IV : Path-dependent options

Step 1: Specify event times and create standard model.

Event times: $\underbrace{t_0}_{\text{initial time}} \; \underbrace{t_1 \ldots t_M}_{\text{should include reset times}}$

Step 2 Extend the model

$$\underbrace{X}_{\text{original state process}} \longrightarrow (X, \underbrace{Y}_{\text{additional state process}})$$

Additional component Y is defined by

(a) Origin (value before the first reset)

(b) Indexes of reset times (that is, indexes of event times when Y changes)

(c) One-step evolutions at reset times:

$$Y_{t_{i+1}} = G_{i+1}\Big(X_{t_{i+1}}, \underset{\substack{\| \\ Y_{t_i}}}{y}\Big)$$

(value before reset)

<u>Step 3</u> : write algorithm.

(a) Basic payoffs (Slices)

(i) From standard implementation (cash, spot, discount, forward)

(ii) Path-dependent component Y.

(b) Operations on Slices:

(i) At given event time: everything.

(ii) Between event times: rollback.

Asset Path-Dependent : up-down-out

Inputs :

L : lower barrier

U : upper barrier

$(t_i)_{i=1,\dots,M}$: barrier times

N : notional

t_0 : initial time

Event times : $t_0 \ \underbrace{t_1 \dots t_M}_{\text{barrier times}}$

Path-dependent state process :

$$Y_{t_m} = \prod_{i=1}^{m} 1\{S_{t_i} \geq L\} \, 1\{U \geq S_{t_i}\}$$

Y_{t_m} = indicator of no barrer event $\leq t_m$.

(a) Origin : $Y_{t_0} = 1$

(b) Reset indices : $\{1, \ldots, M\}$

(c) One-step evolution :

$$Y_{t_m} \text{ (known)} \longrightarrow Y_{t_{m+1}} \text{ (?)}$$

$$Y_{t_{m+1}} = Y_{t_m} * 1\{S_{t_{m+1}} \geq L\} * 1\{\mathcal{T} \geq S_{t_{m+1}}\}$$

before reset

<u>Algorithm</u> :

$$X_{t_M} = N * Y_{t_M}$$

$$X_{t_0} = \mathcal{R}_{t_0}(X_{t_M}) \qquad \blacksquare$$

Asset Path-Dependent: Asian call

Inputs:

K: strike

T: maturity, $T > t_M$

$(t_i)_{i=1,\dots,M}$: average times

t_0: initial time, $t_0 < t_1$

Event times: $t_0 \ \underbrace{t_1 \dots t_M}_{\text{reset times}} \ T$

Path-dependent state process:

Y: historical average

$$Y_{t_m} = \frac{1}{m} \sum_{i=1}^{m} S_{t_i}$$

S_{t_i} stock price at t_i

(a) Origin:

$$Y_{t_0} = 0 \quad \text{(or anything)}$$

(b) reset indices : $\{1, \dots, M\}$

(c) one-step evolution

$$Y_{t_m} \text{ (known)} \longrightarrow Y_{t_{m+1}} \text{ (?)}$$

$$Y_{t_{m+1}} = \underbrace{\frac{1}{m+1}}_{\substack{\#\text{ of reset times}\\ \leq t_{m+1}}} \big(\underbrace{m}_{\#\text{ of reset times } < t_{m+1}} Y_{t_m} + S_{t_{m+1}} \big)$$

Algorithm :

$$X_T = \max(Y_T - K, 0)$$

$$X_{t_0} = \mathcal{R}_{t_0}(X_T) \qquad \blacksquare$$

Interest Rate Path Dependent:

savings account.

Inputs:

N : notional

δt : period

M : # of periods

t_0 : initial time

Event times:

$t_0 \; t_1 \ldots t_M$

$$t_m = t_0 + m\,\delta t$$

Path-dependent state process:

Y_{t_m} : capital on savings account at $t_m + \delta t$.

(a) Origin:

$$Y_{t_0-} = N$$

(b) Reset indices : $\{0, \ldots, M\}$

(c) One-step evolution :

$$Y_{t_m} \text{ (known)} \longrightarrow Y_{t_{m+1}} \text{ (?)}$$

$$Y_{t_{m+1}} = Y_{t_m} \,/\, B(t_{m+1}, t_{m+1} + \delta t)$$

discount factor

<u>Algorithm</u> :

$$X_{t_M} = Y_{t_M} * B(t_M, t_M + \delta t)$$

$$X_{t_0} = \mathcal{R}_{t_0}(X_{t_M}) \qquad \blacksquare$$

Interest Rate Path Dependent:

put on savings account

Inputs:

N : notional

δt : period

R : fixed rate (strike)

M : # of payments

t_0 : initial time

Event times: $t_0 \ \underbrace{t_1 \dots t_{M-1}}$

$t_m = t_0 + m\delta t$ payment times except the last one

Path-dependent state process:

Y_{t_m} : capital on savings account at $t_m + \delta t$.

(a) Origin: $Y_{t_0 -} = N$

(b) Reset indices : $\{0, \ldots, M-1\}$

(c) One-step evolution :

$$Y_{t_m} \text{ (known)} \longrightarrow Y_{t_{m+1}} \text{ (?)}$$

$$Y_{t_{m+1}} = Y_{t_m} \,/\, B(t_{m+1}, t_{m+1} + \delta t)$$

discount factor

<u>Algorithm</u> :

$F(t_M) = N(1 + R\delta t)^N$:

capital at t_M of account paying fixed rate R

$$X(t_{M-1}) = B(t_{M-1}, t_{M-1} + \delta t) * \max\left(F(t_M) - Y_{t_{M-1}}, 0\right)$$

$$X(t_0) = \mathcal{R}_{t_0}\left(X_{t_{M-1}}\right) \qquad \blacksquare$$