T$. Thus S is an open neighbourhood of z and T is an open neighbourhood of the identity, so by Lemma 2.12 $(z^{-1}S) \cap T$ is an open neighbourhood of the identity, which by Lemma 2.14 contains a symmetric open neighbourhood of the identity V. Now $zV \subset S$ and $V \subset T$ and so $zV^2 \subset U$ as required. □

Remark 2.26. Endow $\mathbb{Z}$ with the cofinite topology as in Example 2.9 so that if $V \subset \mathbb{Z}$ is open and non-empty then there is some $a \in \mathbb{N}_0$ such that $x, -x \in V$ for all $x \geqslant a$ (since $\mathbb{Z}\backslash V$ is finite). It follows that if $n \in \mathbb{N}_0$ then $a+n, a, -a, -a-n \in V$ and so $n = (a+n)+(-a) \in V+V$ and $-n = a + (-a-n) \in V + V$, whence $V + V = \mathbb{Z}$. In particular, then, $U := \mathbb{Z}\backslash\{w\}$ which is a neighbourhood of every $z \neq w$ does not contain $z + V + V$ for any non-empty open set V.

In view of Lemma 2.25, $\mathbb{Z}$ with the cofinite topology is *not* a topological group, and since we know from Example 2.9 that it *is* a quasitopological group it follows that the hypothesis of Lemma 2.25 cannot be relaxed to apply to quasitopological groups.

[4] Note that $K^0 = \{1_G\}$ by definition and so is compact since it is finite.

Lemma 2.25 can be used to establish some uniformity in open covers of compact sets. A cover $\mathcal{U}$ is a **refinement** of a cover $\mathcal{V}$ of a set X if $\mathcal{U}$ is a cover of X and each set in $\mathcal{U}$ is contained in some set in $\mathcal{V}$.

Remark 2.27. Refinements are transitive meaning that if $\mathcal{W}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$ then $\mathcal{W}$ is a refinement of $\mathcal{U}$.

Lemma 2.28 (Key Lemma VI). *Suppose that G is a topological group and $K \subset G^n$ is compact for some $n \in \mathbb{N}$, and $\mathcal{U}$ is an open cover of K. Then there is a symmetric open neighbourhood of the identity $U \subset G$ such that $\{x_1U \times \cdots \times x_nU : x \in K\}$ is a refinement of $\mathcal{U}$.*

Proof. First, the structure of the product topology (and Lemma 2.12) means that we can pass to a refinement of $\mathcal{U}$ where for each $x \in K$ there are open neighbourhoods of the identity $U_1^{(x)}, \dots, U_n^{(x)}$ (our notation is a little clumsy here to make the x-dependence explicit) such that $x_1U_1^{(x)} \times \cdots \times x_nU_n^{(x)}$ is in the refinement. The set $\bigcap_{i=1}^n U_i^{(x)}$ is an open neighbourhood of the identity and so by Lemma 2.25 there is a symmetric open neighbourhood of the identity U_x such that $U_x^2 \subset U_i^{(x)}$ for all $1 \leqslant i \leqslant n$. In particular, $\mathcal{V} := \{x_1U_x \times \cdots \times x_nU_x : x \in K\}$ is an open cover of K and a refinement of $\mathcal{U}$.

By compactness of K there is a finite set $F \subset K$ such that $\mathcal{W} := \{x'_1U_{x'} \times \cdots \times x'_nU_{x'} : x' \in F\}$ is a cover of K. Let $U := \bigcap_{x' \in F} U_{x'}$ which is a finite intersection of symmetric open neighbourhoods of the identity and so a symmetric open neighbourhood of the identity. Since $\mathcal{W}$ is a cover of K, for each $x \in K$ there is some $x' \in F$ such that $x \in x'_1U_{x'} \times \cdots \times x'_nU_{x'}$, and hence

$$\begin{aligned} x_1U \times \cdots \times x_nU &\subset x'_1U_{x'}U \times \cdots \times x'_nU_{x'}U \\ &\subset x'_1U_{x'}^2 \times \cdots \times x'_nU_{x'}^2 \subset x'_1U_1^{(x')} \times \cdots \times x'_nU_n^{(x')} \end{aligned}$$

so that $\{x_1U \times \cdots \times x_nU : x \in K\}$ is a refinement of $\mathcal{V}$ which in turn is a refinement of $\mathcal{U}$ as required. □

Remark 2.29. The lemma above is not unrelated to the Generalised Tube Lemma from topology (see *e.g.* [Mun00, Lemma 26.8]), which is also known as Wallace's Theorem.

This proposition highlights an important interplay of compactness and the group structure and has content even in seemingly simple cases:

Corollary 2.30. *Suppose that G is a topological group, A is a compact set and B is an open set containing A. Then there is a symmetric open neighbourhood of the identity U such that $\overline{AU} \subset B$. In particular, every neighbourhood of x contains a closed neighbourhood of x.*

Proof. Apply Lemma 2.28 with $n = 1$ to the open cover $\{B\}$ of A to get an open neighbourhood of the identity, V, such that $AV \subset B$. By Lemma 2.25 there is a symmetric open neighbourhood of the identity U such that $UU^{-1} = U^2 \subset V$, and so by Lemma 2.18 $\overline{AU} \subset AUU^{-1} \subset AV \subset B$ as required.

The last part follows immediately since the given neighbourhood contains an open neighbourhood B of x. The set $\{x\}$ is compact and so there is an open neighbourhood of the identity U with $\overline{xU} \subset B$ as required. □

A topological space is said to be **regular** if for any closed set A and $x \notin A$ there are open sets U and V such that $A \subset U$ and $x \in V$.

Remark 2.31. Since finite sets (and in particular singletons) are compact Corollary 2.30 implies that every topological group is regular, and in particular that every neighbourhood contains a closed neighbourhood. On the other hand an infinite group (*e.g.* $\mathbb{Z}$) endowed with the cofinite topology (as in Example 2.9) shows that this does not hold in quasitopological groups since any non-empty open set there is infinite, while the only infinite closed set is the whole group.

A topological space is said to be **Hausdorff** (or T_2) if for any $x \neq y$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$.

Remark 2.32. A topological space has unique limits (for nets) if and only if it is Hausdorff, so this is a pretty uncontroversial axiom to want.

Remark 2.33. A subspace of a Hausdorff topological space is Hausdorff, so if H is a subgroup of a Hausdorff topological group G then H is a Hausdorff topological group when equipped with the subspace topology.

Remark 2.34. ⚠ At first glance it may look as if regularity is a stronger condition than being Hausdorff, and to make matters worse it is *sometimes* called separation axiom T_3, but in fact while we noted in Remark 2.31 that all topological groups are regular, there are topological groups that are not Hausdorff *e.g.* a non-trivial group with the indiscrete topology. More usually T_3 space means regular Hausdorff.

Corollary 2.35. *Suppose that G is a topological group. Then G is Hausdorff if and only if $\{1_G\}$ is closed (equivalently*[5] *if and only if G is Fréchet).*

Proof. First, if G is Hausdorff then for each $x \neq 1_G$ there is an open set U_x containing x and not containing 1_G. Hence $G\setminus\{1_G\} = \bigcup_{x \in G} U_x$ is open as required.

Conversely, if $\{1_G\}$ is closed then G is Fréchet and so for all $x \neq y$, $\{x\}$ is closed and $\{y\}$ is compact (since it is finite) so G is Hausdorff by Corollary 2.30. □

[5] By Lemma 2.13.

Remark 2.36. Compact subsets of Hausdorff topological spaces are closed. Conversely, the Corollary 2.35 tells us that a topological group G is Hausdorff if the compact set $\{1_G\}$ is closed and *a fortiori* if every compact set is closed.

⚠ An uncountable group (*e.g.* $\mathbb{R}$) endowed with the cocountable topology (as in Example 2.10) has no infinite compact set: if K is infinite then there is (by the Axiom of Countable Choice!) a countably infinite set $S \subset K$, but then $\{S^c \cup \{s\} : s \in S\}$ is an open cover of K with no finite subcover *i.e.* K is *not* compact. Since every finite set is closed in this topology, we conclude that we have a quasitopological group in which every compact set is closed. It is *not*, however, Hausdorff since any two non-empty open sets have a non-trivial intersection.

To some extent the situation in non-Hausdorff topological groups can be recovered by the following lemma.

Lemma 2.37. *Suppose that G is a topological group and K is a compact subset of G. Then $\overline{K}$ is compact.*

Proof. Suppose $\mathcal{U}$ is an open cover of $\overline{K}$ then by for each $x \in K$ there is an open neighbourhood of x in $\mathcal{U}$, call it U_x. By Corollary 2.30 applied to the compact set $\{x\}$ in the open set U_x there is an open neighbourhood of x, call it V_x, such that $\overline{V_x} \subset U_x$. The set $\{V_x : x \in K\}$ is an open cover of K and so by compactness has a finite subcover, say $K \subset V_{x_1} \cup \cdots \cup V_{x_k}$ and hence $\overline{K} \subset U_{x_1} \cup \cdots \cup U_{x_k}$. Thus $\mathcal{U}$ has a finite subcover of $\overline{K}$, and the result is proved. □

Remark 2.38. The reals with the right order topology ($\mathbb{R}_{\mathrm{RO}}$ from Example 2.8) have $\{0\}$ as a compact subset (since it is finite), but $\overline{\{0\}} = (-\infty, 0]$ which is *not* compact since the open cover $\{(a,\infty) : a \in \mathbb{R}\}$ has no finite subcover. In particular, we cannot relax the requirement that G be a topological group to semitopological group in Lemma 2.37.

A topological space X is **locally compact** if every point has a compact neighbourhood.

Example 2.39. $\mathbb{Q}$ is a subgroup of $\mathbb{R}$ (with its usual topology) and so by Proposition 2.21 is a topological group with the subspace topology. However, while $\mathbb{R}$ is locally compact, $\mathbb{Q}$ is *not* locally compact. In particular, unlike the situation in Remark 2.33, locally compactness is *not* preserved on passing to subgroups.

Remark 2.40. We shall mostly be interested in locally compact topologies and this is one of the reasons we do not concern ourselves overly with semitopological and quasitopological groups: there is a theorem of Ellis [Ell57, Theorem 2] which says that any locally compact Hausdorff semitopological group is a topological group.

We shall think of a locally compact topological group is a group that is ‘locally’ not too large – every point has a neighbourhood that is compact – but it might otherwise be massive, for example *any* group with the discrete topology is locally compact.

Lemma 2.41. *Suppose that G is a locally compact topological group and K is a compact set. Then there is a symmetric open neighbourhood of the identity containing K whose closure is compact. In particular, G has a symmetric open neighbourhood of the identity with compact closure.*

Proof. Since G is locally compact there is a compact neighbourhood of the identity L, and hence by Lemma 2.14 there is a symmetric open neighbourhood of the identity $V \subset L$. The union of two compact sets is compact so $K \cup \{1_G\}$ is compact and so by Lemma 2.14 there is a symmetric compact set M containing $K \cup \{1_G\}$. Now, $U := VMV$ is open by Lemma 2.12; it contains the identity since V and M do; it contains K since V contains the identity and M contains K; and it is symmetric since $(VMV)^{-1} = V^{-1}(VM)^{-1} = V^{-1}M^{-1}V^{-1} = VMV$. Finally, $U \subset LML$ which is compact by Lemma 2.23, and hence $\overline{U} \subset \overline{LML}$ is compact by Lemma 2.37. □

We say that G is **compactly generated** to mean that there is a symmetric open neighbourhood of the identity S with compact closure such that $G = \langle S \rangle$.

Remark 2.42. ⚠ A topological space is said to be compactly generated if A is closed if and only if $A \cap K$ is closed for all compact subsets K. This is a distinct notion – for example a discrete group that is not finitely generated is compactly generated in this sense but not in our sense – which we shall not use.

We think of compactly generated groups as 'globally' not too large, a bit like being finitely generated.

Corollary 2.43. *Suppose that G is a locally compact topological group. Then there is a compactly generated open subgroup of G.*

Proof. Apply Lemma 2.41 to get a symmetric open neighbourhood of the identity S with compact closure. Then $\langle S \rangle$ is a compactly generated subgroup of G, and it is open by Corollary 2.19. □

Proposition 2.44. *Suppose that G is a topological group and U is a symmetric neighbourhood of the identity with compact closure. Then there is a finite set T such that $\langle U \rangle = \langle T \rangle U$.*

Proof. Let $V \subset U$ be open and note that $\{tV : t \in \overline{U}^2\}$ is an open cover of $\overline{U}^2$. The latter is compact by Lemma 2.23 and so there is a finite set T such that $\{tV : t \in T\}$ is a cover of $\overline{U}^2$. It follows that $U^2 \subset \overline{U}^2 \subset TV \subset TU$. Now by induction $U^n \subset T^{n-1}U \subset \langle T \rangle U$ for $n \in \mathbb{N}$, and so since U is symmetric and contains the identity, $\bigcup_{n\in\mathbb{N}} U^n$ is a subgroup by the subgroup test. We conclude that $\langle U \rangle \subset \langle T \rangle U$.

Since U is a neighbourhood, $\langle U \rangle$ is closed by Corollary 2.19 and so $\overline{U} \subset \langle U \rangle$, and by algebraic closure $T \subset \overline{U}^2 \subset \langle U \rangle$. We conclude that $\langle T \rangle U \subset \langle U \rangle$, and the result is proved. □

3 The structure-preserving maps

A map $\theta : G \to H$ is a **homomorphism of topological groups** if it is a continuous group homomorphism between topological groups.

Example 3.1. The map $\theta : \mathbb{R} \to S^1; x \mapsto \exp(2\pi i x)$ is a (surjective) continuous homomorphism.

Example 3.2. Suppose that G is a group and $\theta : G \to G$ is the identity map. If the domain is endowed with the discrete topology then θ is a homomorphism of topological groups whatever the topology on the codomain, and if the codomain is endowed with the indiscrete topology then similarly.

This example may seem trivial but leads to a number of counter-examples.

Example 3.3. Suppose that $\theta : \mathbb{Q} \to \mathbb{Q}$ is the identity map, with the domain discrete and the codomain the usual subspace topology inherited from $\mathbb{R}$ (as in Example 2.39). Then the domain is locally compact but the codomain is not, so local compactness is not preserved by surjective topological group homomorphisms.

Example 3.4. Suppose that $\theta : \mathbb{R} \to \mathbb{R}$ is the identity map, with the domain the usual topology on $\mathbb{R}$ and the codomain the indiscrete topology. Then the domain is Hausdorff and the codomain is not, so being Hausdorff is not preserved by surjective topological group homomorphisms.

Remark 3.5. This being said, if there is a surjective topological group homomorphism from a compact topological group G to a topological group H then H is compact. (Of course, this fact is purely topological.)

The group structure makes checking continuity and openness a little easier:

Lemma 3.6. *Suppose that G and H are semitopological groups and $B = (B_i)_{i \in I}$ is a neighbourhood base*[6] *of the identity in H. Then a homomorphism $\theta : G \to H$ is continuous if (and only if) $\theta^{-1}(B_i)$ is a neighbourhood of the identity for all $i \in I$; and a homomorphism $\theta : H \to G$ is open if (and only if) $\theta(B_i)$ is a neighbourhood of the identity for all $i \in I$.*

Proof. Suppose that $U \subset H$ is open and $\theta(y) \in U$. By Lemma 2.12 there is an open neighbourhood of the identity V_y such that $\theta(y)V_y \subset U$. Since B is a neighbourhood base of the identity there is $i \in I$ such that $B_i \subset V_y$ and hence $\theta^{-1}(B_i) \subset \theta^{-1}(V_y)$ so $y\theta^{-1}(B_i) \subset \theta^{-1}(U)$ (using that θ is a homomorphism) and hence $\theta^{-1}(U)$ contains a neighbourhood of y *i.e.* $\theta^{-1}(U)$ is open. In the other direction, since B_i is a neighbourhood of the identity it contains an open neighbourhood of the identity which has an open set as a preimage and the

[6] Defined in Remark 1.4.

identity in this preimage (since homomorphisms map the identity to the identity), whence it is an open neighbourhood of the identity and $\theta^{-1}(B_i)$ is a neighbourhood of the identity.

Now suppose that $U \subset H$ is open and $x \in \theta(U)$ so that there is some $y \in U$ such that $x = \theta(y)$. Since U is open, by Lemma 2.12 there is an open neighbourhood of the identity V_y such that $yV_y \subset U$. Since B is a neighbourhood base of the identity there is $i \in I$ such that $B_i \subset V_y$ and hence $x\theta(B_i) = \theta(yB_i) \subset \theta(U)$ (using that θ is a homomorphism). But $x\theta(B_i)$ is open by hypothesis, so $\theta(U)$ is open as required. In the other direction since B_i is a neighbourhood of the identity it contains an open set containing the identity which has an open image containing the identity (since homomorphisms map the identity to the identity), and hence the image of B_i is a neighbourhood of the identity. □

Topological groups G and H are **isomorphic as topological groups** if there are continuous homomorphisms $\theta : G \to H$ and $\psi : H \to G$ such that $\theta \circ \psi = \iota_H$ and $\psi \circ \theta = \iota_G$ where ι_H and ι_G are the identity maps on H and G respectively.

Example 3.7 (Conjugation)**.** For a topological group G, the map $G \times G \to G; (a, x) \mapsto axa^{-1}$ is a left action of G on G – it is called **conjugation**– and it is (jointly) continuous (as it is a composition of continuous maps).

In particular, for fixed $a \in G$ the map $G \to G; x \mapsto axa^{-1}$ is a continuous map with a continuous inverse $G \to G; x \mapsto a^{-1}xa$, and hence a topological isomorphism. ⚠ The *joint* continuity says more than just this last fact however.

Some useful examples of topological groups and homomorphisms between them arise through products.

Proposition 3.8. *Suppose that $(G_i)_{i\in I}$ is a family of topological groups. Then the direct product of the groups, $\prod_{i\in I} G_i$, with the product topology is a topological group and the projection maps $p_j : \prod_{i\in I} G_i \to G_j; x \mapsto x_j$ for each $j \in I$ are continuous open maps.*

Proof. The key to this is recalling the fact that the open sets in $\prod_{i\in I} G_i$ are unions of sets of the form

$$\prod_{i\in I} U_i \text{ where } \begin{cases} U_i = G_i \text{ for all } i \in I\backslash J \\ U_i \text{ is open in } G_i \text{ for all } i \in J \end{cases} \tag{3.1}$$

where J ranges all finite subsets of I. The image of a set like this under the projection p_j is open and so p_j is an open map, and if $U_j \subset G_j$ is open then $p_j^{-1}(U_j) = \prod_{i\in I} U_i$ where $U_i = G_i$ for $i \neq j$, is open, so the p_js are continuous. If $\prod_{i\in I} U_i$ is as in (3.1) then $(\prod_{i\in I} U_i)^{-1} = \prod_{i\in I} U_i^{-1}$ is also open and hence inversion is continuous, and similarly for multiplication. □

Remark 3.9. We call the topological group above the **topological direct product** of the groups $(G_i)_{i\in I}$.

Quotient groups

Given a topological group G and a subgroup H the quotient map $q : G \to G/H; x \mapsto xH$ naturally induces a topology on G/H – the weakest topology making the quotient map continuous or, more concretely, $U \subset G/H$ is open if and only if $\bigcup U$ is open in G.

Proposition 3.10. *Suppose that G is a topological group and H is a normal subgroup of G. Then G/H is a topological group when endowed with the quotient topology and the quotient map $q : G \to G/H$ is (continuous and) open.*

Proof. The quotient map is continuous by definition; to show it is open it suffices to note that if U is open in G then UH is open by Lemma 2.12 and $q(U) = \{uH : u \in U\}$ so that $\bigcup q(U) = UH$. Thus $\bigcup q(U)$ is open in G, and hence $q(U)$ is open by definition.

Suppose that $U \subset G/H$ is open. First we show that inversion is continuous on G/H:

$$\bigcup U^{-1} = \bigcup \{(xH)^{-1} : xH \in U\} = \left\{x^{-1} : x \in \bigcup U\right\} = \left(\bigcup U\right)^{-1}$$

and so U^{-1} is open in G/H by definition since $\bigcup U$ is open in G and inversion is continuous on G. Second, define

$$W := \left\{(zH, wH) \in (G/H)^2 : (zH)(wH) \in U\right\} \text{ and } V := \left\{(z, w) \in G^2 : zw \in \bigcup U\right\}.$$

Suppose that $(xH, yH) \in W$. Then $xy \in (xH)(yH) \subset \bigcup U$ so $(x, y) \in V$ and since V is open there are open sets $S, T \subset G$ such that $x \in S$, $y \in T$, and $S \times T \subset V$. If $s \in S$ and $t \in T$, then $st \in \bigcup U$, and since the latter is a union of cosets of H we have $(st)H \subset \bigcup U$. Since H is normal we have $(sH)(tH) = (st)H \subset \bigcup U$, and so $SH \times TH \subset V$.

By Lemma 2.12, SH and TH are open sets, and so the sets $S' := \{sH : s \in S\}$ and $T' := \{tH : t \in T\}$ are open in G/H; $xH \in S'$ and $yH \in T'$; and $S' \times T' \subset W$. It follows that W is open as required. □

Remark 3.11. If G is a (locally) compact topological group and H is a normal subgroup of G then G/H is (locally) compact. In the case of compactness this is just that the quotient map is continuous and the continuous image of a compact space is compact. In the locally compact case since the quotient map is open it maps neighbourhoods to neighbourhoods so a compact neighbourhood of the identity in G (which exists by local compatness) is mapped to a compact neighbourhood of the identity in G/H as required.

Example 3.12. The topological group $\mathbb{R}$ has a (normal) subgroup $\mathbb{Z}$ and $\mathbb{R}/\mathbb{Z}$ is a topological group – it is the reals modulo 1. Moreover, the map $\mathbb{R}/\mathbb{Z} \to S^1; x + n\mathbb{Z} \mapsto \exp(2\pi i x)$ is an isomorphism of topological groups.

Remark 3.13. ⚠ The adjunction space formed by taking the topological space $\mathbb{R}$ and identifying all the elements of $\mathbb{Z}$ considered as a topological subspace of $\mathbb{R}$ is also sometimes denoted $\mathbb{R}/\mathbb{Z}$ though this is quite a different object; we shall have no call to refer to it.

Example 3.14. The group $\mathbb{Q}$ is a subgroup of $\mathbb{R}$ with its usual topology, and so $\mathbb{R}/\mathbb{Q}$ is a topological group. If $U \subset \mathbb{R}/\mathbb{Q}$ is open then $\bigcup U$ is open in $\mathbb{R}$ and so if it is non-empty it contains an interval I. However, $\bigcup U$ is a union of cosets of $\mathbb{Q}$ so $\bigcup U = \bigcup U + \mathbb{Q} \supset I + \mathbb{Q} = \mathbb{R}$. It follows that $\mathbb{R}/\mathbb{Q}$ is indiscrete.

⚠Note that the quotient map $q : \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ is *not* closed since *e.g.* $q(\{0\}) = \{\mathbb{Q}\}$ is not closed in $\mathbb{R}/\mathbb{Q}$. This is by way of contrast with the fact that every quotient map between topological groups is open.

Topological closure preserves algebraic structure in a useful way:

Lemma 3.15. *Suppose that G is a topological group and $H \leqslant G$. Then $\overline{H}$ is a subgroup of G. If G is compact then so is $\overline{H}$; if G is locally compact then so is $\overline{H}$; and if H is normal then so is $\overline{H}$.*

Proof. Suppose that $(x, y) \in G^2$ is such that $xy^{-1} \notin \overline{H}$. Then since $(z, w) \mapsto zw^{-1}$ is continuous, there are open sets $S, T \subset G$ such that $x \in S$, $y \in T$ and $ST^{-1} \cap \overline{H} = \varnothing$. Since $\overline{H} \supset H$, and H is a subgroup, if $S \cap H \neq \varnothing$ then $T \cap H = \varnothing$, and hence $\overline{H} \subset G \backslash T$ so that $T \cap \overline{H} = \varnothing$. Similarly, if $S \cap H = \varnothing$ then $S \cap \overline{H} = \varnothing$. It follows that $x \notin \overline{H}$ or $y \notin \overline{H}$ and so $\overline{H}$ is a group.

Closed subsets of compact sets are compact so if G is compact then so is $\overline{H}$; and if G is locally compact then G has a compact neighbourhood of the identity N and hence $N \cap \overline{H}$ is a compact neighbourhood of the identity in $\overline{H}$ and so $\overline{H}$ is locally compact.

Finally, assume that H is normal. Conjugation is continuous and hence $a^{-1}\overline{H}a$ is closed for all $a \in G$, and contains $a^{-1}Ha = H$. Hence it contains the closure of H and so applying the map $x \mapsto axa^{-1}$ we get $a\overline{H}a^{-1} \subset \overline{H}$ *i.e.* $\overline{H}$ is normal. □

Taking quotients gives us a way of introducing the Hausdorff property in topological groups.

Proposition 3.16. *Suppose that G is a topological group and H is a normal subgroup of G. Then $\overline{H}$ is a closed normal subgroup of G and $G/\overline{H}$ is a Hausdorff topological group.*

Proof. $\overline{H}$ is a closed normal subgroup by Lemma 3.15. By Corollary 2.35, $G/\overline{H}$ is Hausdorff if and only if $\{\overline{H}\}$ is closed in $G/\overline{H}$ which, by definition of the quotient topology, is true since $\overline{H}$ is closed in G. □

Since $\{1_G\}$ is a normal subgroup of any topological group (and it is compact) we have the following corollary (with Lemma 2.37 for compactness).

Corollary 3.17. *Suppose that G is a topological group. Then $\overline{\{1_G\}}$ is a compact normal subgroup and $G/\overline{\{1_G\}}$ is a Hausdorff topological group.*

As well as the 'small' normal subgroup $\overline{\{1_G\}}$ (which in some cases is the whole of G!), open subgroups in compact topological groups create large normal subgroups. To establish this we begin with a lemma.

Lemma 3.18. *Suppose that G is a semitopological group and H is a closed subgroup of G of finite index. Then H is open in G and there is an open (and so closed[7]) normal subgroup of G contained in H.*

Proof. Since H has finite index in G there are elements $x_1, \dots, x_m \in G$ such that $G/H = \{x_1H, \dots, x_mH\}$. The sets x_iH are all closed and so $\bigcup\{x_iH : H \neq x_iH\}$ is a finite union of closed sets and so closed. Since G/H is a partition of G it follows that we have $H^c = \bigcup\{x_iH : H \neq x_iH\}$ and hence H is open.

Let $N := \bigcap_{i=1}^m x_iHx_i^{-1}$ which is a finite intersection of open subgroups and so an open subgroup. On the other hand, if $x \in G$ then for each $1 \leqslant j \leqslant m$ there is some $1 \leqslant i \leqslant m$ (depending on j and x) such that $xx_iH = x_jH$. But then $(Hx_i^{-1})x^{-1} = Hx_j^{-1}$ and so $x(x_iHx_i^{-1})x^{-1} \subset x_jHx_j^{-1}$, whence $xNx^{-1} \subset x_jHx_j^{-1}$. However, j was arbitrary and so $xNx^{-1} \subset N$, and N is normal as required. The result is proved. □

Remark 3.19. This result almost deserves the status of one of the 'Key Lemma's we marked in red in §2. The reason we omit it is because we do not make very much use of it restricting ourselves to the next corollary.

Corollary 3.20. *Suppose that G is a compact semitopological group and H is an open subgroup of G. Then H has finite index in G and there is an open (and so closed) normal subgroup of G contained in H.*

Proof. Since H is open the set of left cosets – G/H – is an open over of G. Since G is compact, there is a finite subset of G/H that covers G and since the cosets in G/H are disjoint this must be the whole of G/H so that G/H is finite *i.e.* H has finite index in G. Thus by Lemma 3.18, H contains an open normal subgroup of G. □

The open mapping theorem

Example 3.2 shows that there are continuous bijective group homomorphisms that are not isomorphisms of topological groups. This is by contrast with the purely algebraic situation where any bijective group homomorphism is a group isomorphism (*i.e.* has an inverse that is a homomorphism), but in alignment with the topological situation where continuous bijections need not be homeomorphisms. With a few mild conditions on the topology we can recover with algebraic situation:

[7] By Corollary 2.19.

Theorem 3.21. *Suppose that G is a compactly generated topological group, H is a locally compact Hausdorff topological group, and $\pi : G \to H$ is a continuous bijective homomorphism. Then π is an isomorphism of topological groups.*

Proof. Since the inverse of a bijective group homomorphism is a group homomorphism, it suffices to show that $\pi(C)$ is closed whenever C is closed in G. Let K be the compact closure of the symmetric open neighbourhood generating G. Then K is symmetric (Remark 2.17) and $G = \bigcup_{n \in \mathbb{N}_0} K^n$ (Remark 2.3).

Claim. *There is some $n \in \mathbb{N}$ such that $\pi(K^n)$ is a neighbourhood.*

Proof. For those familiar with the Baire Category Theorem this is particularly straightforward. We shall proceed directly by what is essentially the proof of the BCT for locally compact Hausdorff spaces.

By Lemma 2.23, for each $n \in \mathbb{N}$ the set K^n is compact and so $\pi(K^n)$ is compact. Since H is Hausdorff the sets $\pi(K^n)$ are therefore closed. We construct a nested sequence of closed neighbourhoods inductively: Let U_0 be a compact (and so closed since H is Hausdorff) neighbourhood in H, and for $n \in \mathbb{N}$ let $U_n \subset \pi(K^n)^c \cap U_{n-1}$ be a closed neighbourhood.

This is possible since (by the inductive hypothesis) U_{n-1} is a neighbourhood and so contains an open neighbourhood V_{n-1}. But then $\pi(K^n)^c \cap V_{n-1}$ is open and non-empty since otherwise $\pi(K^n)$ contains a neighbourhood. It follows that $\pi(K^n)^c \cap U_{n-1}$ contains an open neighbourhood and so it contains a closed neighbourhood by Corollary 2.30.

Now by the finite intersection property of the compact space U_0, the set $\bigcap_n U_n$ is non-empty. This contradicts surjectivity of π since $G = \bigcup_{n \in \mathbb{N}_0} K^n$ and the claim is proved. □

Claim. *If $X \subset H$ is compact then $\pi^{-1}(X)$ is compact.*

Proof. Suppose X is compact. By the previous claim $\pi(K^n)$ contains a neighbourhood and the set $\{x\pi(K^n) : x \in H\}$ covers X, so there are elements $x_1, \ldots, x_m$ such that $X \subset \bigcup_{i=1}^m x_i \pi(K^n)$ and hence $\pi^{-1}(X) \subset \bigcup_{i=1}^m \pi^{-1}(x_i)K^n$ (by injectivity of π). $\pi^{-1}(x_i)K^n$ is compact by Lemma 2.23 (and the fact that the continuous image of a compact set is compact), and since a finite union of compact sets is compact it follows that $\pi^{-1}(X)$ is contained in a compact set. Finally, X is closed so $\pi^{-1}(X)$ is closed and H is Hausdorff so a closed a subset of a compact set is compact we have the claim. □

Finally, suppose that $C \subset G$ is closed, and y is a limit point of $\pi(C)$. H is locally compact so y has a compact neighbourhood X. Now $\pi^{-1}(X)$ is compact and so $\pi^{-1}(X) \cap C$ is compact. But then $X \cap \pi(C)$ is compact since π is continuous, and hence closed since H is Hausdorff. But by design $y \in \overline{X \cap \pi(C)} = X \cap \pi(C) \subset \pi(C)$. □

Remark 3.22. The Open Mapping Theorem in functional analysis is the result that if $A : X \to Y$ is a surjective continuous linear operator between Banach spaces X and Y then A is an open mapping. The connection between this and the above result is spelt out in Exercise II.5.

4 Continuous complex-valued functions

Given a topological space X the **support** of a (not necessarily continuous) function $f : X \to \mathbb{C}$, denoted $\operatorname{supp} f$, is the set of $x \in X$ such that $f(x) \neq 0$; f is said to be **compactly supported** if its support is contained in a compact set.

We write $C(X)$ for the complex vector space of continuous complex-valued functions $X \to \mathbb{C}$, and $C_c(X)$ for the subset of functions in $C(X)$ that are compactly supported.

Remark 4.1. ⚠As we have defined it the support of a function f that is compactly supported need not actually *be* a compact set it is simply contained in one. Terminology in the literature is not always completely clear on this point.

Remark 4.2. The set $C_c(X)$ is a subspace of $C(X)$ since the union of two compact sets is compact and the support of the sum of two functions is contained in the union of their supports. More than this, the function

$$\|f\|_\infty := \sup \{|f(x)| : x \in X\}$$

is a norm on $C_c(X)$. It is well-defined since every continuous (complex-valued) function on a compact set is bounded, and the axioms of a norm are easily checked.

⚠In general $\|\cdot\|_\infty$ is *not* a norm on $C(X)$ since we are not assuming the elements of $C(X)$ are bounded.

⚠In general $C_c(X)$ is *not* complete despite the fact that the uniform limit of continuous functions is continuous since this limit function may not be compactly supported.

Remark 4.3. As a normed space $C_c(X)$ is, itself, a topological group (recall Example 1.6).

Remark 4.4. If G is a topological group and $C_c(G)$ contains a non-zero function then G is locally compact: Indeed, if $f \in C_c(G)$ is non-trivial then $\operatorname{supp} f \neq \emptyset$, but $\operatorname{supp} f$ is open (since f is continuous), and $\operatorname{supp} f$ is contained in a compact set K (since f is compactly supported). It follows that K is a compact neighbourhood of some point $x \in G$, and $yx^{-1}K$ is then a compact neighbourhood of y for $y \in G$ *i.e.* G is locally compact. This observation explains why the material of the remainder of the course will almost exclusively concern locally compact topological groups.

Remark 4.5. Traces of Lemma 2.25 can seen through applications of the triangle inequality which are useful to us: For $f \in C(X)$ and $\epsilon > 0$ there is an open cover $\mathcal{U}$ of X such that if $U \in \mathcal{U}$ and $x, y \in U$ then $|f(x) - f(y)| < \epsilon$.

To see this just let $\mathcal{U} := \{f^{-1}(z + (-\epsilon/2, \epsilon/2)) : z \in \mathbb{C}\}$ so that if $U \in \mathcal{U}$ and $x, y \in U$ then there is some $z \in \mathbb{C}$ such that $|f(x) - z| < \epsilon/2$ and $|f(y) - z| < \epsilon/2$ whence by the triangle inequality

$$|f(x) - f(y)| \leqslant |f(x) - z| + |z - f(y)| < \epsilon.$$

Remark 4.6. It is worth recalling the algebra of continuous functions: if $f, g \in C(X)$ then $f + g, fg \in C(X)$. The first of these uses that addition on $\mathbb{C}$ is jointly continuous (coupled with the fact that $X \to \mathbb{C}^2; x \mapsto (f(x), g(x))$ is continuous) and the second that multiplication is jointly continuous. ⚠ The first here is part of the statement that $\mathbb{C}$ under addition is a topological group, while the second is *not* since $\mathbb{C}$ is not a group under multiplication.

⚠ Quotients of continuous functions behave a little differently: if $f, g \in C(X)$ then the support of g is open and there is a continuous function $h : \operatorname{supp} g \to \mathbb{C}$ such that $f = gh$, but in general this need[8] *not* have a continuous extension to the whole of X. However, if $f, g \in C_c(X)$ and $\overline{\operatorname{supp} f} \subset \operatorname{supp} g$ then there is $h \in C_c(X)$ such that $f = gh$.

We do not yet actually know that there are any continuous functions besides the constant functions, and of course for indiscrete topological groups there need not be. Nevertheless the following theorem will be very useful for generating such functions and shows that indiscrete topological groups are the only groups without non-constant continuous functions to $\mathbb{C}$. (See Example 4.11.)

Theorem 4.7. *Suppose that G is a topological group, A is a compact set and B is an open set containing A. Then there is a continuous function $g : G \to [0, 1]$ such that $g(x) = 0$ on for all $x \in A$ and $g(x) = 1$ for all $x \notin B$. Similarly, there is a continuous function $f : G \to [0, 1]$ such that $f(x) = 1$ for all $x \in A$ and* $\operatorname{supp} f \subset B$.

Proof. The proof of this theorem is really a more sophisticated version of the proof of Corollary 2.30. As in the proof there we apply Lemma 2.28 to the open cover $\{B\}$ to get a symmetric open neighbourhood of the identity V such that $AV \subset B$. We may apply Lemma 2.25 twice to get a symmetric open neighbourhood of the identity V_0 such that[9] $V_0^3 \subset V$, and continue iteratively in this manner producing symmetric open neighbourhoods V_i with $V_{i+1}^3 \subset V_i$ for all $i \in \mathbb{N}_0$. In particular, note that $V_{i+1} \subset V_i$ since all the V_is are neighbourhoods of the identity.

We shall 'divide up the space between A and B' in a way that will be indexed by **dyadic rationals**, that is rationals whose denominator is a power of 2. For $i \in \mathbb{N}_0$ we write $D_i := \{q \in [0, 1] : 2^i q \in \mathbb{Z}\}$, so $D := \bigcup_{i=0}^{\infty} D_i$ is the set of dyadic rationals in $[0, 1]$. Note, in particular, that $D_0 \subset D_1 \subset \ldots$ and every element of $D_{i+1} \backslash D_i$ can be written uniquely

[8] Consider, for example, the functions $f(x) = x$ and $g(x) = x^2$ in $C(\mathbb{R})$. Then $h(x) = 1/x$ for all $x \in \operatorname{supp} g$ but h has no continuous extension to $\mathbb{R}$.

[9] Since $1_G \in V_0$ we certainly have $V_0^3 \subset (V_0^2)^2$.

in the form $\frac{1}{2}(q+q')$ where $q < q'$ are *consecutive* elements of D_i. Furthermore, in any two consecutive elements of D_{i+1}, one of them will be an element of D_i and one of $D_{i+1}\backslash D_i$.

For each $q \in D$ we define an open set U_q such that if $q < q'$ are consecutive elements of D_i for some i then $\overline{U_q}V_i \subset U_{q'}$. We proceed inductively on $i \in \mathbb{N}_0$. First, $D_0 = \{0,1\}$; let $U_0 := AV_0$ which is open by Lemma 2.12 and $U_1 := B$ which is open by definition of B. Then by Lemma 2.18 $\overline{U_0}V_0 = \overline{AV_0}V_0 \subset AV_0V_0^{-1}V_0 \subset AV \subset B = U_1$ as required.

Suppose U_q has been defined with the required property for all $q \in D_i$. For $q < q'$ consecutive elements of D_i we define $U_{\frac{1}{2}(q+q')} := \overline{U_q}V_{i+1}$ which is open by Lemma 2.12, and furthermore by Lemma 2.18 we have $\overline{U_{\frac{1}{2}(q+q')}}V_{i+1} \subset \overline{U_q}V_{i+1}V_{i+1}^{-1}V_{i+1} \subset \overline{U_q}V_i \subset U_{q'}$. Now, if $q < q'$ are consecutive elements of D_{i+1} then either $q \in D_i$, $q'' := q + 2^i \in D_i$ and $q' = \frac{1}{2}(q+q')$; or $q' \in D_i$, $q'' := q' - 2^{-i} \in D_i$ and $q = \frac{1}{2}(q'+q'')$. In either case, by design we have $\overline{U_q}V_{i+1} \subset U_{q'}$.

We now forget about the V_is: for each $q \in D$ we have an open set U_q such that (by nesting) whenever $q < q'$ are elements of D we have $\overline{U_q} \subset U_{q'}$. Moreover, $A \subset U_0$ and $U_1 \subset B$. Define a function $g : G \to [0,1]$ by

$$g(x) := \inf\{q \in D : x \in U_q\} \text{ if } x \in U_1 \text{ and } g(x) = 1 \text{ if } x \notin U_1.$$

First note that this is well-defined and really does map into $[0,1]$. Then, since $U_1 \subset B$ we have $g(x) = 1$ for all $x \notin B$; and since $A \subset U_0$ for all $x \in A$ we have $g(x) = 0$ for $x \in A$.

It remains to establish that g is continuous. Since all open subsets of $[0,1]$ are (possibly empty) unions of finite intersections of sets of the form $[0,\alpha)$ and $(\alpha,1]$ for $\alpha \in (0,1)$, we shall show that g is continuous by showing that preimages of sets of this form are open, and we shall do *this* by showing that every point in the preimage is contained in an open neighbourhood.

First, if $x \in g^{-1}([0,\alpha))$ then $g(x) < \alpha$ and so $x \in U_1$ and by the approximation property for infima there is some $q \in D$ such that $g(x) \leqslant q < \alpha$. But then $g(z) \leqslant q < \alpha$ for all $z \in U_q$, and so $g^{-1}([0,\alpha))$ contains the open neighbourhood U_q of x as required.

Secondly, if $x \in g^{-1}((\alpha,1])$ then since D is dense in $[0,1]$ there are $q, q' \in D$ with $\alpha < q < q' < g(x)$. Hence $x \notin U_{q'}$, but $\overline{U_q} \subset U_{q'}$ by nesting and so $x \in \overline{U_q}^c$. Moreover, if $z \in \overline{U_q}^c$ then $z \notin U_q$ and so (either $z \notin U_1$ and $g(z) = 1 > \alpha$ or) $g(z) \geqslant q > \alpha$ and $g^{-1}((\alpha,1])$ contains the open neighbourhood $\overline{U_q}^c$ of x as required.

The first part is proved. For the second put $f := 1 - g$ which is continuous and maps into $[0,1]$. By design $f(x) = 1$ for all $x \in A$ and $\operatorname{supp} f \subset B$. □

Remark 4.8. A topological space X is said to be **completely regular** if for every $x \in X$ and closed set A not containing x there is a continuous function $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $f(a) = 0$ for all $a \in A$. Theorem 4.7 shows that every topological group is completely regular since $\{x\}$ is compact and contained in the open set A^c, so that the Theorem applies to give a continuous function f with $f(x) = 1$ and $f(y) = 0$ for all $y \in (A^c)^c = A$.

⚠ Completely regular spaces need not be Hausdorff (they may not have *any* non-trivial closed sets). A completely regular Hausdorff space is also called a Tychonoff space or a $T_{3^1\!/\!_2}$ space.

Remark 4.9. Theorem 4.7 is very closely related to Urysohn's Lemma which says that if a topological space X is normal[10], meaning any disjoint closed sets A and C are contained in disjoint open sets, then there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ for all $x \in A$ and $f(x) = 0$ for all $x \in C$. Exercise I.7 is about showing that a wide class of topological groups are normal, while in Exercise I.8 an example of a topological group that is not normal is developed.

⚠ As with completely regular spaces, normal spaces need not be Hausdorff. A normal Hausdorff space is also called a T_4 space.

Remark 4.10. ⚠ Theorem 4.7 does *not* assume that G is Hausdorff, so there may not be any non-trivial open sets.

Example 4.11. If G is a topological group that is not indiscrete then G supports a non-constant continuous function into $\mathbb{C}$: Since G is indiscrete there is a non-empty open set U with non-empty complement. Let $x \in U$. Then $\{x\}$ is compact and contained in U and so by Theorem 4.7 there is a continuous function $f : G \to \mathbb{C}$ such that $\operatorname{supp} f \subset U$ and $f(x) = 1$. Since the complement of U is non-empty there is some $y \in G$ such that $f(y) = 0 \neq 1 = f(x)$ and we conclude that f is non-constant.

For us Theorem 4.7 will be crucial in providing a supply of compactly supported functions in locally compact topological groups.

Corollary 4.12. *Suppose that G is a locally compact topological group and $K \subset G$ is compact. Then there is a continuous compactly supported $f : G \to [0,1]$ such that $f(x) = 1$ for all $x \in K$.*

Proof. Since G is locally compact it contains a compact neighbourhood of the identity L; let $H \subset L$ be an open neighbourhood of the identity, and $C \subset H$ a closed neighbourhood of the identity (possible by Corollary 2.30). KH is open by Lemma 2.12 and apply Theorem 4.7 to get a continuous $f : G \to [0,1]$ with $f(x) = 1$ for all $x \in K$ and $\operatorname{supp} f \subset KH \subset KL$ which is compact by Lemma 2.23. ☐

Furthermore, we can product continuous partitions of unity:

Corollary 4.13. *Suppose that G is a (locally compact) topological group, $F : G \to [0,1]$ is continuous, K is a compact set containing the support of F, and $\mathcal{U}$ is an open cover of K. Then there is some $n \in \mathbb{N}$ and continuous compactly supported functions $f_1, \dots, f_n$:*

[10]We shall avoid this terminology because of the potential for confusion with normal subgroups.

$G \to [0,1]$ *such that* $F = f_1 + \cdots + f_n$*; and for each* $1 \leqslant i \leqslant n$ *there is* $U_i \in \mathcal{U}$ *such that* $\operatorname{supp} f_i \subset U_i$.

Proof. Since $\mathcal{U}$ is an open cover of K, for each $x \in K$ there is an open neighbourhood of x, call it $U_x \in \mathcal{U}$, and by Corollary 2.30 there is a closed neighbourhood $V_x \subset U_x$ of x. Since each V_x is a neighbourhood and $\{V_x : x \in K\}$ is a cover of K, compactness tells us that there are elements $x_1, \dots, x_n$ such that $K \subset V_{x_1} \cup \cdots \cup V_{x_n}$. By Lemma 2.37 $\overline{K}$ is compact and so for each i the set $V_{x_i} \cap \overline{K}$ is a closed subset of a compact set and so compact. Apply Theorem 4.7 to $V_{x_i} \cap \overline{K} \subset U_{x_i}$ to get a continuous function $g_i : G \to [0,1]$ such that $g_i(x) = 1$ for all $x \in V_{x_i} \cap \overline{K}$ and $\operatorname{supp} g_i \subset U_{x_i}$.

Since the sets $V_{x_1}, \dots, V_{x_n}$ are closed, $\overline{K} \subset V_{x_1} \cup \cdots \cup V_{x_n}$, and so since the g_is are non-negative we have

$$\overline{\operatorname{supp} F} \subset \overline{K} \subset (V_{x_1} \cap \overline{K}) \cup \cdots \cup (V_{x_n} \cap \overline{K}) \subset \operatorname{supp}(g_1 + \cdots + g_n).$$

Thus (see Remark 4.6) there is $h \in C_c(G)$ such that $F = h(g_1 + \cdots + g_n)$ and since F maps into $[0,1]$ and $g_1(x) + \cdots + g_n(x) \geqslant 1$ on the support of F, we conclude that h maps into $[0,1]$; for $1 \leqslant i \leqslant n$ put $f_i = g_i h$.

It remains to check the properties of the f_is. First, f_i is a continuous function $G \to [0,1]$ by design of h and g_i. Secondly, $F = f_1 + \cdots + f_n$ by design. Finally, $\operatorname{supp} f_i \subset \operatorname{supp} g_i \subset U_{x_i} \in \mathcal{U}$. Moreover, since the f_is are non-negative $\operatorname{supp} f_i \subset K$ so f_i has compact support. The result is proved. □

Remark 4.14. Although we have not required G to be locally compact in the above, if F is not identically 0 then G is necessarily locally compact whence the parenthetical inclusion. (*c.f.* Remark 4.4.)

Integrals of continuous functions

Given a topological space X if $f, g \in C_c(X)$ are both real-valued then we write $f \geqslant g$ if $f(x) \geqslant g(x)$ for all $x \in X$ and $C_c^+(X)$ for the set of $f \in C_c(G)$ such that $f \geqslant 0$, where 0 is the constant 0 function – in words f is **non-negative**.

Remark 4.15. The functions $\mathbb{C} \to \mathbb{R}; z \mapsto \operatorname{Re} z$, $\mathbb{C} \to \mathbb{R}; z \mapsto \operatorname{Im} z$, $\mathbb{R} \to \mathbb{R}_{\geqslant 0}; x \mapsto \max\{x, 0\}$ and $\mathbb{R} \to \mathbb{R}_{\geqslant 0}; x \mapsto \max\{-x, 0\}$ are continuous and so any $f \in C_c(X)$ can be written as $f = f_1 - f_2 + if_3 - if_4$ for $f_1, f_2, f_3, f_4 \in C_c^+(X)$, and this decomposition is unique. We shall frequently have call to understand elements of $C_c(X)$ through this linear combination of elements of $C_c^+(X)$.

We shall be interested in linear functionals $\int : C_c(X) \to \mathbb{C}$ that are **non-negative** which means that if $f \in C_c^+(X)$ then $\int f \geqslant 0$.

Remark 4.16. If $f, g \in C_c(X)$ are real-valued with $f \geqslant g$ and $\int$ is a non-negative linear functional $C_c(X) \to \mathbb{C}$ then $\int f \geqslant \int g$; and if $f \in C_c(G)$ then $\left|\int f\right| \leqslant \int |f|$.

Remark 4.17. The decomposition in Remark 4.15 can be used to show that if $\int$ is a non-negative linear functional then $\overline{\int f} = \int \overline{f}$ for all $f \in C_c(X)$.

Remark 4.18. We think of non-negative linear functionals as integrals and in fact the Riesz-Markov-Kakutani Representation Theorem actually tells us that every non-negative linear map $C_c(X) \to \mathbb{C}$ arises as an integral against a suitably well-behaved measure on X.

Given $F : X^2 \to \mathbb{C}$ and $x \in X$ we write $\int_y F(x, y)$ for the functional $\int$ applied to the function $X \to \mathbb{C}; y \mapsto F(x, y)$ (assuming this function is continuous and compactly supported), and similarly for $y \in X$ and $\int_x F(x, y)$. It will be crucial for us that the order of integration can be interchanged and this is what the next result concerns:

Theorem 4.19 (Fubini's Theorem for continuous functions)**.** *Suppose that G is a locally compact topological group, $\int$ and $\int'$ are non-negative linear functionals $C_c(G) \to \mathbb{C}$, and $F \in C_c(G^2)$. Then the map $x \mapsto \int'_y F(x, y)$ is continuous and compactly supported, so that $\int_x \int'_y F(x, y)$ exists. Similarly $y \mapsto \int_x F(x, y)$ is continuous and compactly supported, so that $\int'_y \int_x F(x, y)$ exists and moreover*

$$\int_x \int'_y F(x, y) = \int'_y \int_x F(x, y).$$

Proof. In view of the decomposition in Remark 4.15 and linearity of $\int$ and $\int'$ it is enough to establish the result for F non-negative.

Since $F \in C_c^+(G^2)$ has support contained in a compact set K, and since the coordinate projection maps $G^2 \to G$ are continuous (and the union of two compact sets is compact) there is a compact set L such that $K \subset L \times L$. It follows that the maps $x \mapsto F(x, y)$ for $y \in G$ and $y \mapsto F(x, y)$ for $x \in G$ are continuous and have support in the compact set L.

We also need an auxiliary 'dominating function' which is a compactly supported continuous function on whose support all of the 'action' happens. For those familiar with the theory of integration, the Dominated Convergence Theorem may come to mind. Concretely, by Corollary 4.12 there is a continuous function $f : G \to [0, 1]$ with $f(x) = 1$ for all $x \in L$ supported in a compact set M.

For $\epsilon > 0$ (by Remark 4.5) let $\mathcal{U}$ be an open cover of $G \times G$ such that $|F(x, y) - F(x', y')| < \epsilon$ for all $(x, y), (x', y') \in U \in \mathcal{U}$. $M \times M$ is compact and so by Lemma 2.28 there is a symmetric open neighbourhood of the identity U in G such that $\mathcal{U}' := \{xU \times yU : x, y \in M\}$ is a refinement of $\mathcal{U}$ (as a cover of $M \times M$ *not* of $G \times G$). First, the support of $\int'_y F(x, y)$ is contained in the (compact) set L and if $x' \in xU$ then by design and non-negativity of $\int'$ we have

$$\int'_y F(x', y) = \int'_y F(x', y) f(y) \leqslant \int'_y (F(x, y) + \epsilon) f(y) = \int'_y F(x, y) + \epsilon \int' f.$$

Since U is symmetric we have $x \in x'U$ and similarly $\int'_y F(x,y) \leqslant \int'_y F(x',y) + \epsilon \int' f$ and hence $|\int'_y F(x',y) - \int'_y F(x,y)| \leqslant \epsilon \int' f$. Since ϵ is arbitrary (and $\int' f$ does not depend on ϵ) it follows that $x \mapsto \int'_y F(x,y)$ is continuous (and compactly supported) and similarly for $y \mapsto \int_x F(x,y)$.

By Corollary 4.13 applied to f supported on the compact set M with the open cover $\{xU : x \in M\}$, there are continuous compactly supported $f_1, \dots, f_n : G \to [0,1]$ such that $f_1 + \dots + f_n = f$ and $\operatorname{supp} f_i \subset x_i U$ for some $x_i \in M$. Now, $F(x,y) = F(x,y)f(x)f(y)$ and $f = f_1 + \dots + f_n$, so

$$F(x,y) = \sum_{i=1}^{n}\sum_{j=1}^{n} F(x,y)f_i(x)f_j(y) \text{ for all } x, y \in G.$$

By design of $\mathcal{U}'$ and $\mathcal{U}$, for $1 \leqslant i, j \leqslant n$ there is $\lambda_{i,j} \geqslant 0$ such that $|F(x,y) - \lambda_{i,j}| < \epsilon$ for all $(x,y) \in \operatorname{supp} f_i \times \operatorname{supp} f_j$. We conclude that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} f_i(x) f_j(y) - \epsilon f(x)f(y) \leqslant F(x,y) \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} f_i(x) f_j(y) + \epsilon f(x)f(y).$$

Since $\int$ and $\int'$ are non-negative linear functionals, we conclude that

$$\left| \int_x \int'_y F(x,y) - \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} \int f_i \int' f_j \right| \leqslant \epsilon \int f \int' f$$

and

$$\left| \int'_y \int_x F(x,y) - \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_{i,j} \int f_i \int' f_j \right| \leqslant \epsilon \int f \int' f.$$

The result is proved by the triangle inequality since ϵ is arbitrary (and $\int f$ and $\int' f$ do not depend on ϵ). □

5 The Haar integral

We now turn to one of the most beautiful aspects of the theory of topological groups. This describes the way the topology and the algebra naturally conspire to produce an integral. Given a topological group G and a function $f \in C(G)$ we write

$$\lambda_x(f)(z) := f(x^{-1}z) \text{ for all } x, z \in G.$$

Remark 5.1. $\lambda_x(f) \in C(G)$ for all $f \in C(G)$ and $x \in G$ (since left multiplication is continuous and the composition of continuous functions is continuous), and λ is a left action meaning $\lambda_{xy}(f) = \lambda_x(\lambda_y(f))$ for all $x, y \in G$ and $\lambda_{1_G}(f) = f$, and the maps λ_x are linear on the vector space $C(G)$.

⚠ Without inversion this is naturally a right action.

Remark 5.2. For a topological group G, λ restricts to an action on the space $C_c(G)$ and this action is isometric with respect to $\|\cdot\|_\infty$ *i.e.* $\|\lambda_x(f)\|_\infty = \|f\|_\infty$ for all $x \in G$.

Lemma 5.3. *Suppose that G is a topological group and $f \in C_c(G)$. Then $G \to C_c(G); x \mapsto \lambda_x(f)$ is continuous.*

Proof. Let $U \subset C_c(G)$ be open and $x \in G$ have $\lambda_x(f) \in U$. Since U is open there is $\epsilon > 0$ such that $\lambda_{x'}(f) \in U$ whenever $\|\lambda_{x'}(f) - \lambda_x(f)\|_\infty < \epsilon$.

Let K be a compact set containing the support of f. As in Remark 4.5 let $\mathcal{U}$ be an open cover of G such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in U \in \mathcal{U}$. Then $\{U^{-1} : U \in \mathcal{U}\}$ is an open cover of K^{-1}. Since inversion is continuous and K is compact, K^{-1} is compact and so by Lemma 2.28 there is a symmetric open neighbourhood of the identity V such that $\{yV : y \in K^{-1}\}$ refines $\{U^{-1} : U \in \mathcal{U}\}$ (as a cover of K^{-1}), and hence $\{V^{-1}y : y \in K\}$ is a refinement of $\mathcal{U}$ (as a cover of K).

Suppose that $v \in V$ and $y \in G$ is such that $\lambda_v(f)(y) - f(y) \neq 0$. Then either $f(y) \neq 0$ so $y \in K$, but then $V^{-1}y$ is a subset of an element of $\mathcal{U}$ and so $|\lambda_v(f)(y) - f(y)| < \epsilon$; or $\lambda_v(f)(y) \neq 0$ so $v^{-1}y \in K$, but then $V(v^{-1}y) = V^{-1}(v^{-1}y)$ is a subset of an element of $\mathcal{U}$ and so again $|\lambda_v(f)(y) - f(y)| < \epsilon$. Since $\lambda_v(f) - f$ is continuous and compactly supported it attains its bounds so $\|\lambda_v(f) - f\|_\infty < \epsilon$. Finally, since λ is an action, the map λ_x is linear, and this action is isometric (Remark 5.2) we have

$$\|\lambda_{xv}(f) - \lambda_x(f)\|_\infty = \|\lambda_x(\lambda_v(f) - f)\|_\infty = \|\lambda_v(f) - f\|_\infty < \epsilon.$$

By Lemma 2.12 xV is an open neighbourhood of x and by design it is contained in the preimage of U. Since x was an arbitrary element of the preimage of U it follows this preimage is open as required. □

Given a topological group G we say that $\int : C_c(G) \to \mathbb{C}$ is a **(left) Haar integral** on G if $\int$ is a non-trivial (meaning not identically zero) non-negative linear map with

$$\int \lambda_x(f) = \int f \text{ for all } x \in G \text{ and } f \in C_c(G).$$

We sometimes call this last property **(left) translation invariance**.

Remark 5.4. Our definition of Haar integral requires $C_c(G)$ to be non-trivial and hence (*c.f.* Remark 4.4) for G to support a Haar integral it must be locally compact. It will turn out in Theorem 5.10 that this is enough to guarantee that there is a Haar integral.

Remark 5.5. There is an analogous notion of right Haar integral which we shall not pursue here.

Example 5.6. If G is a discrete group then it supports a left Haar integral:

$$\int : C_c^+(G) \to \mathbb{C}; f \mapsto \sum_{x \in G} f(x).$$

Remark 5.7. ⚠ Note that this definition does *not* work for non-discrete groups. See Exercise III.1.

The integral of a non-negative continuous function that is not identically 0 is positive, and this already follows from the axioms of a Haar integral. To establish this we begin with a lemma on the comparability of functions:

Lemma 5.8. *Suppose that G is a topological group, $f, g \in C_c^+(G)$ and f is not identically zero. Then there is $n \in \mathbb{N}$, $c_1, \ldots, c_n \geqslant 0$ and $y_1, \ldots, y_n \in G$ such that*

$$g(x) \leqslant \sum_{i=1}^{n} c_i \lambda_{y_i}(f)(x) \text{ for all } x \in G.$$

Proof. Since $f \not\equiv 0$ there is some $x_0 \in G$ such that $f(x_0) > 0$ and hence (by Lemma 2.12) an open neighbourhood of the identity U such that $f(x_0 y) > f(x_0)/2$ for all $y \in U$. Let K be compact containing the support of g. Then $\{xU : x \in K\}$ is an open cover of K and so there are elements $x_1, \ldots, x_n$ such that $x_1 U, \ldots, x_n U$ covers K. But then

$$g(x) \leqslant 2f(x_0)^{-1}\|g\|_\infty \sum_{i=1}^{n} f(x_0 x_i^{-1} x) = 2f(x_0)^{-1}\|g\|_\infty \sum_{i=1}^{n} \lambda_{x_i x_0^{-1}}(f)(x) \text{ for all } x \in G,$$

and the result is proved. □

Corollary 5.9. *Suppose that G is a topological group, $\int$ is a left Haar integral on G, and $f \in C_c^+(G)$ has $\int f = 0$. Then $f \equiv 0$.*

Proof. Suppose that $g \in C_c^+(G)$ so by Lemma 5.8 we have $g \leqslant \sum_{i=1}^{n} c_i \lambda_{y_i}(f)$ for $c_1, \ldots, c_n \geqslant 0$ and $y_1, \ldots, y_n \in G$. Then by linearity, non-negativity, and translation invariance of the Haar integral

$$\int g \leqslant \sum_{i=1}^{n} c_i \int \lambda_{y_i}(f) = \sum_{i=1}^{n} c_i \int f = 0.$$

Since $g \geqslant 0$, non-negativity of the Haar integral implies $\int g \geqslant 0$, and hence $\int g = 0$.

Now, in view of Remark 4.15 we have that $\int h = 0$ for all $h \in C_c(G)$ *i.e.* $\int$ is identically 0 contradicting the non-triviality of the Haar integral. The lemma follows. □

Existence of a Haar Integral

Our first main aim is to establish the following.

Theorem 5.10 (Existence of a Haar integral). *Suppose that G is a locally compact topological group. Then there is a left Haar integral on G.*

We begin by defining a sort of approximation: for $f, \phi \in C_c^+(G)$ with ϕ not identically 0 put

$$(f;\phi) := \inf\left\{\sum_{j=1}^{n} c_j : n \in \mathbb{N}; c_1, \dots, c_n \geqslant 0; y_1, \dots, y_n \in G; \text{ and } f \leqslant \sum_{j=1}^{n} c_j \lambda_{y_j^{-1}}(\phi)\right\}. \quad (5.1)$$

We think of this as a sort of 'covering number' and begin with some basic properties:

Lemma 5.11. *Suppose that $f, g, \phi, \psi \in C_c^+(G)$ with ϕ and ψ are not identically 0. Then*

(i) $(f;\phi)$ *is well-defined;*

(ii) $(\phi;\phi) \leqslant 1$*;*

(iii) $(f;\phi) \leqslant (g;\phi)$ *whenever* $f \leqslant g$*;*

(iv) $(f+g;\phi) \leqslant (f;\phi) + (g;\phi)$*;*

(v) $(\mu f;\phi) = \mu(f;\phi)$ *for* $\mu \geqslant 0$*;*

(vi) $(\lambda_x(f);\phi) = (f;\phi)$ *for all* $x \in G$*;*

(vii) $(f;\psi) \leqslant (f;\phi)(\phi;\psi)$.

Proof. Lemma 5.8 shows that the set on the right of (5.1) is non-empty; it has 0 as a lower bound. (i) follows immediately. For (ii)[11] note that $\phi \leqslant 1.\lambda_{1_G^{-1}}(\phi)$ so that $(\phi;\phi) \leqslant 1$. (iii), (iv), (v), and (vi) are all immediate. Finally, for (vii) suppose $c_1, \dots, c_n \geqslant 0$ are such that $f \leqslant \sum_{j=1}^n c_j \lambda_{y_j^{-1}}(\phi)$, so that by (iii), (iv), (v), and (vi) we have $(f;\psi) \leqslant \sum_{j=1}^n c_j(\phi;\psi)$. The result follows on taking infima. □

To make use of $(\cdot\,;\cdot)$ we need to fix a non-zero reference function $f_0 \in C_c^+(G)$ and for $\phi \in C_c^+(G)$ not identically zero we put

$$I_\phi(f) := \frac{(f;\phi)}{(f_0;\phi)} \leqslant (f;f_0), \quad (5.2)$$

where the inequality follows from Lemma 5.11 (vii).

Many of the properties of Lemma 5.11 translate into properties of I_ϕ. In particular, we have $I_\phi(f_1+f_2) \leqslant I_\phi(f_1) + I_\phi(f_2)$; for suitable ϕ we also have the following converse.

Lemma 5.12. *Suppose that G is a locally compact topological group, $f_1, f_2 \in C_c^+(G)$ and $\epsilon > 0$. Then there is a symmetric open neighbourhood of the identity V such that if $\phi \in C_c^+(G)$ is not identically 0 and has support in V then $I_\phi(f_1) + I_\phi(f_2) \leqslant I_\phi(f_1+f_2) + \epsilon$.*

[11] As it happens it is easy to prove equality here but we do not need it.

Proof. Let K be a compact closed set containing the support of both f_1 and f_2 (possible since the union of two compact sets is compact and the closure of a compact set is compact by Lemma 2.37) and apply Corollary 4.12 to get $F : G \to [0,1]$ continuous, compactly supported, and with $F(x) = 1$ for all $x \in K$.

For $j \in \{1,2\}$ let g_j be continuous such that $(f_1 + f_2 + \epsilon F)g_j = f_j$ (possible in view of Remark 4.6 and use that $\overline{\operatorname{supp} f_i} \subset K \subset \operatorname{supp} F$). By Remark 4.5 (and the fact that the intersection of two open covers is an open cover) there is an open cover $\mathcal{U}$ of G such that if $x, y \in U \in \mathcal{U}$ then $|g_j(x) - g_j(y)| < \epsilon$ for $j \in \{1,2\}$. K is compact; apply Lemma 2.28 to $\mathcal{U}$ to get a symmetric open neighbourhood of the identity V such that $\{yV : y \in K\}$ refines $\mathcal{U}$ (as a cover of K).

Now suppose that $\phi \in C_c^+(G)$ is not identically 0 and has support in V, and that $c_1, \dots, c_n \geqslant 0$ and $y_1, \dots, y_n \in G$ are such that

$$f_1(x) + f_2(x) + \epsilon F(x) \leqslant \sum_{i=1}^{n} c_i \phi(y_i x) \text{ for all } x \in G.$$

If $\phi(y_i x) g_j(x) \neq 0$ then $x \in K$ and $y_i^{-1} \in xV$ (using $V = V^{-1}$), by xV is a subset of a set in $\mathcal{U}$ so $g_j(x) \leqslant g_j(y_i^{-1}) + \epsilon$ and hence

$$f_j(x) \leqslant \sum_{i=1}^{n} c_i \phi(y_i x) g_j(x) \leqslant \sum_{i=1}^{n} c_i (g_j(y_i^{-1}) + \epsilon) \phi(y_i x) \text{ for all } x \in G, j \in \{1,2\}.$$

By Lemma 5.11 (ii),(iii), (iv),(v) & (vi) we have

$$(f_j; \phi) \leqslant \sum_{i=1}^{n} c_i (g_j(y_i^{-1}) + \epsilon) \text{ for all } j \in \{1,2\},$$

but $g_1(y^{-1}) + g_2(y^{-1}) \leqslant 1$ for all $y \in G$, so

$$(f_1; \phi) + (f_2; \phi) \leqslant \sum_{i=1}^{n} c_i (1 + 2\epsilon).$$

Taking infima and then applying Lemma 5.11 (iv) and (v) and the inequality in (5.2) we get

$$\begin{aligned} I_\phi(f_1) + I_\phi(f_2) &\leqslant (1 + 2\epsilon) I_\phi(f_1 + f_2 + \epsilon F) \\ &\leqslant (1 + 2\epsilon)(I_\phi(f_1 + f_2) + \epsilon I_\phi(F)) \\ &\leqslant I_\phi(f_1 + f_2) + (2(f_1 + f_2; f_0) + (F; f_0) + 2\epsilon(F; f_0))\epsilon. \end{aligned}$$

The result follows since $\epsilon > 0$ was arbitrary and F, f_1, f_2 and f_0 do not depend on ϵ. $\square$

With these lemmas we can turn to the main argument.

Proof of Theorem 5.10. By Corollary 4.12 (applied with $K = \{1_G\}$) there is $f_0 \in C_c^+(G)$ with $f_0 \not\equiv 0$. Write F for the set of functions $I : C_c^+(G) \to \mathbb{R}_{\geqslant 0}$ with $I(f) \leqslant (f; f_0)$ for all $f \in C_c^+(G)$ endowed with the product topology *i.e.* the weakest topology such that the maps $F \to [0, (f; f_0)]; I \mapsto I(f)$ are continuous for all $f \in C_c^+(G)$. Since the closed interval $[0, (f; f_0)]$ is compact, F is a product of compact spaces and so compact. Let X be the set of $I \in F$ such that

$$I(f_0) = 1 \tag{5.3}$$

$$I(\mu f) = \mu I(f) \text{ for all } \mu \geqslant 0, f \in C_c^+(G), \tag{5.4}$$

and

$$I(\lambda_x(f)) = I(f) \text{ for all } x \in G, f \in C_c^+(G). \tag{5.5}$$

The set X is closed as an intersection of the preimage of closed sets. Moreover, by Lemma 5.11 $I_\phi \in X$ for any $\phi \in C_c^+(G)$ that is not identically zero: the fact that $I(f) \in [0, (f; f_0)]$ follows from the inequality in (5.2); (5.3) by design; (5.4) by (v); and (5.5) by (vi).

This almost gives us a Haar integral (on non-negative functions) except that in general the elements of X are not additive, meaning we do not in general have $I(f+f') = I(f)+I(f')$. To get this we introduce some further sets: for $\epsilon > 0$ and $f, f' \in C_c^+(G)$ define

$$B(f, f'; \epsilon) := \{I \in X : |I(f + f') - I(f) - I(f')| \leqslant \epsilon\}.$$

As with X, the sets $B(f, f'; \epsilon)$ are closed. We shall show that any finite intersection of such sets is non-empty: For any $f_1, f_1', f_2, f_2', \ldots, f_n, f_n' \in C_c^+(G)$ and $\epsilon_1, \ldots, \epsilon_n > 0$, by Lemma 5.12 there are symmetric open neighbourhoods of the identity $V_1, \ldots, V_n$ such that if $\phi \in C_c^+(G)$ is not identically 0 and is supported in V_i then

$$|I_\phi(f_i + f_i') - I_\phi(f_i) - I_\phi(f_i')| < \epsilon_i. \tag{5.6}$$

Since G is locally compact by Lemma 2.41 there is a symmetric open neighbourhood of the identity H with compact closure; set $V := H \cap \bigcap_{i=1}^n V_i$ which is also a symmetric open neighbourhood of the identity and by Theorem 4.7 there is $\phi \in C^+(G)$ that is not identically 0 with support contained in V, and hence in the compact set $\overline{H}$ which is to say it has compact support. I_ϕ enjoys (5.6) for all $1 \leqslant i \leqslant n$, and we noted before that $I_\phi \in X$, hence $I_\phi \in \bigcap_{i=1}^n B(f_i, f_i', \epsilon_i)$. We conclude that $\{B(f, f'; \epsilon) : f, f' \in C_c^+(G), \epsilon > 0\}$ is a set of closed subsets of F with the finite intersection property, but F is compact and so there is some I in all of these sets. Such an I is additive since $|I(f + f') - I(f) - I(f')| < \epsilon$ for all f, f' and $\epsilon > 0$. It remains to define $\int : C_c(G) \to \mathbb{C}$ by putting

$$\int f := I(f_1) - I(f_2) + iI(f_3) - iI(f_4) \text{ where } f = f_1 - f_2 + if_3 - if_4 \text{ for } f_1, f_2, f_3, f_4 \in C_c^+(G).$$

This decomposition of functions in $C_c(G)$ is unique (noted in Remark 4.15) and so this is well-defined. Moreover, $\int$ is linear since I is additive and enjoys (5.4); it is non-negative since

I is non-negative (and $I(0) = 0$); it is translation invariant by (5.5); and it is non-trivial by (5.3). The result is proved. $\square$

Uniqueness of the Haar integral

Our second main aim is to establish the following result.

Theorem 5.13 (Uniqueness of the Haar Integral)**.** *Suppose that G is a locally compact topological group and $\int$ and $\int'$ are left Haar integrals on G. Then there is some $\lambda > 0$ such that $\int = \lambda \int'$.*

For this we introduce a little more notation: Given a topological group G and $f \in C_c(G)$ we write $\widetilde{f}(x) = \overline{f(x^{-1})}$.

Remark 5.14. $\widetilde{\cdot}$ is a conjugate-linear multiplicative involution[3] on $C_c(G)$, since complex conjugation and $x \mapsto x^{-1}$ are both continuous (and continuous images of compact sets are compact).

Remark 5.15. The reason for making $\widetilde{\cdot}$ conjugate-linear is to make it compatible with a later inner product. (See Remark 7.8.)

Proof of Theorem 5.13. Suppose that $f_0, f_1 \in C_c^+(G)$ are not identically 0 and write K for a compact set containing the support of f_0 and f_1 (which exists since finite unions of compact sets are compact). By Lemma 2.41 there is a symmetric open neighbourhood of the identity, H, with compact closure.

First, by Corollary 4.12 there is a continuous compactly supported function $F : G \to [0, 1]$ with $F(x) = 1$ for all $x \in K\overline{H}$ (this set is compact by Lemma 2.23, and hence the corollary applies).

Now, suppose $\epsilon > 0$ and use Remark 4.5 (and the fact that intersections of open covers are open covers) to get an open cover $\mathcal{U}$ of G such that if $x, y \in U \in \mathcal{U}$ then $|f_i(x) - f_i(y)| < \epsilon$ for $i \in \{0, 1\}$. By Lemma 2.28 applied to $\mathcal{U}$ and the compact set $K\overline{H}$ there is a symmetric open neighbourhood of the identity V such that $\{xV : x \in K\overline{H}\}$ is a refinement of $\mathcal{U}$ (as a cover of $K\overline{H}$), and by Theorem 4.7 there is a continuous function $h : G \to [0, 1]$ that is not identically zero and is supported in $V \cap H$, and in particular supported in $\overline{H}$ so it has compact support.

For $x \in G$, translation invariance of $\int'$ (and Remark 4.17) tells us that

$$\int_y' h(y^{-1}x) = \int_y' \overline{\widetilde{h}(x^{-1}y)} = \overline{\int_y' \widetilde{h}(x^{-1}y)} = \overline{\int_y' \widetilde{h}(y)} = \int' \overline{\widetilde{h}}.$$

For $i \in \{0, 1\}$, the map $x \mapsto \int_y' f_i(x)h(y^{-1}x) = f_i(x) \int' \overline{\widetilde{h}}$ is continuous and is supported in K and so is compactly supported and $\int_x \int_y' f_i(x)h(y^{-1}x)$ exists and equals $\int f_i \int' \overline{\widetilde{h}}$ (by linearity of $\int$). On the other hand the map $(x,y) \mapsto f_i(x)h(y^{-1}x)$ is continuous and supported on $K \times \overline{H}$ and so is compactly supported and hence by Fubini's Theorem (Theorem 4.19), $y \mapsto \int_x f_i(x)h(y^{-1}x)$ exists, and (using translation invariance of $\int$) we have

$$\int f_i \int' \overline{\widetilde{h}} = \int_x \int'_y f_i(x)h(y^{-1}x) = \int'_y \int_x f_i(x)h(y^{-1}x) = \int'_y \int_x f_i(yx)h(x).$$

Since $\{yV : y \in K\}$ refines $\mathcal{U}$ (as a cover of $K\overline{H}$) we have $|f_i(yx) - f_i(y)| < \epsilon$ for $x \in V$ and $y \in K\overline{H}$; and for $x \in H$ and $f_i(yx) \neq 0$ or $f_i(y) \neq 0$ we have $y \in KH$ whence $F(y) = 1$. It follows that

$$f_i(y)h(x) - \epsilon F(y)h(x) \leqslant f_i(yx)h(x) \leqslant f_i(y)h(x) + \epsilon F(y)h(x) \text{ for all } x,y \in G,$$

and so by non-negativity and linearity of $\int$ and $\int'$ we have

$$\int'_y \int_x f_i(y)h(x) - \int'_y \int_x \epsilon F(y)h(x) \leqslant \int'_y \int_x f_i(yx)h(x) \leqslant \int'_y \int_x f_i(y)h(x) + \int'_y \int_x \epsilon F(y)h(x).$$

It follows (using linearity of $\int$) that $|\int' f_i \int h - \int f_i \int' \overline{\widetilde{h}}| \leqslant \epsilon \int' F \int h$, and hence by the triangle inequality (and division, which is valid since $\int f_0, \int f_1 \neq 0$ by Corollary 5.9 as f_0 and f_1 are not identically zero) that

$$\left|\frac{\int' f_0}{\int f_0} - \frac{\int' f_1}{\int f_1}\right| \leqslant \left|\frac{\int' f_0}{\int f_0} - \frac{\int' \overline{\widetilde{h}}}{\int h}\right| + \left|\frac{\int' \overline{\widetilde{h}}}{\int h} - \frac{\int' f_1}{\int f_1}\right| \leqslant \epsilon \int' F \left(\frac{1}{\int f_0} + \frac{1}{\int f_1}\right).$$

Since ϵ was arbitrary (and in particular f_0, f_1, and F do not depend on it) it follows that $\int' f / \int f$ is a constant λ for all $f \in C_c^+(G)$ not identically zero. This constant must be non-zero since $\int'$ is non-trivial, and it must be positive since $\int'$ and $\int$ are non-negative. The result follows from the usual decomposition (Remark 4.15), and the fact that $\int 0, \int' 0 = 0$. $\square$

6 The dual group

Suppose that G is a topological group. We write $\widehat{G}$ for the set of continuous homomorphisms $G \rightarrow S^1$ (where S^1 is as in Example 2.22), and call the elements of $\widehat{G}$ **characters**.

Remark 6.1. The choice of S^1 here may seem a bit mysterious. In fact in this generality it makes more sense to consider continuous group homomorphisms into groups of unitary matrices. We have made the choice above to ensure that $\widehat{G}$ has a group structure (see Proposition 6.3).

Remark 6.2. ⚠ While characters are (by definition) elements of $C(G)$, they are *not* in $C_c(G)$ unless G is compact.

We endow the set $\widehat{G}$ with the **compact-open topology**, that is the topology generated by the sets $\gamma U(K,\epsilon)$ where

$$U(K,\epsilon) := \{\gamma \in \widehat{G} : |\gamma(x) - 1| < \epsilon \text{ for all } x \in K\}$$

and $\epsilon > 0$ and K is a compact subset of G.

Proposition 6.3. *Suppose that G is a topological group. Then $\widehat{G}$ is a Hausdorff Abelian topological group with multiplication and inversion defined by*

$$(\gamma, \gamma') \mapsto (x \mapsto \gamma(x)\gamma'(x)) \text{ and } \gamma \mapsto (x \mapsto \overline{\gamma(x)}),$$

and identity the character taking the constant value 1*. Moreover,* $(U(K,\delta))$ *as K ranges compact subsets of G and $\delta > 0$ is a neighbourhood base of the identity.*

Proof. The fact that $\widehat{G}$ is an Abelian group is an easy check since S^1 is an Abelian group under multiplication and $z^{-1} = \overline{z}$ when $z \in S^1$.

Since $|\gamma(x) - 1| = |\overline{\gamma(x)} - 1|$ the inversion is certainly continuous. Now suppose that $\gamma\lambda \in \mu U(K,\epsilon)$ for some $\mu \in \widehat{G}$. Since $\gamma\lambda\overline{\mu}$ is continuous and K is compact $|\gamma\lambda\overline{\mu} - 1|$ achieves its bounds on K and hence there is some $\delta > 0$ such that $|(\gamma\lambda\overline{\mu})(x) - 1| < \epsilon - \delta$ for all $x \in K$. But then if $\gamma' \in \gamma U(K, \delta/2)$ and $\lambda' \in \lambda U(K, \delta/2)$ we have

$$\begin{aligned} |(\gamma'\lambda'\overline{\mu})(x) - 1| &\leqslant |(\gamma'\lambda'\overline{\mu})(x) - (\gamma\lambda'\overline{\mu})(x)| + |(\gamma\lambda'\overline{\mu})(x) - (\gamma\lambda\overline{\mu})(x)| + |(\gamma\lambda\overline{\mu})(x) - 1| \\ &< \delta/2 + \delta/2 + \epsilon - \delta = \epsilon. \end{aligned}$$

It follows that $\gamma'\lambda' \in \mu U(K,\epsilon)$ and so the preimage of $\gamma\lambda$ contains a neighbourhood of (γ, λ) in $\widehat{G} \times \widehat{G}$ *i.e.* multiplication is jointly continuous. Finally, the topology is Hausdorff since if $\gamma \neq \lambda$ then there is some $x \in G$ such that $\gamma(x) \neq \lambda(x)$; put $\epsilon := |\gamma(x) - \lambda(x)|/2$ and note that $\gamma U(\{x\}, \epsilon)$ and $\lambda U(\{x\}, \epsilon)$ are disjoint open sets containing γ and λ respectively. □

We call the group $\widehat{G}$ endowed with the compact-open topology the **dual group** of G, so that the above proposition tells us that if G is a topological group then its dual group is a Hausdorff Abelian topological group.

We call the identity, denoted $1_{\widehat{G}}$, the **trivial character**.

Example 6.4. When G is a group with the indiscrete topology the only continuous functions are constant and so $\widehat{G}$ is the trivial group with one character taking the constant value 1 (and there is only one topology on a set with one element) so that we have completely determined the topological group $\widehat{G}$.

The topology on G and $\widehat{G}$ are quite closely related:

Proposition 6.5. *Suppose that G is a compact topological group. Then $\widehat{G}$ is discrete.*

Proof. Suppose that $\gamma \neq 1_{\widehat{G}}$ so there is $x \in G$ such that $\gamma(x) \neq 1$. Let $y \in G$ be such that $|\gamma(y) - 1|$ is maximal (which exists since G is compact and $x \mapsto |\gamma(x) - 1|$ is continuous) and note that by assumption this is positive. If $|\gamma(y) - 1| < 1$ then we have

$$\begin{aligned} |\gamma(y^2) - 1| = |\gamma(y)^2 - 1| &= |(2 + (\gamma(y) - 1))||\gamma(y) - 1| \\ &\geqslant (2 - |\gamma(y) - 1|)|\gamma(y) - 1| > |\gamma(y) - 1|. \end{aligned}$$

This is a contradiction, whence $\gamma \notin U(G, 1)$ and $\{1_{\widehat{G}}\}$ is open so the topology is discrete. □

Remark 6.6. Conversely if G is discrete then it turns out $\widehat{G}$ is compact. (See Exercise IV.4.)

Example 6.7 (Finite cyclic groups with the discrete topology)**.** Suppose that G is a finite cyclic group endowed with the discrete topology. Since G is cyclic it is generated by some element x; consider the map

$$\phi : G \to \widehat{G}; x^r \mapsto (G \to S^1; x^l \mapsto \exp(2\pi i r l/|G|)).$$

This map is a well-defined homomorphism since $x^r = x^{r'}$ (resp. $x^l = x^{l'}$) only if $|G| \mid r - r'$ (resp.$|G| \mid l - l'$),

$$\exp(2\pi i(r + r')l/|G|) = \exp(2\pi i r l/|G|) \exp(2\pi i r' l/|G|)$$

and

$$\exp(2\pi i r(l + l')/|G|) = \exp(2\pi i r l/|G|) \exp(2\pi i r l'/|G|).$$

ϕ is injective since if $\exp(2\pi i r l/|G|) = 1$ for all l then $|G| \mid r$ so $x^r = 1_G$. Finally, ϕ is surjective since if $\gamma : G \to S^1$ is a homomorphism then $\gamma(x)^{|G|} = 1$ so $\gamma(x) = \exp(2\pi i r/|G|)$ for some $r \in \mathbb{Z}$, and $\gamma = \phi(x^r)$.

We conclude that $\phi : G \to \widehat{G}$ is a bijective group homomorphism and hence ϕ^{-1} is a group homomorphism. Since G is discrete ϕ is continuous; since G is finite it is compact and so $\widehat{G}$ is discrete by Proposition 6.5 and hence ϕ^{-1} is continuous, so ϕ is an isomorphism of topological groups.

Example 6.4 gave topological reasons for the dual group being trivial, but there can also be algebraic reasons:

Example 6.8 (Non-Abelian finite simple groups)**.** Suppose that G is a non-Abelian finite simple[12] topological group.

Suppose that $\gamma : G \to S^1$ is a homomorphism. Since G is non-Abelian there are elements $x, y \in G$ with $xy \neq yx$, but then $xyx^{-1}y^{-1} \neq 1_G$ while

$$\gamma(xyx^{-1}y^{-1}) = \gamma(x)\gamma(y)\gamma(x)^{-1}\gamma(y)^{-1} = 1$$

[12]It may help to recall that a simple group is a group whose only normal subgroups are the trivial group and the whole group *e.g.* A_n, the alternating group on n elements, when $n \geqslant 5$. (The Abelian finite simple groups are the cyclic groups of prime order and their dual groups are described in Example 6.7.)

since S^1 *is* Abelian. We conclude that the kernel of γ is non-trivial, but all kernels are normal subgroups and since G is simple it follows that $\ker \gamma = G$ *i.e.* γ is trivial. In other words $\widehat{G} = \{1_{\widehat{G}}\}$.

The process of passing from a topological group to its dual group has a corresponding process for continuous homomorphisms:

Proposition 6.9. *Suppose that G and H are topological groups and $\phi : G \to H$ is a continuous homomorphism. Then the map $\phi^* : \widehat{H} \to \widehat{G}; \gamma \mapsto \gamma \circ \phi$ is a well-defined continuous homomorphism. If ϕ is surjective then ϕ^* is injective.*

Proof. Certainly ϕ^* is well-defined as $\gamma \circ \phi$ is a composition of continuous homomorphisms and so a continuous homomorphisms. To see that ϕ^* is continuous be apply Lemma 3.6 to the observations that $(\phi^*)^{-1}(U(K, \delta)) = U(\phi(K), \delta)$ and that if $K \subset G$ is compact then $\phi(K)$ is compact since ϕ is continuous. Finally, if ϕ is surjective and $\phi^*(\gamma) = 1_{\widehat{G}}$ then $\gamma(\phi(x)) = 1$ for all $x \in G$ and hence $\gamma(z) = 1$ for all $z \in H$ *i.e.* $\gamma = 1_{\widehat{H}}$. □

Remark 6.10. If ϕ is the identity map on G then ϕ^* is the identity map on $\widehat{G}$, and if $\phi : G \to H$ and $\psi : H \to K$ are continuous homomorphisms then $(\phi \circ \psi)^* = \psi^* \circ \phi^*$, so that the map taking G to $\widehat{G}$ and ϕ to ϕ^* is a contravariant functor from the category of topological groups with continuous homomorphisms to the category of Hausdorff Abelian topological groups with continuous homomorphisms.

In particular, since every Hausdorff Abelian topological group is, in particular, a topological group the category of Hausdorff Abelian topological groups is closed under taking duals.

Example 6.11. Suppose that H is a finite non-Abelian simple group with the discrete topology, then as described in Example 6.8 we have that $\widehat{H}$ is trivial. Since H is non-Abelian it contains a non-identity element; let G be the subgroup of H generated by such an element. The embedding $j : G \to H; x \mapsto x$ is a continuous injective homomorphism, and $j^* : \widehat{H} \to \widehat{G}$ is the map taking the identity in $\widehat{H}$ (which is the only element) to the identity in $\widehat{G}$. Moreover, since G is cyclic, finite, and discrete, Example 6.7 tells us that $\widehat{G}$ is topologically isomorphic to G and, since G is non-trivial, $\widehat{G}$ is non-trivial. In particular, j^* is *not* surjective despite j being injective. (*c.f.* the last part of Proposition 6.9.)

An important application of our Haar integral is the following result.

Theorem 6.12. *Suppose that G is a locally compact topological group. Then $\widehat{G}$ is locally compact.*

Proof. Let $\int$ be a left Haar integral on G (which exists by Theorem 5.10). Since $\int$ is non-trivial there is $f_0 \in C_c^+(G)$ such that $\int f_0 \neq 0$ and we may rescale so that $\int f_0 = 1$. Write K

for a compact set containing the support of f_0 and U for a compact neighbourhood of the identity.

UK is compact by Lemma 2.23. Apply Corollary 4.12 to get a continuous compactly supported $F : G \to [0,1]$ such that $F(x) = 1$ for all $x \in UK$. Define

$$V := \{\gamma \in \widehat{G} : |\gamma(x) - 1| \leqslant 1/4 \text{ for all } x \in K\},$$

so that V certainly contains, $U(K, 1/4)$, an open neighbourhood of the identity.

Claim. *Suppose that $\kappa, \delta > 0$. Then there is an open neighbourhood of the identity $L_{\delta,\kappa}$ such that if $\left|\int f_0\gamma\right| \geqslant \kappa$ then $|1 - \gamma(y)| < \delta$ for all $y \in L_{\delta,\kappa}$.*

Proof. By Lemma 5.3 there is an open neighbourhood of the identity $L_{\delta,\kappa}$ (which we may assume is contained in U since U is a neighbourhood and so contains an open neighbourhood of the identity) such that $\|\lambda_y(f_0) - f_0\|_\infty < \delta\kappa / \int F$ for all $y \in L_{\delta,\kappa}$. (Note $\int F > 0$ by Corollary 5.9.) For $y \in L_{\delta,\kappa}$, the support of $\lambda_y(f_0) - f_0$ is contained in UK (since $L_{\delta,\kappa} \subset U$) and so

$$\int |\lambda_y(f_0) - f_0| \leqslant \|\lambda_y(f_0) - f_0\|_\infty \int F < \delta\kappa.$$

Now, if $y \in L_{\delta,\kappa}$ then

$$\begin{aligned} |1 - \gamma(y)|\kappa \leqslant \left|(\gamma(y) - 1) \int f_0\gamma\right| &= \left|\int f_0\lambda_{y^{-1}}(\gamma) - \int f_0\gamma\right| \\ &= \left|\int \lambda_y(f_0)\gamma - \int f_0\gamma\right| \leqslant \int |\lambda_y(f_0) - f_0| < \delta\kappa. \end{aligned}$$

Dividing by κ gives the claim. □

We write M for the set of maps $G \to S^1$ endowed with the product topology (*c.f.* the set F considered in the proof of Theorem 5.10) so that M is compact. As sets $\widehat{G}$ is contained in M, but the compact-open topology on $\widehat{G}$ is not, in general, the same as that induced on $\widehat{G}$ as a subspace of M. Our aim is to make use of the compactness on M to show that $\widehat{G}$ is locally compact in the compact-open topology.

First we restrict to homomorphisms: write H for the set of homomorphisms $G \to S^1$, which is a closed subset of M since it is the intersection over all pairs $x, y \in G$ of the set of $f \in M$ such that $f(xy) = f(x)f(y)$. Write

$$C := \bigcap_{\delta > 0, x \in L_{\delta,3/4}} \{f \in H : |f(x) - 1| \leqslant \delta\}$$

which is also closed as an intersection of closed sets. By Lemma 3.6 as sets we have $C \subset \widehat{G}$ since the sets $\{z \in S^1 : |1 - z| \leqslant \delta\}$ form a neighbourhood base of the identity in S^1, and if $f \in C$ then $f^{-1}(\{z \in S^1 : |1 - z| \leqslant \delta\}) \supset L_{\delta,3/4}$ which is a neighbourhood of the identity in G.

If $\gamma \in V$ then $\left|1-\int f_0\gamma\right| \leqslant \int f_0|1-\gamma| \leqslant 1/4$, so by the triangle inequality $|\int f_0\gamma| \geqslant 3/4$ and hence the claim tells us that $\gamma \in C$. Thus (as sets) $V \subset C \subset \widehat{G}$ and so

$$V = \bigcap_{x\in K} \{f \in C : |f(x)-1| \leqslant 1/4\},$$

which is again a closed subset of M.

Our aim is to show that V is compact in the compact-open topology on $\widehat{G}$. This follows if every cover of the form $\mathcal{U} = \{\gamma U(K_\gamma, \delta_\gamma) : \gamma \in V\}$ (where K_γ is compact and $\delta_\gamma > 0$) has a finite subcover. Write $L_\gamma := L_{\delta_\gamma/2,1/2}$ and note that by compactness of K_γ there is a finite set T_γ such that $K_\gamma \subset T_\gamma L_\gamma$. Write

$$U_\gamma := \{f \in M : |f(x)-1| < \delta_\gamma/2 \text{ for all } x \in T_\gamma\}$$

which is an open set in M since T_γ is finite. Suppose that $\lambda \in (\gamma U_\gamma) \cap V$. Then since $\gamma, \lambda \in V$, the triangle inequality gives

$$\begin{aligned}\left|1-\int f_0\overline{\gamma}\lambda\right| \leqslant \int f_0|1-\overline{\gamma}\lambda| &= \int f_0|1-\overline{\gamma}+\overline{\gamma}-\overline{\gamma}\lambda| \\ &\leqslant \int f_0|1-\gamma| + \int f_0|1-\lambda| \leqslant 1/2.\end{aligned}$$

Hence $\left|\int f_0\overline{\gamma}\lambda\right| \geqslant 1/2$ by the triangle inequality again. The claim gives $|1-\overline{\gamma(y)}\lambda(y)| < \delta_\gamma/2$ for all $y \in L_\gamma$. But $\overline{\gamma}\lambda \in U_\gamma$ so we also have $|1-\overline{\gamma(z)}\lambda(z)| < \delta_\gamma/2$ for all $z \in T_\gamma$. Thus, if $x \in K_\gamma$ then there is $z \in T_\gamma$ and $y \in L_\gamma$ such that $x = zy$ and

$$\begin{aligned}|1-\overline{\gamma(x)}\lambda(x)| &\leqslant |1-\overline{\gamma(z)}\lambda(z)| + |\overline{\gamma(z)}\lambda(z)-\overline{\gamma(zy)}\lambda(zy)| \\ &= |1-\overline{\gamma(z)}\lambda(z)| + |1-\overline{\gamma(y)}\lambda(y)| < \delta_\gamma.\end{aligned}$$

We conclude that $\gamma U_\gamma \cap V \subset \gamma U(K_\gamma, \delta_\gamma) \cap V$. Finally $\{\gamma U_\gamma : \gamma \in V\}$ is a cover of V by sets that are open in M. M is compact and V is closed as a subset of M so V is compact as a subset of M, and hence $\{\gamma U_\gamma : \gamma \in V\}$ has a finite subcover which leads to a finite subcover of our original cover $\mathcal{U}$. The result is proved. □

Remark 6.13. Following on from Remark 6.10, the above shows that the category of locally compact Hausdorff Abelian topological groups is closed under taking duals. Pontryagin duality is a strengthening of this showing that in this restricted category taking the dual is a type of 'equivalence'. A crucial part of establishing Pontryagin duality turns out to be showing that characters separate points on these groups and this is the purpose of our next and final section.

7 Characters on locally compact Hausdorff Abelian topological groups separate points

If we are able to distinguish topologically between two points of a topological group, meaning if there exists an open set containing one and not the other, then Theorem 4.7 can be used to provide a continuous function into $\mathbb{C}$ bearing witness to this, by which we mean a continuous function which taking different values at the two points. In this section we ask when these topological witnesses can also be made to respect the group structure. Our first aim is the following, which will be a key ingredient in the main result of the section: Theorem 7.19.

Theorem 7.1. *Suppose that G is a compact Hausdorff Abelian topological group and $x \in G$ is not the identity. Then there is a character $\gamma \in \widehat{G}$ such that $\gamma(x) \neq 1$.*

Remark 7.2. This result is sometimes called the Peter-Weyl Theorem (see *e.g.* [DS11, Corollary 5.3]), though more often that name refers to a more general result.

To prove this we shall need some inner product structure afforded by Haar integrals. First, given a Haar integral $\int$ on a topological group G we define

$$\langle f, g\rangle := \int f\overline{g} \text{ for all } f,g \in C_c(G) \text{ and } \|f\|^2 := \int |f|^2 = \langle f, f\rangle \text{ for all } f \in C_c(G). \tag{7.1}$$

Remark 7.3. Although the notation $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ makes no mention of $\int$ it will always be clear from context which Haar integral we are referring to. In any case in the light of Remark 5.4, where we noted that Haar integrals can only exist for locally compact topological groups, Theorem 5.13 tells us there is little choice.

Lemma 7.4 (Basic properties of the inner product)**.** *Suppose that $\int$ is a left Haar integral on a topological group G. Then*

(i) $\langle \cdot, \cdot \rangle$ is an inner product on $C_c(G)$;

(ii) for each $x \in G$, λ_x is an isometry[13] of $C_c(G)$ with the norm $\|\cdot\|$;

(iii) and if G is compact then there is a constant $C > 0$ (depending on $\int$) such that

$$\int |f| \leqslant C\|f\| \text{ and } \|f\| \leqslant C\|f\|_\infty \text{ for all } f \in C(G).$$

[13]Recall that an isometry ϕ of a (complex) inner product space actually preserves the inner product (meaning $\langle \phi(f), \phi(g)\rangle = \langle f, g\rangle$ for all f, g) by the polarisation identity:

$$\langle f, g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2 + i\|f+ig\|^2 - i\|f-ig\|^2\right) \text{ for all } f, g.$$

Proof. Linearity in the first argument and conjugate-symmetry follow in view of the definition of $\langle \cdot, \cdot \rangle$ and linearity of the Haar integral and Remark 4.17. $\langle f, f \rangle \geqslant 0$ for all $f \in C_c(G)$ since $\int$ is non-negative and it is positive definite by Corollary 5.9. (ii) follows from the fact that the Haar integral is left-invariant so

$$\int f\overline{g} = \int \lambda_x(f\overline{g}) = \int \lambda_x(f)\overline{\lambda_x(g)} \text{ for all } f, g \in C_c(G).$$

Finally, for (iii) since G is compact the constant function 1 and $\|f\|_\infty^2$ are both in $C_c(G)$, and in particular we can put $C := \|1\| > 0$. By the Cauchy-Schwarz inequality (which holds for all inner products) we have

$$\int |f| = \langle 1, |f| \rangle \leqslant \|1\| \||f|\| = C\|f\| \text{ for all } f \in C(G);$$

and by non-negativity of $\int$ we have

$$\|f\|^2 = \int |f|^2 \leqslant \int \|f\|_\infty^2 = C^2 \|f\|_\infty^2 \text{ for all } f \in C(G).$$

The last inequality follows and the result is proved. □

Remark 7.5. The second inequality in (iii) in particular tells us that if G is compact then uniform convergence implies convergence in $\| \cdot \|$.

Remark 7.6. ⚠ The norm with respect to which λ_x is an isometry is different in (ii) than in Remark 5.2.

Remark 7.7. $C_c(G)$ has a completion as a Hilbert space which we shall not discuss explicitly here. One way to construct this is to associate a measure μ to the Haar integral $\int$ (as in Remark 4.18) and then the completion can be identified with $L_2(\mu)$.

Given a left Haar integral $\int$ it can be used to define an operation called **convolution** (implicitly used in the proof of Theorem 5.13) on $C_c(G)$ by

$$f * g(x) := \int_y f(y)g(y^{-1}x) = \langle f, \lambda_x(\widetilde{g}) \rangle. \tag{7.2}$$

Remark 7.8. This definition is the reason for the complex conjugate in the definition of $\widetilde{\cdot}$ before Remark 5.14.

Remark 7.9. If $f, g \in C_c(G)$ are supported in A and B respectively, then $f * g(x) \neq 0$ implies that there is some y with $f(y) \neq 0$ and $g(y^{-1}x) \neq 0$ *i.e.* $y \in A$ and $y^{-1}x \in B$ so $x \in AB$. In other words, the support of $f * g$ is contained in AB.

Remark 7.10. If $f \in C_c(G)$ then $f * \widetilde{f}(1_G) = \|f\|^2$.

Lemma 7.11 (Basic properties of convolution operators)**.** *Suppose that $\int$ is a Haar integral on a locally compact topological group G and $f \in C_c(G)$. Then* *(i)* $C_c(G) \to C_c(G); g \mapsto g * f$ *is a well-defined linear map;*

(ii) $\lambda_t(g * f) = \lambda_t(g) * f$ *for all* $t \in G$ *and* $g \in C_c(G)$*;*

(iii) $\langle g * f, h\rangle = \langle g, h * \widetilde{f}\rangle$ *for all* $g, h \in C_c(G)$*;*

(iv) $h * (g * f) = (h * g) * f$ *for all* $g, h \in C_c(G)$*;*

(v) $\|g * f\|_\infty \leqslant \|g\|\|\widetilde{f}\|$ *for all* $g \in C_c(G)$*.*

Proof. By the first part of Fubini's Theorem (Theorem 4.19) the function $g * f \in C_c(G)$ since $(x, y) \mapsto g(x)f(x^{-1}y)$ is a continuous and compactly supported. Moreover, $\int$ is linear and so it follows that $g \mapsto g * f$ is linear and (i) is proved.

For (ii) note that $\lambda_t(g * f)(x) = g * f(t^{-1}x) = \langle g, \lambda_{t^{-1}x}(\widetilde{f})\rangle = \langle g, \lambda_{t^{-1}}(\lambda_x(\widetilde{f}))\rangle = \langle \lambda_t(g), \lambda_x(\widetilde{f})\rangle = \lambda_t(g) * f(x)$ since λ_t is an isometry w.r.t. $\|\cdot\|$ (by Lemma 7.4 (ii).

For (iii), since the function $(x, y) \mapsto g(x)f(x^{-1}y)\overline{h(y)}$ is continuous and compactly supported, by Fubini's Theorem (Theorem 4.19), linearity of the Haar integral, and Remark 4.17 we have

$$\begin{aligned}\langle g * f, h\rangle &= \int_y \int_x g(x)f(x^{-1}y)\overline{h(y)}\\ &= \int_x g(x) \int_y f(x^{-1}y)\overline{h(y)} = \int_x g(x) \int_y \overline{h(y)\widetilde{f}(y^{-1}x)} = \langle g, h * \widetilde{f}\rangle,\end{aligned}$$

as required.

For (iv) we apply λ_y to the integrand $z \mapsto g(z)f(z^{-1}y^{-1}x)$ using that $\int_z$ is a left Haar integral; then Fubini's Theorem (Theorem 4.19) since $(z, y) \mapsto h(y)g(y^{-1}z)f(z^{-1}x)$ is continuous; and finally linearity of $\int_y$ to see that

$$\begin{aligned}h * (g * f)(x) &= \int_y h(y) \int_z g(z)f(z^{-1}y^{-1}x)\\ &= \int_y h(y) \int_z g(y^{-1}z)f(z^{-1}x) = \int_z \left(\int_y h(y)g(y^{-1}z)\right) f(z^{-1}x) = (h * g) * f(x)\end{aligned}$$

as claimed.

Finally, (v) follows by the Cauchy-Schwarz inequality and the fact that λ_x is an isometry for $\|\cdot\|$ (Lemma 7.4 (ii)): $|g * f(x)| = |\langle g, \lambda_x(\widetilde{f})\rangle| \leqslant \|g\|\|\lambda_x(\widetilde{f})\| = \|g\|\|\widetilde{f}\|$. The result is proved. □

Remark 7.12. We call the linear operators in (i) **convolution operators**.

Remark 7.13. The linearity in (i) and the inequality (v) mean that convolution operators map convergence in $\|\cdot\|$ to uniform convergence *i.e.* if $g_n \to g$ in $\|\cdot\|$ then $g_n * f \to g * f$ uniformly.

Before beginning our main argument we need one more tool that captures what is called compactness of convolution operators. This is a notion from functional analysis, though familiarity with it is not needed.

Proposition 7.14. *Suppose that $\int$ is a left Haar integral on a compact topological group G, $f \in C^+(G)$ and $(g_n)_{n\in\mathbb{N}}$ is a sequence of elements of $C(G)$ with $\|g_n\| \leqslant 1$ (the norm corresponding to the inner product,* not *the uniform norm). Then there is a subsequence $(g_{n_i})_{i\in\mathbb{N}}$ such that $g_{n_i} * f$ converges uniformly to some element of $C(G)$ as $i \to \infty$.*

Proof. For each $j \in \mathbb{N}$ Remark 4.5 gives us an open cover $\mathcal{U}_j$ of G such that if $x, y \in U \in \mathcal{U}_j$ then $|f(x) - f(y)| < 1/j$. Since G is compact apply Lemma 2.28 to get an open neighbourhood of the identity U_j such that $\{xU_j : x \in G\}$ refines $\mathcal{U}_j$. Since G is compact there is a finite cover $\{x_{1,j}U_j, \dots, x_{k(j),j}U_j\}$ which refines $\{xU_j : x \in G\}$. By Lemma 7.4 (v) $g_n * f(x) \in [-\|\widetilde{f}\|, \|\widetilde{f}\|]$, and the latter is sequentially compact. A countable product of sequentially compact spaces is sequentially compact[14] so there is a subsequence $(n_i)_i$ such that $g_{n_i} * f(x_{k,j})$ converges, say to $g(x_{k,j})$, as $i \to \infty$ for all $1 \leqslant k \leqslant k(j)$ and $j \in \mathbb{N}$.

Suppose $\epsilon > 0$ and let $j := \lceil 3\epsilon^{-1}C \rceil$ (where C is as in Lemma 7.4 (iii)). For all $1 \leqslant k \leqslant k(j)$ let M_k be such that $|g_{n_i} * f(x_{k,j}) - g(x_{k,j})| < \epsilon/6$ for all $i \geqslant M_k$; let $M := \max\{M_k : 1 \leqslant k \leqslant k(j)\}$ and suppose that $i, i' \geqslant M$.

For $x \in G$ there is some $1 \leqslant k \leqslant k(j)$ such that $x \in x_{k,j}U_j$ and hence for all $y \in G$ we have $y^{-1}x, y^{-1}x_{k,j} \in y^{-1}x_{k,j}U_j$ which is a subset of an element of $\mathcal{U}_j$, so $|f(y^{-1}x) - f(y^{-1}x_{k,j})| < 1/j$. Thus for $g \in C(G)$ with $\|g\| \leqslant 1$ we have (by Lemma 7.4 (iii))

$$\begin{aligned}|g * f(x) - g * f(x_{k,j})| &= |\langle g, \lambda_x(\widetilde{f}) - \lambda_{x_{k,j}}(\widetilde{f})\rangle| \\ &\leqslant \|\lambda_x(\widetilde{f}) - \lambda_{x_{k,j}}(\widetilde{f})\|_\infty \cdot \int |g| \\ &\leqslant \sup_{y\in G} |f(y^{-1}x) - f(y^{-1}x_{j,k})| C\|g\| \leqslant \frac{1}{j}C \leqslant \epsilon/3.\end{aligned}$$

In particular this holds for $g = g_{n_i}$ and $g = g_{n_{i'}}$, so that

$$\begin{aligned}|g_{n_i} * f(x) - g_{n_{i'}} * f(x)| \leqslant{} & |g_{n_i} * f(x) - g_{n_i} * f(x_{k,j})| + |g_{n_i} * f(x_{k,j}) - g(x_{k,j})| \\ &+ |g(x_{k,j}) - g_{n_{i'}} * f(x_{k,j})| + |g_{n_{i'}} * f(x_{k,j}) - g_{n_{i'}} * f(x)| < \epsilon.\end{aligned}$$

Since $x \in G$ was arbitrary it follows that the sequence of functions $(g_{n_i} * f)_i$ is uniformly Cauchy and so converges to a continuous function on G. The result is proved. ☐

Corollary 7.15. *Suppose that $\int$ is a left Haar integral on a compact topological group G, $f \in C^+(G)$, and W is a space of eigenvectors of the convolution operator $C(G) \to C(G); g \mapsto g * f$ all with eigenvalue $\lambda \neq 0$. Then W is finite dimensional.*

[14]The proof of this is just Cantor's diagonal argument.

Proof. Suppose that W is infinite dimensional. Then in particular there is an infinite sequence of linearly independent vectors in W and by the Gram-Schmidt process[15] there is an orthonormal sequence of vectors $e_1, e_2, \dots \in W$. Since every vector in W has eigenvalue $\lambda \neq 0$ we have $e_n * f = \lambda e_n$ and by Proposition 7.14 there is a subsequence e_{n_i} such that λe_{n_i} converges uniformly to some $e \in C(G)$. Since $\|e_{n_i}\| = 1$ and $\lambda e_{n_i} \to e$ uniformly, Remark 7.5 tells us that $\|e\| = |\lambda| \neq 0$. Moreover,

$$\sum_{i=1}^{k} |\langle e, e_{n_i}\rangle|^2 \leqslant \sum_{i=1}^{k} |\langle e, e_{n_i}\rangle|^2 + \left\| e - \sum_{i=1}^{k} \langle e, e_{n_i}\rangle e_{n_i} \right\|^2 = \|e\|^2, \tag{7.3}$$

but $|\langle e, e_{n_i}\rangle| \to |\lambda| \neq 0$ and so the left of (7.3) tends to infinity as $k \to \infty$. This is a contradiction proving the corollary. □

Remark 7.16. ⚠ We may have $\|e_n\|_\infty \to \infty$ as $n \to \infty$, despite the fact that $\|e_n\| = 1$ for all $n \in \mathbb{N}$.

Remark 7.17. The inequality in (7.3) for a general orthonormal sequence e_{n_i} and vector e is sometimes called **Bessel's inequality**.

Proof of Theorem 7.1. First, by Theorem 5.10 there is a left Haar integral on G; write $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ for the corresponding inner product and norm as in (7.1).

We begin by defining an auxiliary function. Since G is Hausdorff and $x \neq 1_G$ there is a neighbourhood T of the identity that is disjoint from x. By Lemma 2.25 there is a symmetric open neighbourhood of the identity S such that $S^2 \subset T$, and by Theorem 4.7 with the compact set $\{1_G\}$ contained in the open set S, there is an $f \in C^+(G)$ with $f(1_G) = 1$ and which is supported in S. In particular $\widetilde{f} * f(x) = 0$ in view of Remark 7.9 and the fact that $S^{-1}S = S^2 \subset T$ while $x \notin T$.

To ensure that the homomorphism we construct does not have x in the kernel we shall restrict attention to the subspace

$$V := \{g \in C(G) : \langle g, h\rangle = 0 \text{ for all } h \in C(G) \text{ with } h = \lambda_x(h)\}.$$

Suppose that $g \in V$, $h \in C(G)$ has $h = \lambda_x(h)$, and $k \in C(G)$. Then $h*\widetilde{k} = \lambda_x(h)*\widetilde{k} = \lambda_x(h*\widetilde{k})$ (by Lemma 7.11 (ii)), and so $0 = \langle g, h * \widetilde{k}\rangle = \langle g * k, h\rangle$ (by Lemma 7.11 (iii). Thus $g * k \in V$, and *any* convolution operator is a linear operator (by Lemma 7.11 (i)) mapping V into V.

Let $B := \{g \in V : \|g\| \leqslant 1\}$ and note by Lemma 7.11 (v) that $\|g * f\| \leqslant \|\widetilde{f}\|$ for all $g \in B$; let $M := \sup\{\|g * f\| : g \in B\}$ which consequently exists. Note, in particular, that by

[15] Given $v_1, v_2, \dots$ linearly independent, the Gram-Schmidt process defines

$$u_n := v_n - \sum_{k=1}^{n-1} \langle v_n, e_k\rangle e_k \text{ and } e_n := u_n/\|u_n\|,$$

and it can be shown by induction that $e_1, e_2, \dots$ is an orthonormal sequence.

Lemma 7.11 (iii), the Cauchy-Schwarz inequality and the definition of M (since $h * \widetilde{f} \in V$) we have

$$\|h * \widetilde{f}\|^2 = \langle h * \widetilde{f}, h * \widetilde{f}\rangle = \langle (h * \widetilde{f}) * f, h\rangle \leqslant M\|h * \widetilde{f}\|\|h\| \text{ for } h \in V \tag{7.4}$$

and so $\|h * \widetilde{f}\| \leqslant M\|h\|$ for $h \in V$.

Put $g_0 := \widetilde{f} - \lambda_{x^{-1}}(\widetilde{f})$ and suppose $h \in C(G)$ has $h = \lambda_x(h)$. Then by Lemma 7.4 (ii)

$$\langle g_0, h\rangle = \langle \widetilde{f} - \lambda_{x^{-1}}(\widetilde{f}), h\rangle = \langle \widetilde{f}, h\rangle - \langle \lambda_x(\lambda_{x^{-1}}(\widetilde{f})), \lambda_x(h)\rangle = \langle \widetilde{f}, h\rangle - \langle \widetilde{f}, \lambda_x(h)\rangle = 0,$$

so $g_0 \in V$. Moreover,

$$g_0 * f(1_G) = (\widetilde{f} - \lambda_{x^{-1}}(\widetilde{f})) * f(1_G) = \widetilde{f} * f(1_G) - \widetilde{f} * f(x) = \|f\|^2 \neq 0$$

by Remark 7.10 and the design of f. Whence $\|g_0 * f\| \neq 0$. Since the zero function is mapped to the zero function by any convolution operator, we conclude that $g_0 \neq 0$ and by linearity of the convolution operator we have that $\|(g_0/\|g_0\|) * f\| \neq 0$ so $M > 0$.

Let $(g_n)_n$ be a sequence of elements in B such that $\|g_n * f\| \to M$. By Proposition 7.14 there is a subsequence g_{n_i} and some $h \in C(G)$ such that $g_{n_i} * f \to h$ uniformly. Uniform convergence implies convergence in $\|\cdot\|$ (Remark 7.5) and so V is closed under uniform limits (and hence $h \in V$); $\langle g_{n_i} * f, h\rangle \to \|h\|^2$; and $\|g_{n_i} * f\| \to \|h\|$ leading to $\|h\| = M$. Combined with (7.4) and Lemma 7.11 (iii), gives

$$\begin{aligned}\|h * \widetilde{f} - M^2 g_{n_i}\|^2 &= \|h * \widetilde{f}\|^2 - 2M^2 \operatorname{Re}\langle h * \widetilde{f}, g_{n_i}\rangle + M^4\|g_{n_i}\|^2 \\ &\leqslant M^2\|h\|^2 - 2M^2 \operatorname{Re}\langle h, g_{n_i} * f\rangle + M^4 \to 0\end{aligned}$$

as $i \to \infty$. Hence $M^2 g_{n_i} \to h * \widetilde{f}$ in $\|\cdot\|$, and since convergence in $\|\cdot\|$ is mapped to uniform convergence by convolution operations (Remark 7.13) we have $M^2 g_{n_i} * f \to (h * \widetilde{f}) * f$. Uniqueness of limits then ensures $M^2 h = (h * \widetilde{f}) * f = h * (\widetilde{f} * f)$. (The last equality is the associativity proved in Lemma 7.11 (iv).)

Since $h \in V$, $\lambda_x(h) \neq h$ since $h \neq 0$. Consider the subspace

$$W := \left\{\sum_{i=1}^{n} \mu_i \lambda_{t_i}(h) : n \in \mathbb{N}, \mu_1, \ldots, \mu_n \in \mathbb{C}, t_1, \ldots, t_n \in G\right\}$$

which contains h. In view of the design of h, linearity of the convolution operator $\alpha : C(G) \to C(G); g \mapsto g * (\widetilde{f} * f)$ (Lemma 7.11 (i)), and Lemma 7.11 (ii), we have that every W is an eigenvector of α with eigenvalue $M^2 \neq 0$. Moreover, $\widetilde{f} * f \in C^+(G)$ (since $f \in C^+(G)$) and so by Corollary 7.15 $\dim W < \infty$.

We call $U \leqslant C(G)$ an **invariant space** if $\lambda_t(u) \in U$ for all $u \in U$ and all $t \in G$. By design W is an invariant space and moreover, it is finite dimensional and contains $h \neq 0$. We think of the λ_ts as linear maps on W.

Claim. *If U is an invariant space and $U' \leqslant U$ is an eigenspace of λ_t as a linear map $U \to U$, then U' is itself an invariant space.*

Proof. By hypothesis there is some μ such that $U' = \{u \in U : \lambda_t(u) = \mu u\}$ for some μ, and so if $u \in U'$, then since G is Abelian[16], $\lambda_t(\lambda_s(u)) = \lambda_{ts}(u) = \lambda_{st}(u) = \lambda_s(\lambda_t(u)) = \lambda_s(\mu u) = \mu\lambda_s(u)$ so that $\lambda_s(u) \in U'$ for all $s \in G$. □

Since $h \in W$ and $\lambda_x(h) \neq h$ the linear map $\lambda_x : W \to W$ is not the identity and is isometric (Lemma 7.4 (ii)) so[17] it has an eigenspace with eigenvalue distinct from 1. Let $U \leqslant W$ be an eigenspace corresponding to a value which is not 1, and which by the claim is invariant.

Let $U' \leqslant U$ be a non-trivial invariant subspace of smallest dimension. Since U' is finite dimensional (as a subspace of W) λ_t has an eigenvalue, call it $\gamma(t)$, and since U' has a smallest dimension the corresponding eigenspace, which is an invariant space by the claim, must be the whole of U'. In particular $\lambda_t(u) = \gamma(t)u$ for all $u \in U'$ and $t \in G$; and there is $u \in U'$ with $u \neq 0$. Then:

- $\lambda_t(u) \in U'$ by invariance and so $\lambda_s(\lambda_t(u)) = \gamma(s)\lambda_t(u) = \gamma(s)\gamma(t)u$, but on the other hand $\lambda_s(\lambda_t(u)) = \lambda_{st}(u) = \gamma(st)u$. Since $u \neq 0$ we may divide and $\gamma(st) = \gamma(s)\gamma(t)$.
- λ_s is isometric so $|\gamma(s)|\|u\| = \|\gamma(s)u\| = \|\lambda_s(u)\| = \|u\|$ for all $s \in G$. Since $u \neq 0$ we have $|\gamma(s)| = 1$ and in particular γ is a homomorphism into S^1.
- Again, since $u \neq 0$, for $\epsilon > 0$ we have

  $$\{t \in G : |\gamma(t) - 1| < \epsilon\} = \{t \in G : \|\lambda_t(u) - u\|_\infty < \epsilon\|u\|_\infty\}.$$

  The latter set is open since the map $G \to C(G); t \mapsto \lambda_t(u)$ is continuous by Lemma 5.3. The sets $\{z \in S^1 : |z - 1| < \epsilon\}$ form a neighbourhood base of the identity in S^1 (*c.f.* the proof of Theorem 6.12) and hence γ is continuous by Lemma 3.6.
- Finally $\gamma(x) \neq 1$ since $U' \leqslant U$.

The result is proved. □

Remark 7.18. The invariant spaces described above are sometimes called unitary representations.

[16]This is the *only* place where we use that G is Abelian.

[17]This fact is more involved than the others here as it relies on the fact that λ_x on W is isometric and so corresponds to a unitary matrix which can be diagonalised.

Bootstrapping separating characters in compact Hausdorff Abelian topological groups to locally compact Hausdorff Abelian topological groups

Finally we turn to the full separation theorem:

Theorem 7.19. *Suppose that G is a locally compact Hausdorff Abelian topological group and $x \in G$ is not the identity. Then there is a character $\gamma \in \widehat{G}$ such that $\gamma(x) \neq 1$.*

We shall prove this by finding an open subgroup with a compact quotient that does not identify x with the identity and then applying the result for compact groups. To begin we need a lemma which is sometimes called Weil's Lemma.

Lemma 7.20. *Suppose that G is a topological group, $U \subset G$ is a symmetric open neighbourhood of the identity with compact closure, and $x \in G$ is such that $|\langle x\rangle \cap U| = \infty$. Then $\overline{\langle x\rangle}$ is compact.*

Proof. There is no loss in assuming that $G = \overline{\langle x\rangle}$. Let $Z := \langle x\rangle = \{x^z : z \in \mathbb{Z}\}$ and $N := \{x^n : n \in \mathbb{Z}, n \geqslant 1\}$. The hypothesis is that $Z \cap U$ is infinite so $x^{n_i} \in U$ for some sequence of integers n_i with $|n_i| \to \infty$. Hence for $\alpha \in \mathbb{Z}$ then there is some i such that $\alpha + |n_i| \in \mathbb{N}$, whence $x^\alpha = x^{\alpha+|n_i|}x^{-|n_i|} \in N(U \cup U^{-1}) = NU$ *i.e.* $Z \subset NU$ and similarly $Z \subset N^{-1}U$.

Since U is a neighbourhood of the identity and $G = \overline{Z}$, for all $z \in G$ there is $\alpha \in \mathbb{Z}$ such that $x^\alpha \in zU$ and hence $G \subset ZU^{-1} = ZU$. Since $Z \subset NU$ it follows that $G \subset NU^2$, so for each $z \in G$ we may let $n(z) \geqslant 1$ be minimal such that $z \in x^{n(z)}U^2$.

$\overline{U}^2$ is compact by Lemma 2.23 and (since $Z \subset N^{-1}U$ we also have) $G \subset N^{-1}U^2$, so there is some $n_0 \geqslant 1$ such that

$$U^2 \subset \overline{U}^2 \subset \{x^{-1}, x^{-2}, \dots, x^{-n_0}\}U^2.$$

Coupling this with the definition of $n(z)$ we conclude that there is $1 \leqslant i \leqslant n_0$ such that $x^{-n(z)}z \in x^{-i}U^2$, whence $z \in x^{n(z)-i}U^2$. By minimality of $n(z)$ it follows that $n(z) - i \leqslant 0$ and so $n(z) \leqslant n_0$. Hence $G \subset \{x, x^2, \dots, x^{n_0}\}U^2 \subset \{x, x^2, \dots, x^{n_0}\}\overline{U}^2$. Thus G is a finite union of compact spaces, and so compact proving the result. □

The next result makes considerable use of the fact the underlying group is Abelian and while our proof of Theorem 7.1 adapts reasonably easily to give a non-Abelian analogue, the following proposition is much harder to modify.

Proposition 7.21. *Suppose that G is a locally compact Abelian topological group and $x_0 \in G$ is not the identity. Then there is an open subgroup H of G and a closed subgroup $L \leqslant H$ such that H/L is compact and $x_0 \in H \backslash L$.*

Proof. Lemma 2.41 applied with the compact set $\{x_0\}$ gives us U, a symmetric open neighbourhood of the identity in G containing $\{x_0\}$ with compact closure. Let $H := \langle U\rangle$ which is an open subgroup of G by Corollary 2.19. By Proposition 2.44 there is a finite set T such that $H = \langle T\rangle U$.

Let $S \subset T$ be maximal (possibly empty) so that there is $n \in \mathbb{N}^S$, a vector of non-zero natural numbers indexed by S, such that $L := \langle s^{n_s} : s \in S\rangle$ is U**-separated** meaning that if $xU \cap yU \neq \emptyset$ and $x, y \in L$ then $x = y$. This immediately tells us that L is closed since if z is a limit point of L then there must be $x \neq y$ with $x, y \in L$ with $x, y \in Uz$ which cannot happen in view of the separation property. Moreover, since $x_0, 1_G \in U$ and $x_0 \neq 1_G$ we have $x_0 \notin L$.

Let $q : H \to H/L$ be the quotient map which is continuous and open by Proposition 3.10. Since q is a homomorphism $q(U^2)$ is a symmetric set containing the identity; it is open since q is open. $q(\overline{U}^2)$ is a continuous image of a compact (by Lemma 2.23) set and so compact and hence has compact closure by Lemma 2.37. Since $q(U^2) \subset q(\overline{U}^2)$ we conclude that $q(U^2)$ has compact closure.

Suppose that $t \in T$ and suppose that $q(t)$ does not have finite order in H/L. Then $t \notin S$ since $q(s)$ has order dividing n_s for $s \in S$ since $q(s)^{n_s} = 1_{H/L}$ by design of L and the fact q is a homomorphism. If $\langle q(t)\rangle \cap q(U^2)$ were finite then there would be $n_t \geqslant 1$ such that if $q(t)^{zn_t} \in q(U^2)$ and $z \in \mathbb{Z}$ then $z = 0$. But then if $L' := \langle s^{n_s} : s \in S \cup \{t\}\rangle$, and $x', y' \in L'$ has $x'U \cap y'U \neq \emptyset$, then since G is Abelian there are $z, w \in \mathbb{Z}$ and $x, y \in L$ such that $x' = t^{zn_t}x$ and $y' = t^{wn_t}y$. It follows that $y^{-1}t^{(z-w)n_t}x \in U^2$ so (since G is Abelian) $q(t)^{(z-w)n_t} \in q(U^2)$ and hence $z = w$, but then $xU \cap yU \neq \emptyset$ and so $x = y$ by design of L. We conclude that $x' = y'$ *i.e.* L' is U-separated, but this contradicts the maximality of S (since we could just have added t). Thus we conclude that $\langle q(t)\rangle \cap q(U^2)$ is infinite, and hence by Lemma 7.20 $\langle q(t)\rangle$ is contained in a compact set K_t.

Note that if $t \in T$ has finite order then $\langle q(t)\rangle$ is finite and so is also (contained in) a compact set, which we shall denote K_t. Again, since G is Abelian and q is a homomorphism we have $q(\langle T\rangle) = \prod_{t\in T} \langle q(t)\rangle \subset \prod_{t\in T} K_t$ (note the product here is the group operation *not* the Cartesian product) and so by Lemma 2.23 we have $q(\langle T\rangle)$ is contained in a compact set. Finally, q is continuous and so $q(\overline{U})$ is compact and $q(H) = q(\langle T\rangle)q(U)$ is contained in a compact setby Lemma 2.23 again. Finally, H is open and so $q(H)$ is open (Proposition 3.10), and a subgroup so closed (Corollary 2.19). It follows that $q(H)$ is a closed subset of a compact set and hence compact. The result is proved. □

Lemma 7.22. *Suppose that G is an Abelian topological group and H is an open subgroup of G. Then for every $\gamma \in \widehat{H}$ there is $\lambda \in \widehat{G}$ such that $\lambda|_H = \gamma$.*

Proof. The argument here is a typical Zorn's Lemma argument. We begin with the engine for $H \leqslant K \leqslant G$:

Claim. *Suppose that $\gamma \in \widehat{K}$, $x \in G\backslash K$, and K' is the group generated by x and K. Then there is some $\lambda \in \widehat{K'}$ such that $\lambda|_K = \gamma$.*

Proof. Let $k \in \mathbb{N}_0$ be minimal (when $\mathbb{N}_0$ is partially ordered by divisibility) such that $x^k \in K$ (*i.e.* k is the order of xK as an element of G/K with the convention that infinite order is denoted 0), and let w be a kth root of $\gamma(x^k)$ (with the convention that it is 1 if $k = 0$); define $\lambda(x^z h) := w^z\gamma(h)$ for all $z \in \mathbb{Z}$ and $h \in K$. We need to check this is well-defined so that if $x^z h = x^{z'}h'$ then $x^{z-z'} = h'h^{-1} \in K$ and so $k \mid z - z'$ (meaning $z = z'$ if $k = 0$, and hence $h = h'$) whence $w^z\gamma(h) = w^{z'}\gamma(x^{z-z'})\gamma(h) = w^{z'}\gamma(h')$ as required. λ is also visibly a homomorphism and the claim is proved since K is open in K', and so λ is continuous since γ is continuous and K'/K is discrete. □

Let $\mathcal{L}$ be the set of pairs (K, λ) such that $H \leqslant K \leqslant G$ and $\lambda \in \widehat{K}$ has $\lambda|_H = \gamma$. This set is partially ordered by $(K, \lambda) \leqslant (K', \lambda')$ if $K \leqslant K'$ and $\lambda = \lambda'|_K$. If $\mathcal{C}$ is a chain in $\mathcal{L}$ then $K^* := \bigcup\{K : (K, \lambda) \in \mathcal{C}\}$ is a group containing H and all K with $(K, \lambda) \in \mathcal{C}$, and we can define $\lambda^*(x)$ for all $x \in K^*$ by setting $\lambda^*(x) = \lambda(x)$ whenever $(K, \lambda) \in \mathcal{C}$ and $x \in K$. (λ^* is certainly continuous.)

Thus by Zorn's Lemma $\mathcal{L}$ has a maximal element (K, λ) and by the claim if $x \in G\backslash K$ then $\mathcal{L}$ would contain a larger element. This contradiction proves the result. □

Proof of Theorem 7.19. Apply Proposition 7.21 to get an open subgroup $H \leqslant G$, and a closed subgroup $L \leqslant H$ such that H/L is compact and $x \in H\backslash L$. Write $q : H \to H/L$ for the usual quotient map. Since L is closed H/L is Hausdorff by Proposition 3.16 and $q(x) \neq 1_{H/L}$ since $x \notin L$. By Theorem 7.1 there is $\lambda \in \widehat{H/L}$ such that $\lambda(xL) \neq 1$, so then $\lambda \circ q \in \widehat{H}$ has $\lambda \circ q(x) \neq 1$. Finally, apply Lemma 7.22 to this to get a character $\gamma \in \widehat{G}$ such that $\gamma(x) \neq 1$. The result is proved. □

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