

Numerical Solution of Partial Differential Equations

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Lecture 3

Finite difference approximation of a two-point b.v.p.

We illustrate the method of finite difference approximation on a simple two-point boundary-value problem for a second-order linear (ordinary) differential equation:

$$\begin{aligned} -u'' + c(x)u &= f(x), \quad x \in (0, 1), \\ u(0) &= 0, \quad u(1) = 0, \end{aligned} \tag{1}$$

where f and c are real-valued functions, which are defined and continuous on the interval $[0, 1]$ and $c(x) \geq 0$ for all $x \in [0, 1]$.

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Let N be an integer, $N \geq 2$, and let $h = 1/N$ be the mesh-size; the mesh-points are $x_i = ih$, $i = 0, \dots, N$.

We define the set of interior mesh-points:

$$\Omega_h := \{x_i : i = 1, \dots, N - 1\}$$

the set of boundary mesh-points:

$$\Gamma_h := \{x_0, x_N\},$$

and the set of all mesh-points:

$$\bar{\Omega}_h := \Omega_h \cup \Gamma_h.$$

The second step

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Suppose that u is sufficiently smooth (e.g. $u \in C^4([0, 1])$). Then, by Taylor series expansion,

$$\begin{aligned}u(x_{i\pm 1}) &= u(x_i \pm h) \\ &= u(x_i) \pm hu'(x_i) + \frac{h^2}{2}u''(x_i) \pm \frac{h^3}{6}u'''(x_i) + \mathcal{O}(h^4),\end{aligned}$$

so that

$$D_x^+ u(x_i) := \frac{u(x_{i+1}) - u(x_i)}{h} = u'(x_i) + \mathcal{O}(h),$$

$$D_x^- u(x_i) := \frac{u(x_i) - u(x_{i-1})}{h} = u'(x_i) + \mathcal{O}(h),$$

and

$$\begin{aligned}D_x^+ D_x^- u(x_i) &= D_x^- D_x^+ u(x_i) \\ &= \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1}))}{h^2} \\ &= u''(x_i) + \mathcal{O}(h^2).\end{aligned}$$

D_x^+ and D_x^- are called the *forward* and *backward first divided difference* operator, respectively, and $D_x^+ D_x^-$ ($= D_x^- D_x^+$) is called the (symmetric) *second divided difference* operator.

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Thus we replace the second derivative u'' in the differential equation by the second divided difference $D_x^+ D_x^- u(x_i)$; hence,

$$\begin{aligned} -D_x^+ D_x^- u(x_i) + c(x_i)u(x_i) &\approx f(x_i), \quad i = 1, \dots, N-1, \\ u(x_0) = 0, \quad u(x_N) &= 0. \end{aligned} \tag{2}$$

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Now (2) motivates us to seek the approximate solution U as the solution of the system of difference equations:

$$\begin{aligned} -D_x^+ D_x^- U_i + c(x_i)U_i &= f(x_i), \quad i = 1, \dots, N-1, \\ U_0 = 0, \quad U_N &= 0. \end{aligned} \tag{3}$$

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$$AU = F,$$

where A is the $(N - 1) \times (N - 1)$ matrix

$$A = \begin{bmatrix} \frac{2}{h^2} + c(x_1) & -\frac{1}{h^2} & & & & & 0 \\ -\frac{1}{h^2} & \frac{2}{h^2} + c(x_2) & -\frac{1}{h^2} & & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -\frac{1}{h^2} & \frac{2}{h^2} + c(x_{N-2}) & -\frac{1}{h^2} & \\ 0 & & & & -\frac{1}{h^2} & \frac{2}{h^2} + c(x_{N-1}) & \end{bmatrix}$$

$$U = (U_1, U_2, \dots, U_{N-2}, U_{N-1})^T$$

and

$$F = (f(x_1), f(x_2), \dots, f(x_{N-2}), f(x_{N-1}))^T.$$

Existence and uniqueness of a solution

We begin the analysis of the finite difference scheme (3) by showing that it has a unique solution. It suffices to show that the matrix A is non-singular (i.e. $\det A \neq 0$), and therefore invertible.

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We shall develop a technique which we shall, in subsequent sections, extend to the finite difference approximation of PDEs.

For this purpose, we introduce, for two functions V and W defined at the interior mesh-points x_i , $i = 1, \dots, N - 1$, the inner product

$$(V, W)_h = \sum_{i=1}^{N-1} hV_iW_i,$$

which resembles the $L_2((0, 1))$ -inner product

$$(v, w) = \int_0^1 v(x)w(x) dx.$$

The argument is based on mimicking, at the discrete level, the following procedure based on integration-by-parts, noting that the solution of the boundary-value problem (1) satisfies the homogeneous boundary conditions $u(0) = 0$ and $u(1) = 0$:

$$\begin{aligned} \int_0^1 (-u''(x) + c(x)u(x)) u(x) dx &= \int_0^1 |u'(x)|^2 + c(x)|u(x)|^2 dx \\ &\geq \int_0^1 |u'(x)|^2 dx, \end{aligned} \tag{4}$$

because $c(x) \geq 0$ for all $x \in [0, 1]$.

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For the finite difference approximation of the boundary-value problem, if we can show by an analogous argument that the homogeneous system of linear algebraic equations corresponding to $f(x_i) = 0$, $i = 1, \dots, N - 1$, has the trivial solution $U_i = 0$, $i = 0, \dots, N$, as its unique solution, then the desired invertibility of the matrix A will directly follow.

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Our key tool is a summation-by-parts identity, which is the discrete counterpart of the integration-by-parts identity

$$(-u'', u) = (u', u') = \|u'\|_{L_2((0,1))}^2 = \int_0^1 |u'(x)|^2 dx$$

satisfied by the function u , obeying the homogeneous boundary conditions $u(0) = 0$, $u(1) = 0$, used in (4) above.

Summation by parts identity

Lemma

Suppose that V is a function defined at the mesh-points x_i , $i = 0, \dots, N$, and let $V_0 = V_N = 0$; then,

$$(-D_x^+ D_x^- V, V)_h = \sum_{i=1}^N h \left| D_x^- V_i \right|^2. \quad (5)$$

PROOF.

By the definitions of $(\cdot, \cdot)_h$ and $D_x^+ D_x^- V_i$ we have that

$$\begin{aligned}(-D_x^+ D_x^- V, V)_h &= - \sum_{i=1}^{N-1} h (D_x^+ D_x^- V_i) V_i \\ &= - \sum_{i=1}^{N-1} \frac{V_{i+1} - V_i}{h} V_i + \sum_{i=1}^{N-1} \frac{V_i - V_{i-1}}{h} V_i \\ &= - \sum_{i=2}^N \frac{V_i - V_{i-1}}{h} V_{i-1} + \sum_{i=1}^{N-1} \frac{V_i - V_{i-1}}{h} V_i \\ &= - \sum_{i=1}^N \frac{V_i - V_{i-1}}{h} V_{i-1} + \sum_{i=1}^N \frac{V_i - V_{i-1}}{h} V_i \\ &= \sum_{i=1}^N \frac{V_i - V_{i-1}}{h} (V_i - V_{i-1}) = \sum_{i=1}^N h \left| D_x^- V_i \right|^2.\end{aligned}$$

In the transition to the 3rd line we shifted the index in the first sum; in the transition to the 4th line used that $V_0 = V_N = 0$. \square

Returning to the finite difference scheme (3), let V be as in the above lemma and note that as, by hypothesis, $c(x) \geq 0$ for all $x \in [0, 1]$, we have

$$\begin{aligned}(AV, V)_h &= (-D_x^+ D_x^- V + cV, V)_h \\ &= (-D_x^+ D_x^- V, V)_h + (cV, V)_h \\ &\geq \sum_{i=1}^N h \left| D_x^- V_i \right|^2.\end{aligned}\tag{6}$$

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It therefore follows that A is a non-singular matrix, and thereby (3) has a unique solution, $U = A^{-1}F$.

We record this result in the next theorem.

Theorem

Suppose that c and f are continuous real-valued functions defined on the interval $[0, 1]$, and $c(x) \geq 0$ for all $x \in [0, 1]$; then, the finite difference scheme (3) possesses a unique solution U .

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We note in passing that, thanks the Lax–Milgram theorem (cf. the Lecture Notes), the boundary-value problem (1) has a unique (weak) solution under the hypotheses on c and f assumed in the above theorem.

Stability, consistency, and convergence

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To prove this, we define the *discrete L_2 -norm*

$$\|U\|_h := (U, U)_h^{1/2} = \left(\sum_{i=1}^{N-1} h |U_i|^2 \right)^{1/2},$$

and the *discrete Sobolev norm*

$$\|U\|_{1,h} := (\|U\|_h^2 + \|D_x^- U\|_h^2)^{1/2},$$

where

$$\|V\|_h^2 := \sum_{i=1}^N h |V_i|^2.$$

Using this notation, the inequality (6) can be rewritten as follows:

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In fact, by employing a discrete version of the Poincaré–Friedrichs inequality, stated in the next lemma, we shall be able to prove that

$$(AV, V)_h \geq c_0 \|V\|_{1,h}^2,$$

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Lemma (Discrete Poincaré–Friedrichs inequality)

Let V be a function defined on the mesh $\{x_i, i = 0, \dots, N\}$, and such that $V_0 = V_N = 0$; then, there exists a positive constant c_ , independent of V and h , such that*

$$\|V\|_h^2 \leq c_* \|D_x^- V\|_h^2 \quad (8)$$

for all such V .

PROOF. Thanks to the definition of $D_x^- V_i$ and by use of the Cauchy–Schwarz inequality,

$$|V_i|^2 = \left| \sum_{j=1}^i h(D_x^- V_j) \right|^2 \leq \left(\sum_{j=1}^i h \right) \sum_{j=1}^i h |D_x^- V_j|^2 = ih \sum_{j=1}^i h |D_x^- V_j|^2.$$

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Thus, because $\sum_{i=1}^{N-1} i = \frac{1}{2}(N-1)N$ and $Nh = 1$, we have that

$$\begin{aligned} \|V\|_h^2 &= \sum_{i=1}^{N-1} h|V_i|^2 \leq \sum_{i=1}^{N-1} ih^2 \sum_{j=1}^i h |D_x^- V_j|^2 \\ &\leq \frac{1}{2}(N-1)Nh^2 \sum_{j=1}^N h |D_x^- V_j|^2 \\ &\leq \frac{1}{2} \|D_x^- V\|_h^2. \end{aligned}$$

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$$\begin{aligned} \|V\|_h^2 &= \sum_{i=1}^{N-1} h |V_i|^2 \leq \sum_{i=1}^{N-1} ih^2 \sum_{j=1}^i h |D_x^- V_j|^2 \\ &\leq \frac{1}{2}(N-1)Nh^2 \sum_{j=1}^N h |D_x^- V_j|^2 \\ &\leq \frac{1}{2} \|D_x^- V\|_h^2. \end{aligned}$$

We note that the constant $c_* = 1/2$ in the inequality (8). \square

Using the inequality (8) to bound the right-hand side of the inequality (7) from below we obtain

$$(AV, V)_h \geq \frac{1}{c_\star} \|V\|_h^2. \quad (9)$$

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Adding the inequality (7) to the inequality (9) we arrive at the inequality

$$(AV, V)_h \geq (1 + c_\star)^{-1} \left(\|V\|_h^2 + \|D_x^- V\|_h^2 \right).$$

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Letting $c_0 = (1 + c_\star)^{-1}$ it follows that

$$(AV, V)_h \geq c_0 \|V\|_{1,h}^2. \quad (10)$$

Now the stability of the finite difference scheme (3) easily follows.

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PROOF. From (10) and (3) we have that

$$\begin{aligned} c_0 \|U\|_{1,h}^2 &\leq (AU, U)_h = (f, U)_h \leq |(f, U)_h| \\ &\leq \|f\|_h \|U\|_h \leq \|f\|_h \|U\|_{1,h}, \end{aligned}$$

and hence (11). \square

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Obviously $e_0 = 0$, $e_N = 0$, and

$$\begin{aligned} Ae_i &= Au(x_i) - AU_i = Au(x_i) - f(x_i) \\ &= -D_x^+ D_x^- u(x_i) + c(x_i)u(x_i) - f(x_i) \\ &= u''(x_i) - D_x^+ D_x^- u(x_i), \quad i = 1, \dots, N-1. \end{aligned}$$

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Thus,

$$\begin{aligned} Ae_i &= \varphi_i, & i = 1, \dots, N - 1, \\ e_0 &= 0, & e_N = 0, \end{aligned} \tag{12}$$

where $\varphi_i := u''(x_i) - D_x^+ D_x^- u(x_i)$ is the *consistency error* (sometimes also called the *truncation error*).

By applying ineq. (11) to the finite difference scheme (12):

$$\|u - U\|_{1,h} = \|e\|_{1,h} \leq \frac{1}{c_0} \|\varphi\|_h. \quad (13)$$

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Consequently,

$$\|\varphi\|_h = \left(\sum_{i=1}^{N-1} h |\varphi_i|^2 \right)^{1/2} \leq Ch^2. \quad (14)$$

By applying ineq. (11) to the finite difference scheme (12):

$$\|u - U\|_{1,h} = \|e\|_{1,h} \leq \frac{1}{c_0} \|\varphi\|_h. \quad (13)$$

It remains to bound $\|\varphi\|_h$. We showed that, if $u \in C^4([0, 1])$, then

$$\varphi_i = u''(x_i) - D_x^+ D_x^- u(x_i) = \mathcal{O}(h^2),$$

i.e. there exists a positive constant C , independent of h , such that

$$|\varphi_i| \leq Ch^2.$$

Consequently,

$$\|\varphi\|_h = \left(\sum_{i=1}^{N-1} h |\varphi_i|^2 \right)^{1/2} \leq Ch^2. \quad (14)$$

Combining the inequalities (13) and (14), it follows that

$$\|u - U\|_{1,h} \leq \frac{C}{c_0} h^2. \quad (15)$$

In fact, a more careful treatment of the remainder term in the Taylor series expansion on p.4 reveals that

$$\varphi_i = u''(x_i) - D_x^+ D_x^- u(x_i) = -\frac{h^2}{12} u^{IV}(\xi_i), \quad \xi_i \in [x_{i-1}, x_{i+1}].$$

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in inequality (14). Recalling that $c_0 = (1 + c_*)^{-1}$ and $c_* = 1/2$, we deduce that $c_0 = 2/3$. Substituting the values of the constants C and c_0 into inequality (15) it follows that

$$\|u - U\|_{1,h} \leq \frac{1}{8} h^2 \|u^{IV}\|_{C([0,1])}.$$

Thus we have proved the following result.

Theorem

Let $f \in C([0, 1])$, $c \in C([0, 1])$, with $c(x) \geq 0$ for all $x \in [0, 1]$, and suppose that the corresponding (weak) solution of the boundary-value problem (1) belongs to $C^4([0, 1])$; then

$$\|u - U\|_{1,h} \leq \frac{1}{8} h^2 \|u^{IV}\|_{C([0,1])}. \quad (16)$$

Some general observations

The analysis of the finite difference scheme (3) contains the key steps of a general error analysis for finite difference approximations of (elliptic) partial differential equations:

Some general observations

The analysis of the finite difference scheme (3) contains the key steps of a general error analysis for finite difference approximations of (elliptic) partial differential equations:

Consider the finite difference scheme:

$$\begin{aligned}\mathcal{L}_h u &= f_h, & \text{in } \Omega_h, \\ \mathcal{B}_h u &= g_h, & \text{on } \Gamma_h.\end{aligned}$$

(1) The first step is to prove the stability of the scheme in an appropriate mesh-dependent norm. A typical stability result for a finite difference scheme is

$$\|U\|_{\Omega_h} \leq C_1 (\|f_h\|_{\Omega_h} + \|g_h\|_{\Gamma_h}), \quad (17)$$

where $\|\cdot\|_{\Omega_h}$, $\|\cdot\|_{\Omega_h}$ and $\|\cdot\|_{\Gamma_h}$ are mesh-dependent norms involving mesh-points of Ω_h (or $\overline{\Omega_h}$) and Γ_h , respectively, and C_1 is a positive constant, independent of h .

(2) The second step is to estimate the size of the *consistency error*,

$$\begin{aligned}\varphi_{\Omega_h} &:= \mathcal{L}_h u - f_h, & \text{in } \Omega_h, \\ \varphi_{\Gamma_h} &:= \mathcal{B}_h u - g_h, & \text{on } \Gamma_h.\end{aligned}$$

(in the case of the finite difference scheme (1) $\varphi_{\Gamma_h} = 0$, and therefore φ_{Γ_h} never appeared explicitly in our error analysis).

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$$\|\varphi_{\Omega_h}\|_{\Omega_h} + \|\varphi_{\Gamma_h}\|_{\Gamma_h} \rightarrow 0 \quad \text{as } h \rightarrow 0,$$

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for a sufficiently smooth solution u of the boundary-value problem, we say that the scheme is *consistent*. If p is the largest positive integer such that

$$\|\varphi_{\Omega_h}\|_{\Omega_h} + \|\varphi_{\Gamma_h}\|_{\Gamma_h} \leq C_2 h^p \quad \text{as } h \rightarrow 0,$$

(where C_2 is a positive constant independent of h) for all sufficiently smooth u , the scheme is said to have *order of accuracy* (or *order of consistency*) p .

The finite difference scheme is said to provide a *convergent* approximation to the solution u of the boundary-value problem in the norm $\|\cdot\|_{\Omega_h}$, if

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The finite difference scheme is said to provide a *convergent* approximation to the solution u of the boundary-value problem in the norm $||| \cdot |||_{\Omega_h}$, if

$$|||u - U|||_{\Omega_h} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

If q is the largest positive integer such that

$$|||u - U|||_{\Omega_h} \leq Ch^q \quad \text{as } h \rightarrow 0$$

(where C is a positive constant independent of the mesh-size h), then the scheme is said to have *order of convergence* q .

We deduce the following fundamental theorem.

Theorem

Suppose that the finite difference scheme is stable (i.e. the inequality (17) holds for all f_h and g_h and the corresponding numerical solution U) and that the scheme is a consistent approximation of the boundary-value problem; then the finite difference scheme is a convergent approximation of the boundary-value problem, and the order of convergence q is not smaller than the order of accuracy (order of consistency) p .

PROOF. We define the *global error* $e := u - U$. Then,

$$\mathcal{L}_h e = \mathcal{L}_h(u - U) = \mathcal{L}_h u - \mathcal{L}_h U = \mathcal{L}_h u - f_h.$$

Thus

$$\mathcal{L}_h e = \varphi_{\Omega_h},$$

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By stability of the scheme it then follows that

$$|||u - U|||_{\Omega_h} = |||e|||_{\Omega_h} \leq C_1(\|\varphi_{\Omega_h}\|_{\Omega_h} + \|\varphi_{\Gamma_h}\|_{\Gamma_h}),$$

and hence the stated result with $q \geq p$, thanks to the assumed consistency of order p of the scheme. \square

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and hence the stated result with $q \geq p$, thanks to the assumed consistency of order p of the scheme. \square

In other words,

stability + consistency \Rightarrow convergence.

This abstract result is at the heart of the convergence analysis of finite difference approximations of PDEs.