Topological Groups

Sheet I — TT21

I.1. Let $\mathbb{R}_2$ be the additive group of reals with topology defined by letting the open sets be those open sets U in $\mathbb{R}$ (with its usual topology) for which there is some $a \in \mathbb{R}$ with $x \in U$ whenever $x > a$. Show that negation is not continuous, but that addition is continuous on $\mathbb{R}_2$. Show that $\mathbb{R}_2$ is Fréchet. Is it Hausdorff?

I.2. Let $\mathbb{R}_1$ be the additive group of reals with topology defined by letting the open sets be those open sets U in $\mathbb{R}$ for which there is some $a \in \mathbb{R}$ with $x \in U$ whenever $|x| > a$. Show that $\mathbb{R}_1$ is a quasiitopological group but *not* topological group.

I.3. Give an example of two compact sets A and B in the quasitopological group $\mathbb{R}_1$ (from Exercise I.2) such that $A + B$ is not compact.

I.4. Suppose that G is a locally compact Abelian group and $H, K \leqslant G$ are closed. Is it necessarily the case that $H + K$ is closed?

I.5. Give an example of a non-Abelian topological group G with a dense Abelian subgroup. Can the topology on G be Hausdorff?

I.6. Give an example of a topological group G and compact sets $A, B \subset G$ with $A \cap B$ not compact.

I.7. Suppose that G is a compactly generated topological group. Show that for any pair of disjoint closed sets A_0, A_1 there are disjoint open sets U_0, U_1 with $A_0 \subset U_0$ and $A_1 \subset U_1$. What if G is merely assumed to be locally compact?

I.8. View the elements of the topological group $\mathbb{Z}^{\mathbb{R}}$ as functions $\mathbb{R} \to \mathbb{Z}$, and let A_i be the set of $f \in \mathbb{Z}^{\mathbb{R}}$ such that f is injective on $\{x \in \mathbb{R} : f(x) \neq i\}$. Show that A_0 and A_1 are disjoint and closed and that there are no disjoint open sets U_0 and U_1 with $A_0 \subset U_0$ and $A_1 \subset U_1$. Is $\mathbb{Z}^{\mathbb{R}}$ locally compact?

[*Hint: It may help to note that the sets* $U(g) := \{f \in \mathbb{Z}^{\mathbb{R}} : f(s) = g(s) \text{ for all } s \in \operatorname{supp} f\}$ *where* $S \subset \mathbb{R}$ *is finite and* $g : S \to \mathbb{Z}$ *form a base for the topology on* $\mathbb{Z}^{\mathbb{R}}$*, and that* $\mathbb{Z}^{\mathbb{R}}$ *has a countable dense subset* D*. (An example of such a* D *is the set of maps* $\mathbb{R} \to \mathbb{Z}; x \mapsto \lfloor p(x) \rfloor$ *where* p *is a polynomial with rational coefficients and* $\lfloor z \rfloor$ *is largest integer less than or equal to* z*.)*]

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