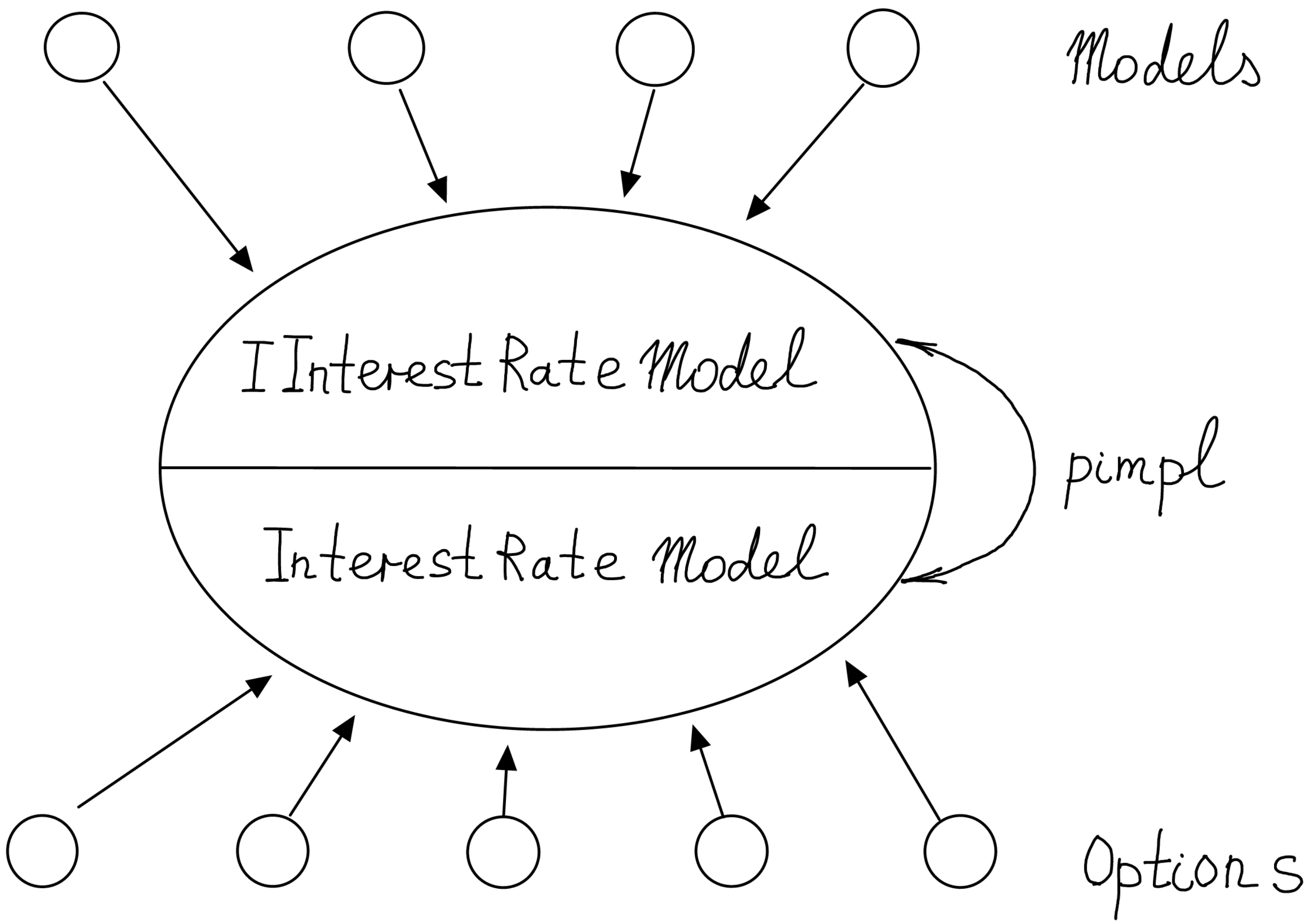
III : Standard and barrier options on interest rates.

Design :



Work with InterestRateModel :

Step 1 : Construct event times

$$\underset{\text{initial time}}{t_0} \quad \underbrace{t_1, \dots, t_M}_{\text{depend on options}}$$

Goal: make # of event times as small as possible

Step 2:

(a) Basic payoffs (Slices)

(i) cash amounts

(ii) discount factors

$$B(i, T)$$

maturity (any time)

index of event time

(b) Operations on Slices:

(i) At given event time: everything.

(ii) Between event times: rollback.

Interest Rate Standard: cap.

Cap = sequence of caplets.

Caplet paid at $t_{n+1} = t_0 + (n+1)\delta t$ is computed at t_n and is given by

$$\underbrace{N}_{\text{notional}} * \max\left(\underbrace{L(t_n, t_{n+1})}_{\text{LIBOR}} \underbrace{\delta t}_{\text{period}} - \underbrace{R}_{\text{cap rate}} \delta t, 0 \right)$$

Inputs:

N : notional

δt : period between payments

M : # of payments

R : cap rate

t_0 : initial time

Event times:

$$\underbrace{t_0}_{\text{initial time}} \quad \underbrace{t_1 \ \dots \ t_{M-1}}_{\text{payment times except the last}}$$

$$t_m = t_0 + m\,\delta t, \quad m = 1, \dots, M-1$$

Algorithm :

Multiply on notional at the end.

1 Boundary condition

$$X_{t_{M-1}} = \max\Big(1 - \underbrace{B(t_{M-1}, t_M)}_{\text{discount factor}} * (1 + R\,\delta t),\ 0\Big)$$

2 Loop $\underset{\text{end}}{t_0} \longleftarrow \underset{\text{begin}}{t_{M-1}}$

$$X_{t_m}\,(?) \longleftarrow X_{t_{m+1}}\ (\text{known})$$

// $X_{t_{m+1}}$: value to continue (value of caplets paid $> t_{m+1}$)

$$\underset{(> t_{m+1})}{X_{t_m}} = \mathcal{R}_{t_m}\Big(\underset{(> t_{m+1})}{X_{t_{m+1}}}\Big)$$

$$\underset{(>t_m)}{X_{t_m}} \mathrel{+}= \max\left(1 - B(t_m, t_{m+1}) * (1 + R\,\delta t),\ 0\right)$$

③ After loop

$$\text{return} \quad \underset{(>t_0)}{X_{t_0}} * \overset{\text{notional}}{N}. \qquad \blacksquare$$

Interest Rate Standard: swaption.

Inputs:

Swap parameters:

{

N : notional

δt : period

R : swap rate

M : # of payments

bPayFloat : if "true" we pay float,
if "false" we pay fixed

}

T : maturity for swaption

t_0 : initial time, $t_0 < T$

Event times: (t_0, T)

Algorithm

$$X(T) = \max\left(0, \text{swap}(T)\right)$$

$$X(t_0) = \mathcal{R}_{t_0}(X(T))$$

Here swap (T) is the value of swap contract issued at T with given parameters and pre-fixed swap rate R.

To compute swap(T) assume first that

$$bPayFloat = \text{"true"}$$

Then

$$\begin{aligned} swap(T) &= couponBond(T) \\ &\quad - savingsAccount(T) \end{aligned}$$

where

(a) couponBond (T) = the value at issue time T of coupon bond paying coupon

$$C = N R \delta t$$

at $t_m = T + m\delta t$, $m = 1, \dots, M$,

+ notional payment N at t_M.

We deduce that

$$\text{coupon Bond}(T) = C \sum_{m=1}^{M} B(T, t_m) + N\, B(T, t_M)$$

(b) savings Account (T) = value at issue time T of float interest payments

LIBOR

$$N\, L(t_{m-1}, t_m)\, \delta t$$

at $t_m = T + m\delta t$, $m = 1, \dots, M$

+ notional payment N at T.

By replication,

$$\text{savings Account}(T) = N.$$

```
if (bPayFloat = false)
{
        swap(T) *= -1;
}
```

Interest Rate Standard:

Futures on LIBOR.

At maturity T the futures price $F(T)$ is given by

LIBOR

$$F(T) = 1 - L(T, T+\Delta)$$

Cash flow:

long		short
0	$\longrightarrow$	$F(t_{n+1}) - F(t_n)$
t_n		t_{n+1}

Inputs:

- Δ: LIBOR period
- M: # of futures times
- T: maturity
- t_0: initial time

Event times: $t_0 \; t_1 \; \ldots \; t_M = T$

$$t_m = t_0 + m\,\delta t \quad , \quad m = 1, \ldots, M$$

$$\delta t = \frac{T - t_0}{M}$$

(1) Boundary condition

$$F(T) = 1 - L(T, T+\Delta)$$

We compute LIBOR from

$$(1 + L(T, T+\Delta)\Delta)\, \underbrace{B(T, T+\Delta)}_{\text{discount factor}} = 1$$

(2) Loop $\underset{\text{end}}{t_0} \longleftarrow \underset{\text{begin}}{t_M}$

// $F_{t_{m+1}}$: futures price at t_{m+1}

$$F_{t_m} (?) \longleftarrow F_{t_{m+1}} \text{ (known)}$$

$$F_{t_n} = \frac{1}{B(t_n, t_{n+1})} \mathcal{R}_{t_n}\left(F_{t_{n+1}}\right)$$

because

$$0 = \mathcal{R}_{t_n}\left(F_{t_{n+1}} - F_{t_n}\right)$$

$$= \mathcal{R}_{t_n}\left(F_{t_{n+1}}\right) - B(t_n, t_{n+1}) F_{t_n}$$

(3) After loop.

return F_{t_0}. $\blacksquare$

Interest Rate Standard:

Cancellable Collar.

Collar = Cap − Floor

Caplet paid at t_{m+1} =

$$N \max\left(L(t_m, t_{m+1})\,\delta t - C\,\delta t,\ 0\right)$$

notional LIBOR at t_m for t_{m+1} period cap rate

Floret paid at t_{m+1} =

$$N \max\left(F\,\delta t - L(t_m, t_{m+1})\,\delta t,\ 0\right)$$

floor rate

Inputs:

N: notional

M: # of payments

δt: period

C : cap rate

F : floor rate

t_0 : initial time = issue time

<u>Event times</u>: $t_0 \underbrace{t_1 \ \dots \ t_{M-1}}_{\text{payment times except the last}}$

$$t_m = t_0 + m\delta t, \quad m = 1, \dots, M-1$$

<u>Algorithm</u>:

We multiply on notional at the end.

1. Boundary condition

$$X_{t_{M-1}} = \max\big(1 - \underbrace{B(t_{M-1}, t_M)}_{\text{discount factor}}(1 + C\delta t), 0\big) - \max\big(B(t_{M-1}, t_M)(1 + F\delta t) - 1, 0\big)$$

2. Loop $\underset{\text{end}}{t_0} \longleftarrow \underset{\text{begin}}{t_{M-1}}$

$$X_{t_m}(?) \longleftarrow X_{t_{m+1}} \text{ (known)}$$

// $X_{t_{m+1}}$: value to continue (value of payments $> t_{m+1}$ if no exercise $\leq t_{m+1}$)

$$X_{t_{m+1}} = \max(0, X_{t_{m+1}})$$

$$X_{t_m} = \mathcal{R}_{t_m}(X_{t_{m+1}})$$

$$X_{t_m} \mathrel{+}= \max\left(1 - B(t_m, t_{m+1})(1 + C\delta t), 0\right)$$

$$X_{t_m} \mathrel{-}= \max\left(B(t_m, t_{m+1})(1 + F\delta t) - 1, 0\right)$$

3 After loop

$$X_{t_0} \mathrel{*}= N$$

return X_{t_0} $\blacksquare$

Interest Rate Standard:

Down-out Cap

Inputs:

N : notional

δt : period between payments

M : # of payments

C : cap rate

K : lower barrier for LIBOR.

t_0 : initial time

Event times: $t_0 \underbrace{t_1 \ldots t_{M-1}}_{\text{payment times except the last}}$

$$t_m = t_0 + m\delta t, \quad m = 1, \ldots, M-1.$$

Algorithm:

We multiply on notional at the end.

(1) Boundary condition

$$X_{t_{M-1}} = \max\left(1 - B(t_{M-1}, t_M) * (1 + c\,\delta t),\ 0\right)$$

(2) Loop $\quad t_0 \longleftarrow t_{M-1}$ begin

end

$$X_{t_m}\ (?) \longleftarrow X_{t_{m+1}}\ (\text{known})$$

// $X_{t_{m+1}}$: value to continue (value of caplets $> t_{m+1}$ if barrier was not crossed $\leq t_{m+1}$)

$$X_{t_{m+1}}^{*} = 1_{\{U \geq B(t_{m+1}, t_{m+2})\}}$$

where U is the upper barrier for discount factors:

$$U = \frac{1}{1 + K\,\delta t}$$

$$X_{t_m} = \mathcal{R}_{t_m}\left(X_{t_{m+1}} \right)$$

$$X_{t_m} \mathrel{+}= \max\left(1 - B(t_m, t_m + \delta t) * (1 + C\delta t), 0 \right)$$

(3) After loop.

$$X_{t_0} \mathrel{*}= N$$

return X_{t_0}. $\blacksquare$