Second-moment method and thresholds:

1. Show that all (finite) graphs of the following types are strictly balanced: complete graphs, cycles, trees, complete bipartite graphs, connected regular graphs.

   Give an example of a graph that is balanced but not strictly balanced.

2. Let T be a tree with k vertices. What is the threshold for $G(n,p)$ to contain a copy of T?

   Show that, for each fixed k, there is a function $p(n)$ such that the probability that $G(n,p(n))$ has a component of size exactly k tends to 1 as $n \to \infty$.

3. What is the threshold for $G(n,p)$ to contain a cycle of length k, for fixed k?

   Find a threshold function for the property of containing a cycle. [*Careful! It doesn't quite follow immediately from the first part.*]

The Local Lemma:

4. Show that it is possible to colour the edges of K_n with $k = \lceil 3\sqrt{n} \rceil$ colours so that no triangle has all its edges the same colour.

5. Let $H = (V, E)$ be a hypergraph. Suppose the vertices are k-coloured uniformly at random; each $v \in V$ receives each colour with probability $1/k$, and the colours of different vertices are independent. For $e \in E$, let A_e be the event that edge e is monochromatic.

   Show that if $|e \cap f| \leqslant 1$, then A_e and A_f are independent.

   Is it true that A_e is independent of the collection $\{A_f : |e \cap f| \leqslant 1\}$?

6. A k-SAT (more properly k-CNF) formula is an expression such as (for $k = 3$)

   $$(x_1 \text{ OR } x_4 \text{ OR } \overline{x_6}) \text{ AND } (\overline{x_1} \text{ OR } x_2 \text{ OR } x_5) \\ \text{ AND } (\overline{x_2} \text{ OR } \overline{x_3} \text{ OR } \overline{x_4}) \text{ AND } (\overline{x_4} \text{ OR } \overline{x_5} \text{ OR } x_6),$$

   where the variables x_i take values TRUE or FALSE, $\overline{x_i}$ means NOT x_i, and k variables or their negations are ORd together in each clause. A formula is *satisfiable* if there is an assignment of values to the variables making the expression true.

   (a) Use the first-moment method to show that each k-SAT formula with fewer than 2^k clauses is satisfiable.

   (b) Use the Symmetric Local Lemma to show that each k-SAT formula in which no variable lies in more than $2^{k-2}/k$ clauses is satisfiable.

7. Let $G = (V, E)$ be a graph with maximum degree Δ, and let $V_1, V_2, \ldots, V_s$ be a collection of disjoint subsets of V, each of size $k \geqslant 2e\Delta$. Show that G has an independent set which contains a vertex from each set V_i. [There is a hint over the page.]

8. Let $G = (V, E)$ be a graph, and suppose that for each $v \in V$ there is a list $S(v)$ of at least $2er$ colours, where r is a positive integer. Suppose also that for each $v \in V$ and each $c \in S(v)$, there are at most r neighbours u of v such that $c \in S(u)$.

   Prove that there is a proper colouring of G under which each vertex v receives a colour from its list $S(v)$.

9. Let $W(k)$ be the smallest integer n such that any two-colouring of the set $\{1, 2, \ldots, n\}$ contains a monochromatic arithmetic progression of length k.

   (a) Use the first-moment method to show that $W(k) \geqslant 2^{k/2}$.

   (b) Use the Local Lemma to show that $W(k) \geqslant \frac{2^k}{2ek}\big(1 + o(1)\big)$.

Bonus questions (compulsory for MFoCS students, optional for others):

10. Let $X_1, X_2, \ldots$ be a sequence of random variables each taking non-negative integer values, and let $k \geqslant 0$ be fixed. Suppose that for each $r \geqslant 0$ we have $\lim_{n\to\infty} \mathbb{E}_r[X_n] = \lambda_r < \infty$, and that $\lambda_{r+k}/r! \to 0$ as $r \to \infty$. show that

$$\mathbb{P}(X_n = k) \to \frac{1}{k!}\sum_{r=0}^{\infty}(-1)^r \frac{\lambda_{r+k}}{r!}.$$

   [*Hint: Use a result from last time. Be careful exchanging sums and limits! You may wish to just/first do the case* $k = 0$.]

   In applications we often have a limiting distribution X in mind; this result shows that under certain assumptions, if the moments of X_n tend to those of X, then $\mathbb{P}(X_n = k) \to \mathbb{P}(X = k)$.

11. Let $c > 0$ be constant. Suppose that $p = p(n)$ satisfies $n^4p^6 \to c$ as $n \to \infty$, and let X_n denote the number of copies of K_4 in $G(n, p)$. Show that $\mathbb{P}(X_n > 0) \to 1 - e^{-c/24}$ as $n \to \infty$.

   [*Hint: use the result of the previous question.*]

   What can you say about the distribution of the number of copies of K_4? Can you generalize this to (certain) other graphs?

Hint for Question 7: pick one vertex X_i from each V_i. You could consider a 'bad' event E_{uv} for each edge uv with u and v in different sets V_i.

If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk