**HT 2020** **C8.4 Probabilistic Combinatorics** **Problem Sheet 0**

**Hints/Solutions**

---

1. (a) Apply the mean value theorem to $f(x) = e^x - 1 - x$; (b) use (a); (c) expand $e^k$. For (d) use (c); and note also that, if $0 \leqslant j < k \leqslant n$, then $\frac{n-j}{k-j} \geqslant \frac{n}{k}$.

2. (a) For $k$ fixed, $f(n) = \Theta(g(n))$ (since $f(n) \leqslant g(n)$ and $f(n) \geqslant g(n)/k^k$ for $n \geqslant k$ by 1(d)). For $k = k(n) \to \infty$, $f(n) = o(g(n))$ (since $f(n) \leqslant g(n)/k!$).

   (b) $f(n) = o(g(n))$ (take logs and note $\log\log n = o(\log n)$). This is just a version of 'exponentials grow faster than powers'.

3. Using part of 1 (d), for the given $n$,

$$\binom{n}{k} 2^{1-\binom{k}{2}} \leqslant \left(\frac{en}{k}\right)^k 2^{1-\binom{k}{2}} \leqslant \left(\frac{e \cdot e^{-1} k 2^{\frac{k}{2}}}{k}\right)^k 2^{1-\frac{k^2}{2}+\frac{k}{2}} = 2^{1+\frac{k}{2}}$$

   and the last term is $o(n)$ as $k \to \infty$.

4. Colour the edges of $K_n$ independently, each red with probability $p$ and blue otherwise. Let $X$ be the number of red $K_k$s and $Y$ the number of blue $K_\ell$s. Find $\mathbb{E}[X + Y]$ and use the fact that $\mathbb{P}(X + Y \leqslant \mathbb{E}[X + Y]) > 0$. N.B. It's not enough to argue that the events $A = \{X \leqslant \mathbb{E}[X]\}$ and $B = \{Y \leqslant \mathbb{E}[Y]\}$ both have positive probability!

5. Pick a 3-colouring of the vertices uniformly at random. Call an edge $e$ *bad* if $e$ gets at most 2 colours. Then $\mathbb{P}(e \text{ is bad}) \leqslant 3\left(\frac{2}{3}\right)^r$, and so the expected number of bad edges is $< 1$. (The result is ok even if $r$ is 1 or 2, since then $H$ has no edges.)

6. If $F$ is finite, pick uniformly at random a $0, 1$ string of length $t$, where $t \geqslant \max_i c_i$. Let $A_i$ be the event that the initial $c_i$ bits form the $i$th codeword. Then $\mathbb{P}(A_i) = 2^{-c_i}$. But the events are disjoint, so …

   $F$ may be infinite; the same argument works using a random infinite sequence. (Or note that its enough to prove the final bound for all finite subsets of $F$.)

*Bonus question (for MFoCS students)*

7. Consider the first displayed equation (0.1). Fix a realisation (i.e., an outcome, i.e., a point $\omega$ in the probability space $\Omega$ we are working in). Let $K$ be the set of $i$ such that $A_i$ holds, and let $k = |K|$.

   Suppose $k \geqslant 1$. LHS is 0. RHS is

$$\sum_{r=0}^{k}(-1)^r S_r = \sum_{r=0}^{k}(-1)^r \sum_{A\subseteq K, |A|=r} 1 = \sum_{r=0}^{k}(-1)^r \binom{k}{r} = (1-1)^k = 0.$$

   Suppose $k = 0$. LHS is 1. RHS is $(-1)^0 S_0 = 1$.

   Thus (0.1) holds, and taking expectations gives (0.2).

   For the alternating inequalities, again consider the RHS in (0.1). Arguing as before, it suffices to check alternating inequalities for $\sum_{r\geqslant 0}(-1)^r\binom{k}{r}$. If $k = 0$, the LHS is 1 and $\sum_{r=0}^{m}$ is 1 for each $m \geqslant 0$. Suppose that $k \geqslant 1$, so the LHS is 0.

   If $m \geqslant k$ then $\sum_{r=0}^{m}(-1)^r\binom{k}{r} = 0$.

   **Method 1.** Let $0 \leqslant m \leqslant (k+1)/2$. For $r \leqslant (k+1)/2$, $\binom{k}{r}$ increases, and so $\sum_{r=0}^{m}(-1)^r\binom{k}{r}$ is $\geqslant 0$ for $m$ even and $\leqslant 0$ for $m$ odd, as required.

   Let $(k+1)/2 < m < k$. We may use

$$\sum_{r=0}^{m}(-1)^r\binom{k}{r} = -\sum_{r=m+1}^{k}(-1)^r\binom{k}{r} = -(-1)^k \sum_{s=0}^{k-m-1}(-1)^s\binom{k}{s}$$

   (setting $s = k - r$) to see from the previous case that the alternating inequalities hold for such $m$.

   **Method 2.** (The slick way.) Notice that

$$\sum_{r=0}^{m}(-1)^r\binom{k}{r} = (-1)^m\binom{k-1}{m},$$

   which easily follows from $\binom{k}{r} = \binom{k-1}{r-1} + \binom{k-1}{r}$.

   Either way, we have the alternating inequalities for $\sum_{r\geqslant 0}(-1)^r S_r$ in (0.1), and taking expectations gives the corresponding result for (0.2).

**If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk**