# Topological Groups

## Sheet III — TT21

III.1. Suppose that $G$ is a topological group. Show that the map $C_c(G) \to \mathbb{C}; f \mapsto \sum_{x \in G} f(x)$ is a Haar integral (if and) only if there is a finite open normal $H \leqslant G$.

III.2. Let $G := (-1, 1)$ be endowed with the usual topology inherited from $\mathbb{R}$ and the binary operation $x \circ y := (x+y)/(1+xy)$. Show that $G$ with this operation is a locally compact Hausdorff Abelian topological group and that the map

$$C_c(-1,1) \to \mathbb{C}; f \mapsto \int_{-1}^{1} \frac{f(x)}{1-x^2} dx$$

is a left Haar integral on $G$.

III.3. Suppose that $G$ is a topological group. We say that $\int : C_c(G) \to \mathbb{C}$ is a **right Haar integral** if it is a non-negative non-trivial linear map with

$$\int_x f(xt) = \int f \text{ for all } t \in G \text{ and } f \in C_c(G).$$

Let $G$ be the set of matrices of the form $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ where $x > 0$ and $y \in \mathbb{R}$. Show that $G$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$ and that the map

$$C_c(G) \to \mathbb{C}; f \mapsto \int_{-\infty}^{\infty} \int_{0}^{\infty} f\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \frac{1}{x^2} dx dy$$

is a left Haar integral on $G$. Show that

$$C_c(G) \to \mathbb{C}; f \mapsto \int_{-\infty}^{\infty} \int_{0}^{\infty} f\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \frac{1}{x} dx dy$$

is a right Haar integral.

III.4. Suppose that $G$ is a locally compact topological group. Show that there is a unique continuous homomorphism $\Delta : G \to \mathbb{R}_{>0}$ such that for any left Haar integral $\int$ on $G$ we have

$$\int_x f(xt) = \Delta(t) \int f \text{ for all } t \in G \text{ and } f \in C_c(G).$$

III.5. We say that a topological group is **unimodular** if the function $\Delta$ in Exercise III.4 is identically 1. Show that if $G$ is a topological group that is either compact, discrete, or locally compact Abelian then it is unimodular. Give an example of a group that is not unimodular. Give an example of a non-Abelian, non-compact and non-discrete group that *is* unimodular.


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