First-moment method:

1. Show that for $r \geqslant 2$, any graph G contains an r-partite subgraph H with $e(H) \geqslant \frac{r-1}{r} e(G)$.

2. A *dominating set* in a graph $G = (V, E)$ is a set $U \subseteq V$ such that every vertex $v \in V$ is either in U or has a neighbour in U.

   Suppose that $|V| = n$ and that G has minimum degree $\delta \geqslant 2$. Choose a subset X of V by including each vertex independently with probability p. Let Y be the set of all vertices which are not in X and which have no neighbour in X.

   Show that $\mathbb{E}[|X \cup Y|] \leqslant np + ne^{-p(\delta+1)}$. What can you say about the set $X \cup Y$?

   By optimizing over p, show that the graph G has a dominating set which contains at most $n\frac{1+\log(\delta+1)}{\delta+1}$ vertices.

3. For $n, r \in \mathbb{N}$, $1 < r < n$, let $z(r, n)$ be the largest possible number of 0 entries in an $n \times n$ matrix which has no $r \times r$ submatrix whose entries are all 0. (Here a submatrix is obtained by selecting any r rows and any r columns; the rows/columns need not be consecutive.)

   Consider a random matrix in which each entry is 0 with probability p and 1 with probability $1-p$, independently. What is the expected number of entries which are 0? What is the expected number of $r \times r$ submatrices which are "all 0"?

   Deduce that $z(r, n) > pn^2 - p^{r^2} n^{2r}$.

   Optimize over p to find the best lower bound on $z(r, n)$ that you can, for fixed r and large n.

4. Let G be a graph with n vertices, and let d_v denote the degree of vertex v.

   (i) Consider a random ordering of $V = V(G)$ (chosen uniformly from all $n!$ possibilities). What is the probability that v precedes all its neighbours in the ordering?

   (ii) Show that G has an independent set of size at least $\sum_{v \in V} \frac{1}{d_v+1}$.

   (iii) Deduce that any graph with n vertices and m edges has an independent set of size at least $\frac{n^2}{2m+n}$.

5. (Harder!) Let G be a bipartite graph with n vertices. Suppose each vertex v has a list $S(v)$ of more than $\log_2 n$ colours associated to it. Show that there is a proper colouring of G in which each vertex v receives a colour from its list $S(v)$.

PTO

Second-moment method and thresholds:

6. Let $p = p(n) = \frac{\log n + f(n)}{n}$. Show that if $f(n) \to \infty$ then the probability that $G(n,p)$ contains an isolated vertex tends to 0, and that if $f(n) \to -\infty$ then this probability tends to 1. [*Hint. Apply the first and second moment methods to the number X of isolated vertices. We may assume (why?) that $|f(n)|$ is not too large, say $|f(n)| \leqslant \log n$. Also, it may be useful to note that $1 - p = e^{-p+O(p^2)}$ as $p \to 0$.*]

   (This shows in particular that $p^*(n) = \log n/n$ is a threshold function for $G(n,p)$ to have minimum degree at least 1.)

7. Let $S_{n,p}$ be a random subset of $\{1, 2, \dots, n\}$ chosen by including each element independently with probability p.

   (i) Show that $p = n^{-2/3}$ is a threshold function for the property "$S_{n,p}$ contains an arithmetic progression of length 3".

   (ii) Show that for $k \geqslant 3$ fixed, $p = n^{-2/k}$ is a threshold function for $S_{n,p}$ to contain an arithmetic progression of length k.

Asymptotics:

8. Use Stirling's Formula to show that if $n, k \to \infty$ with $k^2 = o(n)$, then

$$\binom{n}{k} \sim \frac{1}{\sqrt{2\pi k}} \left(\frac{en}{k}\right)^k.$$

Bonus questions (MFoCS students should try these, optional for others):

The r^{th} (falling) *factorial moment* of a random variable X is defined to be

$$\mathbb{E}_r[X] = \mathbb{E}[X(X-1)\cdots(X-r+1)].$$

9. Let X be a random variable taking values in $\{0, 1, \dots, n\}$. Show that

$$\mathbb{P}(X = 0) = \sum_{r=0}^{n} (-1)^r \frac{\mathbb{E}_r[X]}{r!},$$

   and that the sum satisfies the alternating inequalities. [Hint: write X as a sum of indicator variables.]

   For you to think about: what can you say if X is unbounded, taking values in the non-negative integers?

10. In the setting of the previous question, state and prove an analogous formula for $\mathbb{P}(X = k)$ in terms of factorial moments.

If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk