Topological Groups

Sheet IV — TT21

IV.1. Show that $\widehat{\mathbb{Z}}$ is isomorphic to $\mathbb{T}$ as topological group and similarly for $\widehat{\mathbb{T}}$ and $\mathbb{Z}$.

IV.2. Show that $\mathbb{R}$ and $\widehat{\mathbb{R}}$ are isomorphic as topological groups.

IV.3. Suppose that G is a topological group. Write $H(G)$ for the set of continuous homomorphisms $G \to \mathbb{C}^*$, where $\mathbb{C}^*$ is the group of non-zero complex numbers under multiplication with its usual topology. Show that $H(G)$ is a topological group in the compact-open topology, that is where the open sets are generated by sets of the form $\phi\widetilde{U}(K,\epsilon)$ where $\phi \in H(G)$ and $\widetilde{U}(K,\epsilon) := \{\phi \in H(G) : |\phi(x) - 1| < \epsilon \text{ for all } x \in K\}$ where K is compact and $\epsilon > 0$. On the other hand show that we may have G locally compact and Hausdorff while $H(G)$ is *not* locally compact.

IV.4. Suppose that G is a discrete topological group. Show that $\widehat{G}$ is compact.

IV.5. Suppose that G is a compactly generated Abelian topological group and $\gamma : G \to \mathbb{C}^*$ (not necessarily a homomorphism) is continuous and has

$$\left|\frac{\gamma(x+y)}{\gamma(x)\gamma(y)} - 1\right| < \epsilon < \frac{1}{3} \text{ for all } x, y \in G.$$

Then there is $\lambda \in \widehat{G}$ such that $|\lambda(x) - \gamma(x)| = O(\epsilon)$ for all $x \in G$.

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