Topological Groups

Sheet II — TT21

II.1. Suppose that G is a topological group and $H \leqslant G$ is compact. Show that the quotient map $q : G \to G/H$ is closed.

II.2. Show that the only continuous homomorphisms $\mathbb{R} \to \mathbb{R}$ (where $\mathbb{R}$ has its usual topology) are of the form $\mathbb{R} \to \mathbb{R}; x \mapsto \alpha x$ where $\alpha \in \mathbb{R}$.

II.3. Let $\mathbb{Q}_{>0}$ (resp. $\mathbb{R}_{>0}$) be the multiplicative group of positive rationals (resp. reals) with the subspace topology inherited from $\mathbb{R}$. Show that if $\phi : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ is a continuous homomorphism then there is a continuous homomorphism $\widetilde{\phi} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $\widetilde{\phi}(q) = \phi(q)$ for all $q \in \mathbb{Q}_{>0}$.

II.4. Suppose that α is a real irrational. Show that the map

$$\psi : \mathbb{R} \to S^1 \times S^1; t \mapsto (\exp(2\pi i t), \exp(2\pi i \alpha t))$$

is a continuous injective homomorphism, but that the restriction $\tilde{\psi} : \mathbb{R} \to \operatorname{Im} \psi; t \mapsto \psi(t)$ is *not* a topological isomorphism.

II.5. A topological group is said to be **σ-compact** if it is a countable union of compact sets. Give an example of a σ-compact group that is not compactly generated. Are σ-compact groups necessarily locally compact? Show that if G is σ-compact, H is a locally compact Hausdorff topological group, and $\phi : G \to H$ is a continuous surjective homomorphism then ϕ is an open map.

II.6. Suppose that G is a locally compact topological group. Show that there is a locally compact Hausdorff topological group H such that $C_c(G)$ is isometrically isomorphic (as a normed space) to $C_c(H)$.

II.7. Given a compact metric space T with metric d write $\operatorname{Isom}(T)$ for the set of isometries of T, that is maps $g : T \to T$ with $d(g(x), g(y)) = d(x, y)$ for all $x, y \in T$, and define a metric on $\operatorname{Isom}(T)$ by $d_\infty(g, h) := \sup\{d(g(x), h(x)) : x \in T\}$. Show that $\operatorname{Isom}(T)$ is a compact topological group when endowed with the topology induced by the metric d_∞.

II.8. Given an example of a compact metric space T, a group G (which need not have a topology), and a transitive faithful isometric action of G on T where G is *not* a compact subspace of $\operatorname{Isom}(T)$. (It may help to recall that an action of G on T is transitive if for every $x, y \in T$ there is $g \in G$ such that $gx = y$, and it is faithful if $gx = x$ for all $x \in T$ implies $g = 1_G$.)

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